§3 Factorization theorem of Poincaré polynomials

$R \subset V$: reduced irreducible root sys. of V. $\ell := \operatorname{rank} R = \dim V$

$W = W(R) \subset GL(V)$: Weyl group

$\Delta = \{\alpha_1, \dots, \alpha_\ell\} \subset R$: simple roots. $s_i := s_{\alpha_i} \in W$

$\Rightarrow W = \langle s_1, \dots, s_\ell \rangle_{\mathrm{grp}} \subset GL(V)$

$$s_{\alpha_i}(v) = v - \langle v, \alpha_i^\vee \rangle \alpha_i = v - \frac{2(v,\alpha_i)}{(\alpha_i,\alpha_i)}\alpha_i$$

Dfn. $w \in W$ $\ell(w) := \min\{k \mid \exists\, w = s_{i_1} \cdots s_{i_k}\}$: length of w

$w = s_{i_1} \cdots s_{i_{\ell(w)}}$: minimal presentation of w □

E.g. $R = A_2 \supset \Delta = \{\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_2 - \varepsilon_3\}$

$W = \mathfrak{S}_3 \ni s_1 = (12),\ s_2 = (23)$

$\ell(e) = 0,\ \ell((12)) = \ell(23) = 1,\ \ell(123) = \ell(s_1 s_2) = 2 = \ell(132)$

$(13) = s_1 s_2 s_1 = s_2 s_1 s_2 \quad \ell(13) = 3$ □

$\Pi := R \cap \mathbb{N}\Delta = \{\alpha = \sum_{i=1}^{\ell} c_i \alpha_i \in R \mid c_i \in \mathbb{N}\}$

: positive roots (w.r.t. Δ)

Fact. $R = \Pi \sqcup (-\Pi)$ □

E.g. $A_2 = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\} \sqcup \{-\alpha_1, -\alpha_2, -\alpha_1 - \alpha_2\}$ □

Thm 1. $W(t) := \sum_{w \in W} t^{\ell(w)}$: Poincaré polynom. of W.

$$W(t) = \prod_{\alpha \in \Pi} \frac{1 - t^{1 + ht(\alpha)}}{1 - t^{ht(\alpha)}}$$

但し $\Pi \ni \alpha = \sum_{i=1}^{\ell} c_i \alpha_i$ に対して $ht(\alpha) = \sum_{i=1}^{\ell} c_i$ □

E.g. $R = A_2$ $W(t) = 1 \times t^0 + 2 \times t^1 + 2 \times t^2 + 1 \times t^3 = 1 + 2t + 2t^2 + t^3$

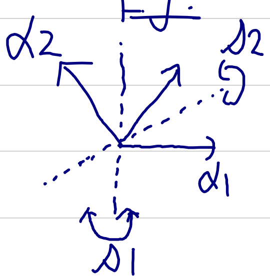
$$= (1+t)(1+t+t^2) = \frac{1-t^2}{\underset{\alpha_1}{1-t}} \, \frac{1-t^2}{\underset{\alpha_2}{1-t}} \, \frac{1-t^3}{\underset{\alpha_1+\alpha_2}{1-t^2}}$$

Proof by I.G. Macdonald, "The Poincaré series of a Coxeter group" Math. Ann. 199 (1972), 161-174.

Dfn. $w \in W$ $\quad \Pi(w) := \Pi \cap w^{-1}(-\Pi)$ $\square$

Fact. $\#\Pi(w) = \ell(w)$ $\square$

Eg. $R = A_2$. $\Pi(e) = \Pi \cap (-\Pi) = \phi$. $\quad \#\Pi(e) = 0 = \ell(e)$



$\Pi(s_1) = \Pi \cap (-\{-\alpha_1, \alpha_1+\alpha_2, \alpha_2\}) = \{\alpha_1\}$ $\quad \#\Pi(s_1) = 1 = \ell(s_1)$

$\Pi(132) = \Pi(s_2 s_1) = \Pi \cap (-\{-(\alpha_1+\alpha_2), \alpha_1, -\alpha_2\}) = \{\alpha_1+\alpha_2, \alpha_2\}$

$\Pi(13) = \Pi(s_1 s_2 s_1) = \Pi \cap (-\{-\alpha_2, -\alpha_1, -(\alpha_1+\alpha_2)\}) = \Pi$ $\square$

Thm. 2. [Macdonald]

$$\sum_{w \in W} \prod_{\alpha \in \Pi} \frac{1 - u_\alpha e^{-w\alpha}}{1 - e^{-w\alpha}} = \sum_{w \in W} \prod_{\alpha \in \Pi(w)} u_\alpha$$

- $\{u_\alpha \mid \alpha \in R\}$: set of commuting variables
- for $v_1, v_2 \in V$ $\quad e^{v_1+v_2} = e^{v_1} e^{v_2}$ $\square$

Eg. $R = A_2$ $\quad \Pi = \{\alpha_1, \alpha_2, \alpha_3 = \alpha_1 + \alpha_2\}$

$u_i := u_{\alpha_i}$, $\quad x := e^{\alpha_1}$ $\quad y := e^{\alpha_2}$

$\text{LHS} = \sum_{w \in S_3} f(w)$

$$f(e) = \frac{1-u_1 e^{-\alpha_1}}{1-e^{-\alpha_1}} \frac{1-u_2 e^{-\alpha_2}}{1-e^{-\alpha_2}} \frac{1-u_3 e^{-\alpha_3}}{1-e^{-\alpha_3}} = \frac{1-u_1/x}{1-1/x} \frac{1-u_2/y}{1-1/y} \frac{1-u_3/(xy)}{1-1/xy}$$

$$f(s_1) = \frac{1-u_1 e^{\alpha_1}}{1-e^{\alpha_1}} \frac{1-u_2 e^{-\alpha_3}}{1-e^{-\alpha_3}} \frac{1-u_3 e^{-\alpha_2}}{1-e^{-\alpha_2}} = \frac{1-u_1 x}{1-x} \frac{1-u_2/xy}{1-1/xy} \frac{1-u_3/y}{1-1/y}$$

$$f(s_2) = \frac{1-u_1 e^{-\alpha_3}}{1-e^{-\alpha_3}} \frac{1-u_2 e^{\alpha_2}}{1-e^{\alpha_2}} \frac{1-u_3 e^{-\alpha_1}}{1-e^{-\alpha_1}} = \frac{1-u_1/xy}{1-1/xy} \frac{1-u_2 y}{1-y} \frac{1-u_3/x}{1-1/x}$$

$$f(s_1 s_2) = \frac{1-u_1 e^{-\alpha_2}}{1-e^{-\alpha_2}} \frac{1-u_2 e^{\alpha_3}}{1-e^{\alpha_3}} \frac{1-u_3 e^{\alpha_1}}{1-e^{\alpha_1}} = \frac{1-u_1/y}{1-1/y} \frac{1-u_2 xy}{1-xy} \frac{1-u_3 x}{1-x}$$

$$f(s_2 s_1) = \frac{1-u_1 e^{\alpha_3}}{1-e^{\alpha_3}} \frac{1-u_2 e^{-\alpha_1}}{1-e^{-\alpha_1}} \frac{1-u_3 e^{\alpha_2}}{1-e^{\alpha_2}} = \frac{1-u_1 xy}{1-xy} \frac{1-u_2/x}{1-1/x} \frac{1-u_3 y}{1-y}$$

$$f(s_1 s_2 s_1) = \frac{1-u_1 e^{\alpha_2}}{1-e^{\alpha_2}} \frac{1-u_2 e^{\alpha_1}}{1-e^{\alpha_1}} \frac{1-u_3 e^{\alpha_3}}{1-e^{\alpha_3}} = \frac{1-u_1 y}{1-y} \frac{1-u_2 x}{1-x} \frac{1-u_3 xy}{1-xy}$$

$$\begin{aligned}
\mathrm{LHS} \times (1-x)(1-y)(1-xy) &= (u_1 - x)(u_2 - y)(u_3 - xy) \\
&+ (1-u_1 x)(u_2 - xy)(u_3 - y) \\
&+ (u_1 - xy)(1-u_2 y)(u_3 - x) \\
&+ (u_1 - y)(1-u_2 xy)(1-u_3 x) \\
&+ (1-u_1 xy)(u_2 - x)(1-u_3 y) \\
&+ (1-u_1 y)(1-u_2 x)(1-u_3 xy)
\end{aligned}$$

1: $u_1u_2u_3 + u_2u_3 + u_1u_3 + u_1 + u_2 + 1 = \mathrm{RHS}$

x^1: $-u_2u_3 - u_1u_2u_3 - u_1 - u_1u_3 - 1 - u_2 = -\mathrm{RHS}$

y^1: $\cdots = -\mathrm{RHS}$

$$\begin{aligned}
x=1:\ & \left.\begin{aligned}&(u_1-1)(u_2-y)(u_3-y) \\ &+ (1-u_1)(u_2-y)(u_3-y)\end{aligned}\right\} = 0 \\
& \left.\begin{aligned}&+ (u_1-y)(1-u_2 y)(u_3-1) \\ &+ (u_1-y)(1-u_2 y)(1-u_3)\end{aligned}\right\} = 0 \\
& \left.\begin{aligned}&+ (1-u_1 y)(u_2-1)(1-u_3 y) \\ &+ (1-u_1 y)(1-u_2)(1-u_3 y)\end{aligned}\right\} = 0 \\
& \qquad = 0
\end{aligned}$$

$y=1$: $\cdots = 0$ $\Rightarrow$ $\mathrm{LHS} = (1-x)(1-y)\cdot \mathrm{RHS} \cdot (1 + c\cdot xy)$

x^2y^2: $-1 - u_1 - u_2 - u_2u_3 - u_1u_3 - u_1u_2u_3 = -\mathrm{RHS}$ $\therefore c = -1$ $\square$

Thm. 2 $\Rightarrow$ Thm. 1 は $\forall \alpha \in R$ $u_\alpha = t$ とすると示せる. (次回)