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§2. continued

$R \subset V$: reduced irreducible root sys. of rank ℓ $(= \dim V)$

$W \subset GL(V)$: Weyl group

$d_1, \dots, d_\ell \in \mathbb{Z}_{>0}$: degrees of W

Prp. $w \in W < GL(V)$ $\quad \det(1 - tw) = (1 - a_1 t) \cdots (1 - a_\ell t)$, $a_i \in \mathbb{C}$

$$\frac{1}{|W|} \sum_{w \in W} \frac{1}{\det(1 - tw)} = \prod_{i=1}^{\ell} \frac{1}{1 - t^{d_i}} \qquad \square$$

Cor. $d_1 + \cdots + d_\ell = \ell + \frac{1}{2}|R|$, $\quad d_1 \cdots d_\ell = |W|$ $\qquad \square$

$\Delta = \{\alpha_1, \dots, \alpha_\ell\} \subset R$: simple roots $\quad s_i := s_{\alpha_i} \in W$

$c_\Delta := s_1 \cdots s_\ell \in W$: <u>Coxeter elem.</u> assoc. to Δ

Lem. $\forall \Delta, \Delta' \subset R$ $\quad c_\Delta$ and $c_{\Delta'}$ are W-conjugate $\quad \square$

order of the element c_Δ of the group W

Dfn. $h = h(R) := \mathrm{ord}(c_\Delta) \in \mathbb{Z}_{>1}$: Coxeter # of R

$\mathrm{EigenVal}(c_\Delta) = \{\zeta_h^{m_1}, \dots, \zeta_h^{m_\ell}\} \subset \mathbb{C}$ $\quad \zeta_h = \exp(2\pi\sqrt{-1}/h)$

$0 \leqq m_1 \leqq \cdots \leqq m_\ell < h$: <u>exponents</u> of R

E.g. $R = A_\ell$, $\Delta = \{\alpha_1 = \varepsilon_1 - \varepsilon_2, \dots, \alpha_\ell = \varepsilon_\ell - \varepsilon_{\ell+1}\} \subset V \subsetneq \mathbb{R}^{\ell+1}$

$V = \left\{ \begin{pmatrix} v_1 \\ \vdots \\ v_{n+1} \end{pmatrix} \,\middle|\, v_1 + \cdots + v_{n+1} = 0 \right\}$

$GL(V) \hookrightarrow GL(\mathbb{R}^{\ell+1})$

$c_\Delta \mapsto P = \begin{bmatrix} 0 & & & 1 \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}$

$\ell = 1$: $c_\Delta = s_1 \mapsto \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}$. $\quad \ell = 2$: $c_\Delta = s_1 s_2 = \begin{bmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 0 & 1 \\ & 1 & 0 \end{bmatrix} = \begin{bmatrix} & 0 & 1 \\ 1 & 0 & \\ 0 & 1 & \end{bmatrix}$

$\therefore$ $h = \mathrm{ord}(P) = \ell + 1$

$\det(t - P) = t^{\ell+1} - 1$ $\qquad \mathrm{EigenVal}(P) = \{\zeta_{\ell+1}^{i} \mid i = 0, \dots, \ell\}$

eigenvec. $\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \notin V$

$\therefore$ exponents of A_ℓ

$= 1, 2, \dots, \ell$ $\quad (m_i = i)$

Thm. (1) $h = |R|/\ell$

(2) $m_i > 0 \quad \forall i,$

$\{h - m_i\}_{i=1}^{\ell} = \{m_i\}_{i=1}^{\ell}$ (so $\sum_{i=1}^{\ell} m_i = \ell h/2$)

$m_1 = 1, \quad m_\ell = h-1$

(3) $d_i = m_i + 1 \quad \forall i.$ (so $|W| = \prod_{i=1}^{\ell} (m_i + 1)$) $\square$

E.g. $R = A_\ell$

$h = \ell + 1 \qquad |R| = \#\{\varepsilon_i - \varepsilon_j \mid 1 \le i \ne j \le \ell+1\} = (\ell+1)\cdot\ell = h\ell$

$\{d_1, \dots, d_\ell\} = \{2, 3, \dots, \ell+1\} = \{1+1, 2+1, \dots, \ell+1\}$

$\prod_{i=1}^{\ell} (m_i + 1) = (\ell+1)! = |\mathfrak{S}_{\ell+1}|$ $\square$