2022 . 5 . 6

§2. W-invariant polynomials

$R \subset V$: reduced irreducible root sys. of type X_ℓ $\leftarrow A_\ell \sim G_2$, $\ell = \mathrm{rank}\, R = \dim V$

$W := \langle s_\alpha \mid \alpha \in R \rangle_{grp} \subset \mathrm{Aut}(V, (\cdot|\cdot))$: Weyl group, fin

$\leftarrow$ isometry group for the inner product $(x|y) := \sum_{\alpha \in R} \langle x, \alpha^\vee \rangle \langle y, \alpha^\vee \rangle$

$W \curvearrowright V$, $\rightsquigarrow$ $W \curvearrowright V^* := \mathrm{Hom}(V, \mathbb{R})$

g $\quad v \mapsto g.v$ $\qquad f \mapsto (g.f)(v) := f(g^{-1}.v)$

$S(V^*)$: symmetric algebra of V^*

- $S(V^*) = \bigoplus_{n \geq 0} S^n(V^*)$

  $S^n(V^*) = (\underbrace{V^* \otimes \cdots \otimes V^*}_{n})/\mathfrak{S}_n$ $\leftarrow$ permuting tensor factors, as $\mathbb{R}$-linear space.

  $x_1, \ldots, x_\ell$: basis of V^* $\quad (\ell = \dim V^* = \dim V)$

  $S^n(V^*) \ni \overline{x_{i_1} \otimes \cdots \otimes x_{i_n}} =: x_{i_1} \cdots x_{i_n}$ form a basis of $S^n(V^*)$

  $\dim S^n(V^*) = \binom{n+\ell-1}{\ell-1}$

  $S^0(V^*) = \mathbb{R}$

- multiplication

  $\bullet : S^m(V^*) \times S^n(V^*) \to S^{m+n}(V^*)$

  $x_{i_1} \cdots x_{i_m} \bullet x_{j_1} \cdots x_{j_n} = x_{i_1} \cdots x_{i_m} x_{j_1} \cdots x_{j_n}$

- $(S(V^*), \bullet, 1)$ is a commutative $\mathbb{R}$-alg. with $\mathbb{N}$-grading

  $\uparrow \mathbb{R} = S^0(V^*)$

$S(V^*) \cong \mathbb{R}[x_1, \ldots, x_\ell]$ as $\mathbb{R}$-alg.

$S^n(V^*) \cong \{\text{polynom. of deg. } n\}$

$W \curvearrowright V^* \rightsquigarrow W \curvearrowright S(V^*)$

$g.(x_{i_1} \cdots x_{i_n}) := (g.x_{i_1}) \cdots (g.x_{i_n})$

$S(V^*)^W := \{ f \in S(V^*) \mid \forall g \in W,\ g.f = f \}$: subalg. of $S(V^*)$
: W-inv. polynomial ring

Thm. [Chevalley] $\exists f_1, \dots, f_\ell \in S(V^*)^W$,
homogeneous, alg. independent & generate $S(V^*)^W$. $\square$

Prp. $\{\deg f_i \mid i = 1, \dots, \ell\}$ is indep. of the choice of generators $\{f_1, \dots, f_\ell\}$ $\square$

Dfn. $d_i := \deg f_i$, $d_1 \leq \cdots \leq d_\ell$: degree of $W = W(R)$. $\square$

E.g. $R = A_\ell$. $W = \mathfrak{S}_{\ell+1} \curvearrowright V = \left\{ \begin{pmatrix} a_1 \\ \vdots \\ a_{\ell+1} \end{pmatrix} \in \mathbb{R}^{\ell+1} \;\middle|\; a_1 + \cdots + a_{\ell+1} = 0 \right\}$

$\rotatebox{90}{$\in$}$ $g : \varepsilon_i \mapsto \varepsilon_{g(i)}$

$\cup$
$R = \{ \varepsilon_i - \varepsilon_j \mid 1 \leq i \neq j \leq \ell+1 \}$
$\cup$
$\Delta = \{ \alpha_1 = \varepsilon_1 - \varepsilon_2, \dots, \alpha_\ell = \varepsilon_\ell - \varepsilon_{\ell+1} \}$

$x_i := \varepsilon_i^* \in (\mathbb{R}^{\ell+1})^*$

$S(V^*) \twoheadleftarrow \mathbb{R}[x_1, \dots, x_{\ell+1}] = S((\mathbb{R}^{\ell+1})^*) \curvearrowleft W$

$\| \quad \mathbb{R}[x_1, \dots, x_{\ell+1}]/(x_1 + \cdots + x_{\ell+1})$ $\qquad x_i \mapsto x_{g^{-1}(i)}$

$S(V^*)^W$ = symmetric polynomial ring of $x_1, \dots, x_{\ell+1}$ mod $x_1 + \cdots + x_{\ell+1}$

(1) elementary sym. polynom.
$e_r := \sum_{1 \leq i_1 < \cdots < i_r \leq \ell} x_{i_1} \cdots x_{i_r}$ $\quad (= e_1 = h_1 = p_1)$
$(e_1 = x_1 + \cdots + x_\ell,\ e_2 = x_1 x_2 + \cdots + x_{\ell-1} x_\ell, \dots, e_\ell = x_1 \cdots x_\ell)$

(2) complete sym. polynom.
$h_r := \sum_{1 \leq i_1 \leq \cdots \leq i_r \leq \ell} x_{i_1} \cdots x_{i_r}$ $\quad , \dots$
$(h_1 = e_1,\ h_2 = x_1^2 + x_1 x_2 + \cdots,\ h_\ell = x_1^\ell + x_1^{\ell-1} x_2 + \cdots)$

(3) power-sum. sym. polynom. $\qquad$ degrees $= 2, 3, \dots, \ell+1$
$p_r := \sum_{i=1}^{\ell} x_i^\ell$ $\qquad$ $d_i = i+1$ $(1 \leq i \leq \ell)$ $\square$