V: $\mathbb{R}$-lin. sp. $V^* = \mathrm{Hom}(V, \mathbb{R})$

$\langle -,- \rangle : V \times V^* \to \mathbb{R}$, $\langle v, f \rangle := f(v)$

Dfn. $R \subset V$, subset, is a <u>root system</u> of V

$:\Leftrightarrow$ (RS1) $\# R < \infty$, $0 \notin R$, $\mathrm{span}_{\mathbb{R}} R = V$

(RS2) $\forall \alpha \in R$ $\exists! \alpha^\vee \in V^*$ s.t. $\langle \alpha, \alpha^\vee \rangle = 2$ and

$s_\alpha(R) = R$ w/ $s_\alpha \in \mathrm{End}(V)$,

$s_\alpha(v) := v - \langle v, \alpha^\vee \rangle \alpha$

(RS3) $\forall \alpha, \beta \in R$ $\langle \alpha, \beta^\vee \rangle \subset \mathbb{Z}$ $\square$

Dfn. & Rmk.

- $\alpha \in R$: root, $\mathrm{rank}\, R := \dim V$
- s_α is a reflection, i.e. $V = V^+ \oplus V^-$, $\begin{cases} V^- = \mathbb{R}\alpha, & s_\alpha(\alpha) = -\alpha \\ s_\alpha|_{V^+} = \mathrm{id}_{V^+} \end{cases}$

  Also, $s_\alpha \in A(R) := \{ a \in \mathrm{Aut}(V) \mid a(R) = R \} \subset \mathrm{Aut}(R)$
- $W(R) := \langle s_\alpha \mid \alpha \in R \rangle_{\text{gp.}} \subset A(R)$: <u>Weyl group</u> $\square$

Lem.4. $(\cdot|\cdot) : V \times V \to \mathbb{R}$ $\quad (x|y) := \sum_{\alpha \in R} \langle x, \alpha^\vee \rangle \langle y, \alpha^\vee \rangle$

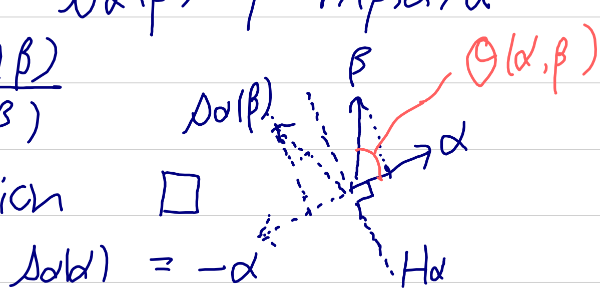
; positive (non-deg.) sym. bilin. form

; $A(R)$-inv. $[:\Leftrightarrow \forall a \in A(R)\ (a(x)|a(y)) = (x|y)]$ $\square$

$\alpha, \beta \in R$ $\quad n(\alpha, \beta) := \langle \alpha, \beta^\vee \rangle \in \mathbb{Z}$ RS3

$n(\alpha, \alpha) = 2$, $s_\alpha(\beta) = \beta - n(\beta, \alpha)\alpha$

Lem.5 $\quad n(\alpha, \beta) = \dfrac{2(\alpha|\beta)}{(\beta|\beta)}$



∵ s_α is a reflection $\square$

<u>Cor.</u> $n(\alpha,\beta)\cdot n(\beta,\alpha) = 4\cos^2\theta(\alpha,\beta) \in \{0,1,2,3,4\}$

$|\alpha| := \sqrt{(\alpha|\alpha)}$

$n(\alpha,\beta)$	$n(\beta,\alpha)$	$\theta(\alpha,\beta)$	length	$\mathrm{ord}(s_\alpha s_\beta)$	
0	0	$\pi/2$		2	$A_1 \times A_1$
1	1	$\pi/3$	$\lvert\alpha\rvert = \lvert\beta\rvert$	3	A_2
-1	-1	$2\pi/3$	"	3	
1	2	$\pi/4$		4	B_2
-1	-2	$3\pi/4$		4	
1	3	$\pi/6$		6	G_2
-1	-3	$5\pi/6$		6	
2	2	$\alpha = \beta$			non-reduced
-2	-2	$\alpha = -\beta$			
1	4	$\beta = 2\alpha$			
-1	-4	$\beta = -2\alpha$			