2022 · 4 · 8

Bourbaki "Lie groups and Lie algebras" Chap. 6

$\leftarrow \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$

$V$ : $\mathbb{R}$-lin. sp. $\quad V^* := \mathrm{Hom}(V, \mathbb{R})$ : dual sp. of $V$

$\langle -,- \rangle : V \times V^* \to \mathbb{R}$, $\quad \langle v, f \rangle := f(v)$ bilinear form

$\leftarrow \mathrm{End}_{\mathbb{R}}(V) = \mathrm{Hom}_{\mathbb{R}}(V, V)$

$(a, f) \in V \times V^*$; $a \neq 0, f \neq 0$. $\quad s_{a,f} \in \mathrm{End}(V)$

$$s_{a,f}(v) := v - \langle v, f \rangle a = v - f(v)a$$

Lem. 1 $\quad \mathrm{rank}(1 - s_{a,f}) = 1$

$\uparrow$ $\mathrm{id}_V \in \mathrm{End}(V)$

(:) $\quad \forall v \in V \quad (1 - s_{a,f})(v) = v - (v - f(v)a) = f(v)a$

$\therefore \mathrm{Im}(1 - s_{a,f}) = \mathbb{R}a \neq 0 \qquad \square$

Lem. 2 $\quad (s_{a,f})^2 = 1 \iff \langle a, f \rangle = 2$

(:) $\quad (s_{a,f})^2(v) = s_{a,f}(v - f(v)a) = (v - f(v)a) - f(v - f(v)a)a$

$= v - 2f(v)a + f(v)f(a)a = v + f(v)(f(a) - 2)a$

$\therefore (s_{a,f})^2 = 1 \iff f(v)(f(a) - 2) = 0 \ \forall v \in V \iff f(a) = 2 \qquad \square$

Dfn $\quad s \in \mathrm{Aut}(V)$ is a reflection $:\iff \mathrm{rank}(1 - s) = 1$ & $s^2 = 1 \quad \square$

$\uparrow$ $\mathrm{Aut}_{\mathbb{R}}(V) = \{ a \in \mathrm{End}(V) \mid \text{invertible} \}$

By Lem. 1 & 2, $\quad \langle a, f \rangle = 2 \Rightarrow s_{a,f}$ is a reflection

Lem. 3 $\quad s$ : reflection of $V \Rightarrow V = V_s^+ \oplus V_s^-$, $\dim V_s^- = 1$,

$V_s^{\pm} := \{ v \in V \mid s(v) = \pm v \}$

(:) $\quad s^2 = 1 \Rightarrow \mathrm{EigenVal}(s) = \{\pm 1\}$, $V = \mathrm{EigenSp}(s; 1) \oplus \mathrm{EigenSp}(s; -1)$

$V_s^{\pm} = \mathrm{EigenSp}(s; \pm 1) \quad \therefore V = V_s^+ \oplus V_s^-$

$V_s^+ = \mathrm{Ker}(1 - s)$, $\quad V/\mathrm{Ker}(1-s) \xrightarrow{\sim} \mathrm{Im}(1-s) \quad \therefore \dim V_s^-$

$\qquad \qquad \qquad \qquad \| \qquad \qquad \qquad \qquad \qquad \qquad = \dim \mathrm{Im}(1-s)$

$\qquad \qquad \qquad V/V_s^+ \ \simeq \ V_s^- \qquad \qquad \qquad = \mathrm{rk}(1 - s) = 1$

<u>Dfn</u> $R \subset V$, subset, is a <u>root system</u> of $V$

$\iff$ (RS1) $\#R < \infty$, $0 \notin R$, $\mathrm{span}_{\mathbb{R}} R = V$

(RS2) $\forall \alpha \in R$ $\exists \alpha^\vee \in V^*$ ← reflection

s.t. $\langle \alpha, \alpha^\vee \rangle = 2$ and $s_{\alpha, \alpha^\vee}(R) = R$

← this condition determines $\alpha^\vee$ uniquely

(RS3) $\forall \alpha \in R$ $\alpha^\vee(R) \subset \mathbb{Z}$ □

- $\alpha \in R$ is called a <u>root</u>. $\mathrm{rank}\, R := \dim V$
- $\forall \alpha \in R$ $s_\alpha := s_{\alpha, \alpha^\vee}$ : the <u>simple reflection</u> assoc. to $\alpha$

By Lem. 3. $V = V_{s_\alpha}^+ \oplus V_{s_\alpha}^-$, $V_{s_\alpha}^- = \mathbb{R}\alpha$,

$s_\alpha(v) = v$ for $v \in V_{s_\alpha}^+$, $s_\alpha(\alpha) = -\alpha$

Also, $s_\alpha \in A(R) := \{a \in \mathrm{Aut}(V) \mid a(R) = R\} \subset \mathrm{Aut}(R) \cong \mathfrak{S}_{\#R}$ ← symmetric group

- $W(R) := \langle s_\alpha \mid \alpha \in R \rangle_{\text{gp.}} \subset A(R)$ : <u>Weyl group</u>

<u>Claim 1.</u> A root system $R$ of rank 1 is of the form

$$R = \{\pm\alpha\} \quad \text{or} \quad R = \{\pm\alpha, \pm 2\alpha\}$$

(∵) $V = \mathbb{R}e$ $V^* = \mathbb{R}e^*$, $\langle e, e^* \rangle = e^*(e) = 1$

(RS1) $\iff$ $R = \{a_1 e, \dots, a_n e\}$, $a_i \in \mathbb{R} \setminus \{0\}$. $a_i \neq a_j$ for $i \neq j$.

(RS2) $\Rightarrow$ $R \ni \alpha_i = a_i e$ $\alpha_i^\vee = (2/a_i)e^*$

(RS3) $\iff$ $\forall i, j$ $\alpha_i^\vee(\alpha_j) \in \mathbb{Z}$ $\iff$ $\forall i, j$ $2a_j / a_i \in \mathbb{Z}$

$\iff$ $\forall i \neq j$ $2a_i / a_j, 2a_j / a_i \in \mathbb{Z}$

$\iff$ $\forall i \neq j$ $a_i / a_j = \pm 1, \pm 2, \pm \frac{1}{2}$

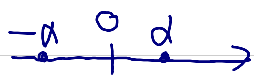
(RS2) $\Rightarrow$ $s_\alpha(\alpha) \in R$ $\iff$ $-\alpha \in R$

$\Rightarrow$ $R = \{\pm a_1, \dots, \pm a_m\}$

$\therefore$ $R = \{\pm\alpha\}$ or $R = \{\pm\alpha, \pm 2\alpha\}$ or $R = \{\pm\alpha, \pm\frac{1}{2}\alpha\}$ □

**Dfn.** A root sys. $R$ is reduced $:\Leftrightarrow$ $\forall \alpha \in R$ is indivisible,
i.e., $\nexists \beta \in R \setminus \{\pm\alpha\}$ $\alpha \in \mathbb{Z}\beta$ $\square$

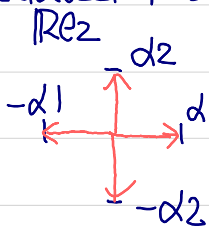
By Claim 1, a reduced root sys. of rank 1 is of the form
$\{\pm\alpha\}$ ($A_1$ root sys.)



$V = \mathbb{R}e$

Claim 2. A reduced root sys. of rank 2 is of the form

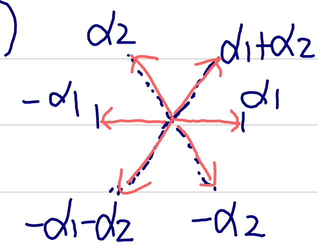
$(A_1 \times A_1)$




$V = \mathbb{R}e_1 \oplus \mathbb{R}e_2$

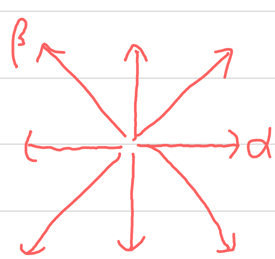
$R = \{\pm\alpha_1, \pm\alpha_2\}$
$\langle \alpha_1, \alpha_2^\vee \rangle = 0$

$(A_2)$



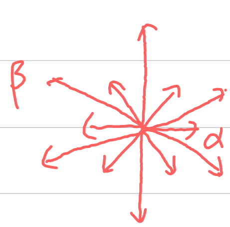

$R = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1+\alpha_2)\}$
$\langle \alpha_1, \alpha_2^\vee \rangle = -1$

$(B_2)$




$R = \{\pm\alpha, \pm\beta, \pm(\beta+\alpha), \pm(\beta+2\alpha)\}$
$\langle \alpha, \beta^\vee \rangle = -1$ $\langle \beta, \alpha^\vee \rangle = -2$

$(G_2)$




$R = \{\pm\alpha, \pm\beta, \pm(\beta+\alpha), \pm(\beta+2\alpha), \pm(\beta+3\alpha),$
$\pm(2\beta+3\alpha)\}$
$\langle \alpha, \beta^\vee \rangle = -1$, $\langle \beta, \alpha^\vee \rangle = -3$

HW show Claim 2.