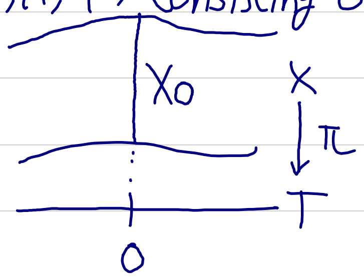
§ Deformations of schemes

$\mathbb{k}$: field (of char. $=0$)
X_0 : algebraic $\mathbb{k}$-scheme ($=$ scheme$/\mathbb{k}$, of fin type)
T : alg. $\mathbb{k}$-sch. $0 \in T$: closed pt.

<u>Dfn.</u> A deformation of X_0 over T is $X = (X, \pi, \varphi)$ consisting of

$$\begin{cases} X : \text{alg. } \mathbb{k}\text{-sch.} \\ \pi : X \longrightarrow T \quad : \text{flat surj.} \\ \varphi : X_0 \xrightarrow{\sim} \pi^{-1}(0) \end{cases}$$



<u>Dfn.</u> $T \overset{i}{\hookrightarrow} T'$ closed emb. of alg. $\mathbb{k}$-sch.
An extension of a deform. X_0 over T' is (X', ϕ) consisting of

$$\begin{cases} X', T' : \text{alg. } \mathbb{k}\text{-sch.} \\ \pi' : X' \longrightarrow T' : \text{flat surj.} \\ \phi : X \xrightarrow{\sim} X' \times_{T'} T = X' \times_{\pi', T', i} T \end{cases}$$

Two extensions $(X_1', \phi_1) \sim (X_2', \phi_2)$
$:\Leftrightarrow \exists\ X_1' \underset{\psi}{\xrightarrow{\sim}} X_2'$ isom. of T'-sch. s.t.

$$\begin{tikzcd} X \arrow[r, "id"] \arrow[d, "\phi_1"'] \arrow[dr, phantom, "\circlearrowleft"] & X \arrow[d, "\phi_2"] \\ X_1' \times_{T'} T \arrow[r, "\psi \times id_T"'] & X_2' \times_{T'} T \end{tikzcd}$$

ε : "dual number" $\mathbb{k}[\varepsilon] := \mathbb{k}[\varepsilon]/(\varepsilon^2)$, $D_\varepsilon^{(1)} := \operatorname{Spec} \mathbb{k}[\varepsilon]$

$T[\varepsilon] := T \times_{\operatorname{Spec} \mathbb{k}} D_\varepsilon^{(1)}$ (for $T = \operatorname{Spec} A$, $T[\varepsilon] = \operatorname{Spec} A[\varepsilon]$
$A[\varepsilon] := A \otimes_{\mathbb{k}} \mathbb{k}[\varepsilon]$)

$\mathbb{k}[\varepsilon] \twoheadrightarrow \mathbb{k}$, $\varepsilon \mapsto 0$ induces cls. emb. $T \hookrightarrow T[\varepsilon]$

X : deform. of X_0 over T.
$D(X/T; T[\varepsilon]) := \{\text{extensions of } X \text{ over } T[\varepsilon]\}/\sim$

Prp. (並河「複素シンプレクティック代数多様体」命題1.3.2)

X_0 : reduced, X : deform. of X_0 over T

$$D(X/T ; T[\varepsilon]) = \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_{X/T}, \mathcal{O}_X)$$

$\Omega^1_{X/T}$: sheaf of $\mathcal{O}_X$-modules of relative Kähler diffs. (1-forms) [Hartshorne II.8]

(affine case) $X = \mathrm{Spec}\,B \to T = \mathrm{Spec}\,A \iff A \to B$ ring hom.

$\iff B$: A-alg.

$$\Omega^1_{B/A} := \left(\bigoplus_{b \in B} B \cdot db\right)/S \quad : B\text{-module}$$

$$S := \langle d(b+b') - db - db',$$
$$d(bb') - b \cdot db' - b' \cdot db,$$
$$d(ab) - a \cdot db \quad (b, b' \in B, a \in A) \rangle_{B\text{-mod.}}$$

$$d : B = \Omega^0_{B/A} \to \Omega^1_{B/A} \quad ; A\text{-mod. hom.}$$
$$b \mapsto db \qquad \text{(de Rham diff.)}$$

$d \in \mathrm{Der}_A(B, \Omega^1_{B/A})$, i.e. $d(bb') = b\,db' + b'\,db$

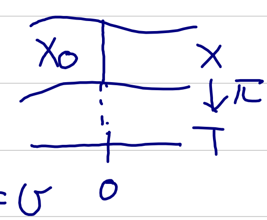
$(\Omega^1_{A/B}, d)$ is universal. i.e. $\forall M \in B\text{-Mod.}$

$$\mathrm{Hom}_{B\text{-Mod}}(\Omega^1_{B/A}, M) = \mathrm{Der}_A(B, M)$$
$$f \mapsto d' = f \circ d$$

$$\Theta_{X/T} := \mathcal{H}om_{\mathcal{O}_X}(\Omega^1_{X/T}, \mathcal{O}_X)$$

Cor. (Kodaira-Spencer map)

X_0, X/T : as in Prp.



$\forall\, v \in T_0 T =$ (tangent sp. of T at $0 \in T$)

$\rightsquigarrow \phi_v : D^{(1)} = \mathrm{Spec}\,\mathbb{k}[\varepsilon] \to T$, $0 \mapsto 0$, $(\phi_v)_*(\partial_\varepsilon) = v$

$\rightsquigarrow X_v := X \underset{\pi, T, \phi_v}{\times} D^{(1)}_\varepsilon$ is a deformation of X_0 over $D^{(1)}$

(1st order infinitesimal deform)

Prp.

$\therefore [X_v] \in D(X_0/\mathbb{k}, D^{(1)}) = \mathrm{Ext}^1_{\mathcal{O}_{X_0}}(\Omega^1_{X_0}, \mathcal{O}_{X_0}) \rightsquigarrow \kappa_{KS} : T_0 T \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_{X_0}}(\Omega^1_{X_0}, \mathcal{O}_{X_0})$

§ Functors of Artin rings [Schlessinger, 1968]

ring := com. unital assoc. ring

A ring is Artin if $\forall$descending chain of ideals stabilizes.

E.g. $\mathbb{K}, \mathbb{K}[\varepsilon], \mathbb{K}[h]/(h^m)$ $(m \in \mathbb{Z}_{>0})$, $\mathbb{K}[h_1, \dots, h_\ell]/(h_1, \dots, h_\ell)^m$

Fact. $\forall$ Artin ring $- = \oplus_{i=1}^{n} A_i$, $A_i = (A_i, m_i)$: local ring

m_i : nilpotent, $\forall N \in \mathbb{Z}_{>0}$ $\dim_{A_i/m_i}(A_i/m_i^N) < \infty$

$\mathbb{K}$: field

$\varphi : A \to B$, $\varphi(m_A) \subset m_B$

$Art_{\mathbb{K}}$: cat. of Artin local $\mathbb{K}$-alg. and local homs.

: in particular, an obj. is a local ring (A, m) w/ $A/m \cong \mathbb{K}$, m is nilpotent.

$\mathrm{Spec} A \ni O \leftrightsquigarrow m$

Dfn. A functor of Artin rings is $F : Art_{\mathbb{K}} \to Sets$ s.t. $F(\mathbb{K}) = \{pt.\}$

E.g. X_0 : alg. $\mathbb{K}$-sch.

$DX_0(A) := \{$ deformations of X_0 over $\mathrm{Spec} A \} / \sim$

We cannot expect representability of functors of Artin rings, but can expect pro-representability

w.r.t. m-adic top.

$\widehat{Art}_{\mathbb{K}}$: cat. of complete local $\mathbb{K}$-alg. w/ residue field $= \mathbb{K}$ and local homs.

So $\forall (R, m) \in \widehat{Art}_{\mathbb{K}}$, $\forall n \in \mathbb{N}$, $R_n := R/m^{n+1} \in Art_{\mathbb{K}}$, $R \cong \varprojlim_n R_n$

E.g. $\underbrace{\mathbb{K}[[h]]}_{m = (h)} = \varprojlim_{\ell} \mathbb{K}[h]/(h^\ell)$

For $R \in \widehat{Art}_{\mathbb{K}}$, $h_R := \mathrm{Hom}_{loc.\mathbb{K}\text{-alg.}}(R, -)$

Thm. [Schlessinger; 並河, 定理3.1.3] F: functor of Artin rings

(1) If F satisfies (H1)–(H3), then $\qquad R_n = R/m_R^{n+1}$

$\exists R \in \widehat{\mathrm{Art}}_k$, $\exists \xi = (\xi_n)_{n\in\mathbb{N}}$, $\xi_n \in F(R_n)$

s.t. $\phi_\xi : h_R \to F$, $(f : R \to A) \mapsto F(f_n)(\xi_n)$

$$\begin{array}{c} \Downarrow \\ \exists n \quad R \xrightarrow{f} A \\ (\because A:\mathrm{Artin}) \quad \searrow \; \circlearrowleft \; \nearrow \exists f_n \\ R_n \end{array}$$

(indep. of n)

satifies (i) ϕ_ξ smooth $\left(:\Longleftrightarrow \; \forall B \twoheadrightarrow A \text{ in } \mathrm{Art}_k \quad h_R(B) \to h_R(A) \times_{F(A)} F(B) \text{ surj.} \right)$

(ii) $\phi_\xi(k[\varepsilon]) : h_R(k[\varepsilon]) \to F(k[\varepsilon])$ bij.

R is called a projective hull of F.

ξ 〃 a formal semiuversal family of F.

(2) If F satisfies (H1)–(H4), then ϕ_ξ is isom.

R is called to pro-represent F,

ξ 〃 the formal universal family of F.

(H1) $\forall$ morph $A' \xrightarrow{\phi} A$ and $\forall$ small extension $B \xrightarrow{\psi} A$ in Art_k

$\Phi_{A',A,B} : F(A' \times_A B) \to F(A') \times_{F(A)} F(B)$ is surj.

↖ fiber prod in Art_k. $\{(a',b) \in A' \times B \mid \phi(a') = \psi(b)\}$

$\psi : B \to A$ in Art_k is a small extension

$:\Longleftrightarrow$ ψ surj. & $(\mathrm{Ker}\,\psi) \cdot m_B = 0$

<u>Fact.</u> $\forall$surj in Art_k is a fin. compos. of small extensions.

<u>E.g.</u> $J_\ell := k[h]/(h^{\ell+1})$ $\quad (J_2 \xrightarrow{\psi} k) = (J_2 \xrightarrow{\psi_2} J_1 \xrightarrow{\psi_1} k)$

$\mathrm{Ker}\,\psi_1 \cdot m_{J_1} = (h)\cdot(h) = 0$ in J_1

$\mathrm{Ker}\,\psi_2 \cdot m_{J_2} = (h^2)\cdot(h) = 0$ in J_2

$\mathrm{Ker}\,\psi \cdot m_{J_2} = (h)\cdot(h) \neq 0$

(H2) $\Phi_{A', \mathbb{k}, \mathbb{k}[\varepsilon]}$ is bij.
(H3) $\dim_{\mathbb{k}} F(\mathbb{k}[\varepsilon]) < \infty$
(H4) $\forall \varphi: B \to A$ small ext. in $\mathrm{Art}_{\mathbb{k}}$. $\Phi_{B, A, B}$ is bij.

<u>Thm.</u> [Schlessinger; 並河. 定理3.1.8]
X_0: alg. $\mathbb{k}$-sch. $\Rightarrow D_{X_0}$ satisfies (H1) and (H2)
If D_{X_0} satisfies (H3) and
(∗) $\forall \varphi: B \to A$ small ext. in $\mathrm{Art}_{\mathbb{k}}$.
$\forall \Upsilon$: deform. of X_0 over $\operatorname{Spec} B$
$X := \Upsilon \times_{\operatorname{Spec} B} \operatorname{Spec} A$: restriction of Υ to $\operatorname{Spec} A$
: deform. of X_0 over $\operatorname{Spec} A$
$\forall a \in \mathrm{Aut}(X; \mathrm{id}|_{X_0})$ lifts to $\exists \tilde{a} \in \mathrm{Aut}(\Upsilon; \mathrm{id}|_{X_0})$
rt $a \in \mathrm{Aut}\, X$, s.t.

$\pi: X \to T$, $0 \in T$

$$\begin{tikzcd} X_0 \arrow[r, "\mathrm{id}"] \arrow[d, "\phi"', "s"] & X_0 \arrow[d, "s"', "\phi"] \\ \pi^{-1}(0) \arrow[r, phantom, "\circlearrowright"] \arrow[d, equal] & \pi^{-1}(0) \arrow[d, equal] \\ X \times_T \operatorname{Spec} \mathbb{k} \arrow[r, "a \times_T \mathrm{id}"] & X \times_T \operatorname{Spec} \mathbb{k} \end{tikzcd}$$

<u>Rmk.</u> $\exists$ similar result for affine Poisson schemes
[並河. 定理3.1.9]