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Gerstenhaber 1963 "On the deformation of rings and algebras"

$\mathbb{k}$: field

$A = (V, \mu)$ assoc. $\mathbb{K}$-alg. $\dim V = n < \infty$. $\mu : V \otimes V \to V$ mult.

$(e_i)_{i=1}^n$: basis of V

$e_i e_j = \mu(e_i, e_j) = \sum_{k=1}^n c_{ij}^k e_k$ $\quad c_{ij}^k \in \mathbb{k}$ str const.

μ assoc. mult. $\iff (e_i e_j) e_k = e_i (e_j e_k)$

$\iff \sum_\ell c_{ij}^\ell c_{\ell k}^m = \sum_\ell c_{i\ell}^m c_{jk}^\ell \quad \forall m \quad (\#)$

"moduli space" of assoc. alg. str. on V $\mathcal{M} = S/G$

$S := \{ (c_{ij}^k)_{i,j,k} \in \mathbb{k}^{\oplus n^3} \mid (\#) \} \curvearrowleft \mathrm{Aut}_{\mathbb{k}}(\mathbb{k}^{\oplus n}) =: G$

"tangent space" $T_{[A]}\mathcal{M}$ at $[A] \in \mathcal{M}$

$\uparrow$ class of $A = (V, \mu) \in S$

consider assoc. mult. $*$ on $V \otimes \mathbb{k}[h]/(h^2) = V[h]/(h^2)$

extending $\cdot = \mu$

$$a * b = ab + h f(a, b) + O(h^2) \qquad a, b \in V,$$

$*$ assoc. $\iff (a*b)*c = a*(b*c)$ $\qquad f \in \mathrm{Hom}(V^{\otimes 2}, V)$

$\iff (ab + h f(a,b)) * c = a * (bc + h f(b,c))$ mod h^2

$\iff f(ab, c) + f(a,b)c = f(a, bc) + a f(b,c)$ $\quad (\#1)$

should identify $*$ and another $*$

which are equiv. under action of

$T = \mathrm{id} + h \cdot g \in \mathrm{Aut}(V[h]/(h^2))$

$g \in \mathrm{End}_{\mathbb{k}}(V)$

$*' \cong * :\Leftrightarrow a*'b = T^{-1}(T(a) * T(b))$

$\Leftrightarrow ab + h f'(a,b) = T^{-1}((a+h\cdot g(a)) * (b+h\cdot g(b)))$ mod h^2

$= T^{-1}(ab + h f(a,b) + h\, a\cdot g(b) + h\cdot g(a)b)$

$= ab + h[f(a,b) + g(a)b + a\, g(b)] - h\, g(ab)$

$\Leftrightarrow f'(a,b) - f(a,b) = a\, g(b) - g(ab) + g(a)b$ mod h^2

(#0)

$T_{[A]}M := \left\{ \begin{array}{l} \text{equiv. cls. of 1-st} \\ \text{order deform. of } \mu. \end{array} \right\} = \{ f \in Hom(V^{\otimes 2}, V) \mid (\#1) \}$

$/ \{ \mu\circ(id \otimes g) - g\circ\mu + \mu(g\otimes id) \mid g \in Hom(V, V) \}$

$= \operatorname{Ker} d_A^2 / \operatorname{Im} d_A^1$

$$Hom_{\mathbb{k}}(A, A) \xrightarrow[(\#0)]{d_A^1} Hom_{\mathbb{k}}(A^{\otimes 2}, A) \xrightarrow[(\#1)]{d_A^2} Hom(A^{\otimes 3}, A)$$

$d_A^1(g)(a,b) = g(a)b + a\, g(b) - g(ab)$

$d_A^2(f)(a,b,c) = a\cdot f(b,c) - f(ab,c) + f(a,bc) - f(a,b)c$

$\rightsquigarrow$ Hochschild cochain complex $C^*_{Hoch}(A) = C^*_{Hoch}(A, A)$

$C^n_{Hoch}(A) = Hom_{\mathbb{k}}(A^{\otimes n}, A)$, $d_A^n : C^n \to C^{n+1}$ $(n \geq 0)$

$(d_A^n f)(a_0, \dots, a_n) = a_0 f(a_1, \dots, a_n)$

$+ \sum_{i=1}^n (-1)^i f(a_0, \dots, a_{i-1}a_i, \dots, a_n)$

$+ (-1)^{n+1} f(a_0, \dots, a_{n-1}) a_n$

Hochschild cohomology $HH^n(A) := H^n(C^*_{Hoch}(A), d_A) = Z^n / B^n$

$HH^0(A) = Z^0 = \operatorname{Ker} d_A^0 = \{ a \in Hom(\mathbb{k}, A) = A \mid \forall b$

$b\cdot a - ab = 0 \} = Z(A)$

$HH^1(A) = \{ g \in Hom(A, A) \mid g(ab) = a\, g(b) + g(a)b \}$

$/ \{ [a, -] \in Hom(A, A) \}$

$= Der_{\mathbb{k}}(A, A) / \{\text{inner deriv.}\} \cong \{\text{exterior deriv.}\} = \Omega_{A/\mathbb{k}}^1$

$HH^2(A) = \{\text{equiv. cls. of 1st order deform.}\} = T_{[A]}\mathcal{M}$

<u>Slogan.</u> $HH^3(A) = \{\text{obstructions to deformations of } A\}$

Consider formal deformation of $\bullet = \mu$

$$a * b = ab + \sum_{n=1}^{\infty} h^n \mu_n(a,b) \qquad \mu_n \in \mathrm{Hom}(A^{\otimes 2}, A)$$

$$* \text{ assoc.} \iff \forall n \sum_{\substack{l+m=n \\ l,m\geq 0}} [\mu_l(\mu_m(a,b),c) - \mu_l(a,\mu_m(b,c))] = 0$$

$$\iff O_n(a,b,c) := \sum_{l+m=n,\ l,m>0} [\mu_l(\mu_m(a,b),c) - \mu_l(a,\mu_m(b,c))]$$

$$\forall n \quad O_n = d^2 \mu_n \quad (\#n)$$

$$n=2: \quad O_1(a,b,c) = \mu_1(\mu_1(a,b),c) - \mu_1(a,\mu_1(b,c)) = \mathrm{Associator}(\mu_1)$$

$$\therefore * \text{ assoc.} \Rightarrow O_1 \in B^3 = \mathrm{Im}\, d^2$$

$$\Rightarrow [O_1] = 0 \in HH^3(A)$$

$[O_1] \in HH^3(A)$: 1st obstruction element.

$\mathrm{Associator}(f)$ is quadratic in $f \in \mathrm{Hom}(A^{\otimes 2}, A)$

$\rightsquigarrow$ by linearization. $f, g \in \mathrm{Hom}(A^{\otimes 2}, A)$

define $[f,g] \in \mathrm{Hom}(A^{\otimes 3}, A)$ by

$$[f,g](a,b,c) = f(g(a,b),c) - f(a,g(b,c)) + g(f(a,b),c) - g(a,f(b,c))$$

(then $[f,f] = 2 \cdot \mathrm{Associator}(f)$)

Lem. $f, g \in Z^2 = \operatorname{Ker} d^2 \Rightarrow [f, g] \in Z^3$

$\because$ direct calculation.

Thm. [Gerstenhaber. §5]

If $\mu_1, \dots, \mu_{n-1}$ satisfy $(\#k)$ for $k \leq n-1$,

then $\mathcal{O}_n \in Z^3$, and

$[\mathcal{O}_n] = 0 \Leftrightarrow \exists \mu_n$, s.t. $(\#n)$ holds.

(i.e. n-th order deformation exists)

Cor. $\forall n \ [\mathcal{O}_n] = 0 \Rightarrow \exists$ formal deformation of A.

$HH^3(A) = 0 \Rightarrow$ 〃

[Schlessinger-Stasheff: 1985]

generalization of $[\cdot,\cdot]$

$f \in C^m = \mathrm{Hom}(A^{\otimes m}, A)$, $g \in \mathrm{Hom}(A^{\otimes n}, A)$

$1 \leq i \leq m$. $f \circ_i g \in C^{m+n-1}$

$(f \circ_i g)(a_1, \dots, a_{m+n-1}) := f(a_1, \dots, a_{i-1}, g(a_i, \dots, a_{i+n-1}), a_{i+n}, \dots, a_{m+n-1})$

$f \circ g := \sum_{i=1}^{m} (-1)^{(n-1)(i+1)} f \circ_i g$

$[f, g] := f \circ g - (-1)^{(m-1)(n-1)} g \circ f$

<u>Prp.</u> [G. 1963]

$\mathfrak{g}_{Hoch} := (\overline{C}^*_{Hoch}(A), [-,-], d_A)$: DGLA

$\overline{C}^m := C^{m+1}$

$$\begin{array}{ccc} [-,-] : \overline{C}^m \otimes \overline{C}^n & \longrightarrow & \overline{C}^{m+n} \\ \| & & \| \\ C^{m+1} \otimes C^{n+1} & & C^{m+n+1} \end{array}$$

<u>Lem.</u> $A = (V, \mu)$, $\nu \in C^2_{Hoch}(A)$

$\mu + \nu$ assoc. $\Longleftrightarrow$ $d_A \nu + \frac{1}{2}[\nu, \nu] = 0$

<u>Dfn.</u> $\mathfrak{g}$: DGLA $\quad (t) \subset \mathbb{K}[[t]]$ max. ideal

Sol. of Maurer-Cartan

$L := \mathfrak{g} \otimes (t) \subset \mathfrak{g} \otimes \mathbb{K}[[t]]$: DGLA

$MC(\mathfrak{g}) = \{ f \in L^1 \mid df + \frac{1}{2}[f, f] = 0 \}$

$L^1 = \mathfrak{g}^1 \otimes (t) \ni f = f_1 t + f_2 t^2 + \cdots$

$G(\mathfrak{g}) := \exp(L^0) = \exp(\mathfrak{g}^0 \otimes (t))$

acts on L^1 by

$\mu \in \mathfrak{g}^1$, $\ell \in L^1$, $g = \exp(x) \in L^1$

$$g.\ell = \ell + [x, \mu+\ell] + \frac{1}{2}[x,[x,\mu+\ell]] + \cdots + \frac{1}{n!}(\mathrm{ad}\,x)^n(\mu+\ell)$$
$$= \exp(x)\cdot(\mu+\ell)\cdot\exp(-x) - \mu$$

Lem: $G(\mathfrak{g}) \curvearrowright MC(\mathfrak{g})$

Dfn. $\mathrm{Def}(\mathfrak{g}) := |\,C(\mathfrak{g})/G(\mathfrak{g})$

… it is a nice "functor of Artin rings".

… can be regarded as moduli space of formal deformations controlled by $\mathfrak{g}$