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京都大学 数学教室 談話会

Geometric construction of derived Hall algebra

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available from https://www.math.nagoya-u.ac.jp/~yanagida/index-j.html

0 Introduction

The derived Hall algebra introduced by Toën (2006) is a version of Ringel-Hall algebra. Roughly it is a "Hall algebra for complexes".

In the case of ordinary Ringel-Hall algebra, we know Lusztig's geometric formulation using the theory of derived categories of constructible sheaves on moduli spaces of quiver representations, which are realized as Artin stacks.

I will explain a geometric formulation of derived Hall algebras using the theory of derived categories of constructible sheaves on moduli spaces of complexes of Quiver representations, which are realized as geometric derived stacks.

Based on my preprint
S. Yanagida, "Geometric derived Hall algebra", arXiv:1912.05442.

See also
柳田伸太郎, 「幾何学的導来Hall代数」代数学シンポジウム講演集 (2020).

Contents

1 Ringel-Hall algebra

A: an $\mathbb{F}_q$-linear abelian category of finite global dim. with $\mathrm{Ext}^i_{\mathsf{A}}(\cdot,\cdot)$ finite dim.

$\mathrm{Iso}(\mathsf{A})$: the set of isomorphism classes $[M]$ of objects M in A.

$\mathbb{C}_c(\mathsf{A})$: the linear space of $\mathbb{C}$-valued functions on $\mathrm{Iso}(\mathsf{A})$ with finite supports.

$1_{[M]}$: the characteristic function of $[M] \in \mathrm{Iso}(\mathsf{A})$, forming a basis of $\mathbb{C}_c(\mathsf{A})$.

Theorem (Ringel (1990)). $\mathbf{R}(\mathsf{A}) := (\mathbb{C}_c(\mathsf{A}), *, 1_{[0]})$ is a unital assoc. $\mathbb{C}$-algebra with

$$1_{[L]} * 1_{[M]} := \nu^{\chi(L,M)} \sum_{[N]\in\mathrm{Iso}(\mathsf{A})} g^N_{L,M} 1_{[N]},$$

$\nu := q^{1/2} \in \mathbb{C}$, $\chi(\cdot,\cdot) := \sum_{i\geq 0}(-1)^i \dim_{\mathbb{F}_q} \mathrm{Ext}^i_{\mathsf{A}}(\cdot,\cdot)$ the Euler form, and

$$g^N_{L,M} := e^N_{L,M} a_L^{-1} a_M^{-1},$$
$$e^N_{L,M} := |\{0 \to M \to N \to L \to 0 \mid \text{exact in } \mathsf{A}\}|, \quad a_M := |\mathrm{Aut}(M)|.$$

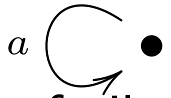
This is the Ringel-Hall algebra of A.

Another definition of $g^N_{L,M}$: counting the number of certain subobjects $M' \subset N$.

$$g^N_{L,M} = |\{M' \in \mathrm{Ob}(\mathsf{A}) \mid M' \subset N,\ M' \simeq M,\ N/M' \simeq L\}|.$$

Theorem (Green (1995), Xiao (1997)). If A is hereditary (global dimension ≤ 1), then the Ringel-Hall algebra $\mathbf{R}(\mathsf{A})$ has a structure of Hopf algebra.

Example: classical Hall algebra

- Q_{Jor}: the Jordan quiver  .
- $\mathsf{Rep}^{\mathsf{nil}}_{\mathbb{F}_q} Q_{\mathsf{Jor}}$: the category of nilpotent representations of Q_{Jor} over $\mathbb{F}_q$.
  $\rightsquigarrow$ Hopf algebra $\mathbf{H}_{\mathsf{cl}} := \mathbf{R}(\mathsf{Rep}^{\mathsf{nil}}_{\mathbb{F}_q} Q_{\mathsf{Jor}})$, called the classical Hall algebra.
- For a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$, $\lambda_i \in \mathbb{N}$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$,

$$I_\lambda := (\mathbb{F}_q^{|\lambda|}, J_\lambda) \in \mathrm{Ob}(\mathsf{A}), \quad J_\lambda := J_{\lambda_1} \oplus J_{\lambda_2} \oplus \cdots \oplus J_{\lambda_r} \in \mathrm{End}(\mathbb{F}_q^{|\lambda|})$$

  with $|\lambda| := \sum_{i=1}^r \lambda_i$ and J_n the Jordan matrix of 0 diagonals and size n.
- The underlying linear space of $\mathbf{H}_{\mathsf{cl}}$ is $\bigoplus_{\lambda \in \mathrm{Par}} \mathbb{C}[I_\lambda]$.
- $\mathbf{H}_{\mathsf{cl}} \simeq \mathbb{C}\big[[I_{(1)}], [I_{(1,1)}], [I_{(1,1,1)}], \ldots\big]$ as $\mathbb{C}$-algebra.

Fact (Steiniz, Hall, Macdonald, ...). $\mathbf{H}_{\mathsf{cl}}$ is isomorphic to the ring of symmetric functions $\Lambda = \mathbb{C}[x_1, x_2, \ldots,]^{\mathfrak{S}_\infty}$.

(Lecture, Day 1)

The extended Ringel-Hall algebra $\widetilde{\mathbf{R}}(\mathsf{A}) := \mathbb{C}K_0(\mathsf{A}) \otimes_{\mathbb{C}} \mathbf{R}(\mathsf{A})$

- $k_\alpha \in \mathbb{C}K_0(\mathsf{A})$: the element associated to $\alpha \in K_0(\mathsf{A})$.
- $k_\alpha * [M] = \nu^{(\alpha,\overline{M})_a}[M] * k_\alpha$, $(\alpha,\beta)_a := \chi(\alpha,\beta) + \chi(\beta,\alpha)$, $\overline{M} \in K_0(\mathsf{A})$.

Ringel's realization of Borel subalgebras of quantum groups

- Q: a quiver without loops. $\mathrm{Rep}^{\mathrm{nilp}}_{\mathbb{F}_q} Q$: category of nilpotent representations of Q.
  $\rightsquigarrow$ the extended Ringel-Hall algebra $\widetilde{\mathbf{R}}(\mathrm{Rep}^{\mathrm{nilp}}_{\mathbb{F}_q} Q)$.
- $\underline{Q}$: underlying graph. $\rightsquigarrow$ $A_{\underline{Q}}$: symmetric generalized Cartan matrix. $\rightsquigarrow$
  $U_\nu(\mathfrak{g}_{\underline{Q}}) = \langle E_i, F_i, K_i^{\pm 1}\rangle$: quantum group of the Kac-Moody Lie algebra $\mathfrak{g}_{\underline{Q}}$.
  $U_\nu(\mathfrak{b}_{\underline{Q}}) = \langle E_i, K_i^{\pm 1}\rangle \subset U_\nu(\mathfrak{g}_{\underline{Q}})$: Borel subalgebra.

Theorem (Ringel, 1990)**.** There is an algebra embedding

$$U_\nu(\mathfrak{b}_{\underline{Q}}) \hookrightarrow \widetilde{\mathbf{R}}(\mathrm{Rep}^{\mathrm{nilp}}_{\mathbb{F}_q} Q), \quad E_i \longmapsto [S_i],\ K_i \longmapsto k_{\overline{S_i}}.$$

If Q is of finite type ($\Longleftrightarrow$ $\underline{Q}$ is Dynkin), then it is an isomorphism.

(Lecture, Day 2–3)

1.2 Lusztig's geometric construction of Ringel-Hall algebras

A geometric reformulation of the Ringel-Hall algebra $\mathbf{R}(\mathrm{Rep}_{\mathbb{F}_q} Q)$ for a quiver Q.
Fix an algebraically closed field $k = \overline{\mathbb{F}_q}$.

- $Q = (I, H)$: a quiver without loops. I: vertex set. H: arrow set.
  $h \in H$ connects the start $s(h) \in I$ and the target $t(h) \in I$.
- The moduli space of representations of Q over k of dimension $\alpha \in \mathbb{N}^I$:

$$\mathcal{M}_Q^\alpha := E_\alpha / G_\alpha, \quad E_\alpha := \bigoplus_{h \in H} \mathrm{Hom}(k^{\alpha_{s(h)}}, k^{\alpha_{t(h)}}), \quad G_\alpha := \prod_{i \in I} \mathrm{GL}(\alpha_i, k),$$

  regarded as an algebraic stack (quotient stack). It is the moduli space since

$$\{\text{isom. classes of reps. of } Q \text{ of dimension } \alpha\} = \{G_\alpha\text{-orbits in } E_\alpha\}.$$

- Recalling $g^N_{L,M} = |\{M' \subset N \mid M' \simeq M, N/M' \simeq L\}|$ in $\mathbf{R}(\mathsf{A})$, we consider

$$\begin{array}{ccc} \mathcal{G}_Q^{\alpha,\beta} & \xrightarrow{c} & \mathcal{M}_Q^{\alpha+\beta} \\ {\scriptstyle p}\downarrow & & \\ \mathcal{M}_Q^\alpha \times \mathcal{M}_Q^\beta & & \end{array} \qquad \begin{array}{ccc} (M \subset N) & \longmapsto & N \\ \downarrow & & \\ (N/M, M) & & \end{array}$$

  with $\mathcal{G}_Q^{\alpha,\beta}$ the moduli space parametrizing $(M \subset N)$, $\dim M = \beta$, $\dim N = \alpha + \beta$.

- Using the above diagram. define the induction functor as

$$\mu\colon \mathsf{D}^b(\mathcal{M}_Q^\alpha \times \mathcal{M}_Q^\beta) \longrightarrow \mathsf{D}^b(\mathcal{M}_Q^{\alpha+\beta}), \quad F \longmapsto c_! p^*(F)[\dim p].$$

Here, for a quotient stack E/G ($\mathcal{G}_Q^{\alpha,\beta}$ is also a quotient stack),

$$\mathsf{D}^b(E/G) := \mathsf{D}^b_G(E)\colon\ G\text{-equiv. derived category of } \overline{\mathbb{Q}}_l\text{-constructible complexes,}$$

and $c_!$, p^* are derived functors (p needs a technical modification).

- On $\mathcal{M}_Q := \bigsqcup_{\alpha\in\mathbb{N}^I} \mathcal{M}_Q^\alpha$, μ induces an associative operation

$$F_1 \star F_2 := \mu(F_1 \boxtimes F_2) \quad \text{for } F_1, F_2 \in \mathsf{D}^b(\mathcal{M}_Q).$$

- $\mathcal{M}_Q$ can be written in the form $\mathcal{M}^0 \otimes_{\mathbb{F}_q} k$ with $\mathcal{M}^0$ defined over $\mathbb{F}_q$.
  Using the sheaf-function dictionary, for $F \in \mathsf{D}^b(\mathcal{M}_Q)$ attached with a Weil structure, we have a constructible function $\mathrm{Tr}(F)$ on $\mathcal{M}^0(\mathbb{F}_q)$ by

$$\mathrm{Tr}(F)\colon \mathcal{M}^0(\mathbb{F}_q) \longrightarrow \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}, \quad x \longmapsto \sum_{i\geq 0} (-1)^i \,\mathrm{Tr}(\mathrm{Frob}, H^i(F)\big|_x),$$

- $\star$ can be restricted to a certain $\mathsf{Q}_Q \subset \mathsf{D}^b(\mathcal{M}_Q)^{ss}$ consisting of semisimple objects with Weil structure, and have $(\{\mathrm{Tr}(F) \mid F \in \mathsf{Q}_Q\}, \star) \simeq \mathbf{R}(\mathsf{Rep}_{\mathbb{F}_q} Q)$.

1.3 Bridgeland-Hall algebra

For a Dynkin quiver $Q = Q(ADE)$, we have a realization of the Borel subalgebra of $U_\nu(\mathfrak{b}_{\underline{Q}})$ by the extended Ringel-Hall algebra $\widetilde{\mathbf{R}}(\mathsf{A})$ for $\mathsf{A} = \mathsf{Rep}_{\mathbb{F}_q} Q)$.

$$\begin{array}{ccc} U_\nu(\mathfrak{b}_{\underline{Q}}) & \xrightarrow{\sim} & \widetilde{\mathbf{R}}(\mathsf{Rep}_{\mathbb{F}_q} Q) \\ \big\downarrow & & \big\downarrow \\ U_\nu(\mathfrak{g}_{\underline{Q}}) & \dashrightarrow^{\sim} & ? \end{array} \qquad\qquad \begin{array}{ccc} E_i, K_i & \longmapsto & [S_i], k_{\overline{S_i}} \\ & & \\ E_i, K_i, F_i & \dashrightarrow & ? \end{array}$$

Theorem (Bridgeland, 2013)**.** Let $\mathsf{P} \subset \mathsf{A}$ be the full subcategory of projective objects. Using the category $\mathsf{C}_2(\mathsf{P})$ of two-periodic complexes $M^\bullet$

$$M^1 \underset{d^0}{\overset{d^1}{\rightleftarrows}} M^0 \qquad\qquad M^i \in \mathrm{Ob}(\mathsf{P}),\ d^{i+1} \circ d^i = 0,$$

one can construct the Bridgeland-Hall algebra by non-commutative (Ore) localization

$$\mathbf{BH}(\mathsf{A}) := \mathbf{R}[C_2(\mathsf{P})]\big[[M^\bullet] \mid H^*(M^\bullet) = 0\big],$$

which is isomorphic to $U_\nu(\mathfrak{g}_{\underline{Q}})$ in the case $\mathsf{A} = \mathsf{Rep}_{\mathbb{F}_q} Q$.

(Lecture, Day 3–4)

Motivation of today's talk

- The original motivation is to make an analogue of Lusztig's geometric construction of Ringel-Hall algebras for Bridgeland-Hall algebras.
  I have not reached this goal yet.
  (The difficult point is to have a geometric interpretation of the non-commutative localization.)
- Aside to Bridgeland-Hall algebras, there are several versions of Hall algebras of complexes.
  Among them, I will explain Toën's derived Hall algebras, which seems to admit an analogous geometric construction.
- A geometric construction of Hall algebra of complexes, in any sense, would take the following steps.
  1. The moduli spaces of complexes of representations/modules,
  2. The derived categories and derived functors of $\underline{\mathbb{Q}}_l$-constructible complexes over the moduli spaces.
  3. Sheaf-function dictionary, the theory of weights, perverse t-structures, ...
- I will explain the item 1 and 2 for derived Hall algebras.
  (3 is not yet established.)

2 Derived Hall algebra

Toën introduced an analogue of Ringel-Hall algebra of complexes using the category of DG modules over a DG category.

2.1 DG category

A DG category over a commutative ring k is a category D whose morphism set is equipped with the structure of differential graded k-module and whose composition of morphisms is a homomorphism of differential graded k-modules.

$$\mathrm{Hom}_{\mathsf{D}}(M,N) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}^n_{\mathsf{D}}(M,N), \quad d\colon \mathrm{Hom}^n_{\mathsf{D}}(\cdot,\cdot) \longrightarrow \mathrm{Hom}^{n+1}_{\mathsf{D}}(\cdot,\cdot),\ d^2 = 0.$$

Example: The DG category $\mathsf{C}_{\mathsf{dg}}(\mathsf{A})$ of complexes over an additive category A.

- Recall that in the Ringel-Hall algebra $\mathbf{R}(\mathsf{A})$ for an abelian category A, the structure constant $g^N_{L,M}$ counts pairs $M \subset N$ of an object and its subobject.
- For a DG category D, we can do an analogous counting, using the model structure of the category M(D) of DG modules over D.
- Rough idea: instead of counting subobjects, count cofibrations up to homotopy.

Model structure

- A model structure on a category C consists of 3 classes of morphisms: fibrations, cofibrations, and weak equivalences, which are subject to certain axioms.
- It is designed to provide a natural setting of homotopy theory.
- Localization of C by weak equivalences gives the homotopy category $\mathrm{Ho}(\mathsf{C})$.

Examples:

1. $\mathsf{C}(k)$: the category of complexes of modules over a commutative ring k.
   It has a projective model structure with
   - A fibration is defined to be an epimorphism of complexes.
   - A weak equivalence is defined to be a quasi-isomorphism.
2. For a DG category D over k, a DG D-module is a DG functor $\mathsf{D}^{\mathsf{op}} \to \mathsf{C}_{\mathsf{dg}}(k)$.
   $\mathsf{M}(\mathsf{D})$: the category of DG D^{op}-modules.
   It has a model structure induced by the projective model structure of C.

2.2 The diagram of correspondence

Let D be a DG category over $k = \mathbb{F}_q$.

- $\mathsf{P}(\mathsf{D}) \subset \mathsf{M}(\mathsf{D})$: the full subcategory of perfect objects.
- $\mathsf{M}(\mathsf{D})^I := \mathsf{Fun}(I, \mathsf{M}(\mathsf{D}))$ with $I = \Delta^1$ the 1-simplex. It has the model structure induced levelwise by that of $\mathsf{M}(\mathsf{D})$.

We have a diagram (of left Quillen functors)

$$\begin{array}{ccc} \mathsf{M}(\mathsf{D})^I & \xrightarrow{\;c\;} & \mathsf{M}(\mathsf{D}) \\ {\scriptstyle p}\big\downarrow & & \\ \mathsf{M}(\mathsf{D}) \times \mathsf{M}(\mathsf{D}) & & \end{array} \qquad \begin{array}{ccc} (x \to y) & \longmapsto & y \\ \big\downarrow & & \\ (y \coprod_x 0, x) & & \end{array}$$

Restricting to the subcategories of cofibrant and perfect objects and of equivalences,

$$\begin{array}{ccc} w\big(\mathsf{P}(\mathsf{D})^I\big)^{\mathsf{cof}} & \xrightarrow{\;c\;} & w\mathsf{P}(\mathsf{D})^{\mathsf{cof}} \\ {\scriptstyle p}\big\downarrow & & \\ w\mathsf{P}(\mathsf{D})^{\mathsf{cof}} \times w\mathsf{P}(\mathsf{D})^{\mathsf{cof}} & & \end{array} \qquad \begin{array}{ccc} (x \rightarrowtail y) & \longmapsto & y \\ \big\downarrow & & \\ (y \coprod_x 0, y) & & \end{array}$$

Simplicial sets and the homotopy category of spaces

- Given a category C, the nerve construction yields a simplicial set $\mathrm{N}(\mathsf{C}) \in \mathsf{sSet}$.
- $\mathsf{sSet} := \mathsf{Fun}(\mathbf{\Delta}^{\mathsf{op}}, \mathsf{Set})$: the category of simplicial sets and simplicial maps.
  It has the Kan model structure where a fibration is a Kan fibration and a weak equivalence is a homotopy equivalence of geom. realizations.
- $\mathcal{H} := \mathrm{Ho}(\mathsf{sSet})$: the homotopy category of spaces. $[\cdot] : \mathsf{sSet} \to \mathcal{H}$.
  An object $X \in \mathrm{Ob}(\mathcal{H})$ is called a homotopy type.
  CG: the category of compactly generated Hausdorff spaces.
  The standard Quillen adjunction $|\ | : \mathsf{sSet} \rightleftarrows \mathsf{CG} : \mathrm{Sing}$ yields $\mathrm{Ho}(\mathsf{sSet}) \simeq \mathrm{Ho}(\mathsf{CG})$.

Define $X^{(0)}(\mathsf{D}), X^{(1)}(\mathsf{D}) \in \mathcal{H}$ by

$$X^{(0)}(\mathsf{D}) := \left[\mathrm{N}(w\mathsf{P}(\mathsf{D})^{\mathsf{cof}})\right], \quad X^{(1)}(\mathsf{D}) := \left[\mathrm{N}\big(w(\mathsf{P}(\mathsf{D})^{I})^{\mathsf{cof}}\big)\right].$$

Then we have the diagram of homotopy types

$$\begin{array}{ccc} X^{(1)}(\mathsf{D}) & \xrightarrow{\ c\ } & X^{0}(\mathsf{D}) \\ \downarrow{\scriptstyle p} & & \\ X^{(0)}(\mathsf{D}) \times X^{(0)}(\mathsf{D}) & & \end{array}$$

Lemma. If the DG category D is locally finite, then

1. p is proper ($:\iff$ for each $y \in \pi_0(Y)$, $|\{x \in \pi_0(X) \mid f(x) = y\}| < \infty$).
2. The homotopy types $X^{(i)}(\mathsf{D}) \in \mathcal{H}$ are locally finite.

Here we used:

Definition. A DG category D is called locally finite if the complex $\mathrm{Hom}_{\mathsf{D}}(M, N)$ is cohomologically bounded with finite-dimensional cohomology groups for any $M, N \in \mathsf{D}$.

Definition. A homotopy type $X \in \mathrm{Ob}(\mathcal{H})$ is called locally finite if for any $x \in X$ the group $\pi_i(X, x)$ is finite and there exists an $n \in \mathbb{N}$ such that $\pi_i(X, x)$ is trivial for $i > n$.
$\mathcal{H}^{\mathsf{lf}}$: the full subcategory of $\mathcal{H}$ spanned by locally finite objects

2.3 The definition of derived Hall algebra

For $X \in \mathcal{H}^{\mathrm{lf}}$, we denote $\mathbb{C}_c(X) := \{\alpha\colon \pi_0(X) \to \mathbb{C} \mid \text{having finite support}\}$.
For a proper morphism $f : X \to Y$ in $\mathcal{H}^{\mathrm{lf}}$, define $f^* : \mathbb{C}_c(Y) \to \mathbb{C}_c(X)$ by

$$f^*(\alpha)(x) := \alpha(f(x)) \quad (\alpha \in \mathbb{C}_c(Y),\ x \in \pi_0(X)).$$

Also, for a morphism $f : X \to Y$ in $\mathcal{H}^{\mathrm{lf}}$, define $f_! : \mathbb{C}_c(X) \to \mathbb{C}_c(Y)$ by

$$f_!(\alpha)(y) := \sum_{x \in \pi_0(X),\, f(x)=y} \alpha(x) \cdot \prod_{i>0} \Big(|\pi_i(X,x)|^{(-1)^i} \, |\pi_i(Y,y)|^{(-1)^{i+1}} \Big).$$

Theorem (Toën 2006)**.** Let D be a locally finite DG category over $\mathbb{F}_q$. Then

$$H(\mathsf{D}) = \mathbb{C}_c(X^{(0)}(\mathsf{D}))$$

has a structure of a unital associative $\mathbb{Q}$-algebra with the multiplication

$$\mu := c_! \circ p^* : H(\mathsf{D}) \otimes_{\mathbb{Q}} H(\mathsf{D}) \longrightarrow H(\mathsf{D}).$$

We call $H(\mathsf{D})$ the derived Hall algebra of D.

2.4 An example of derived Hall algebra

The derived Hall algebra $\mathbf{DH}_{\mathsf{cl}}$ for the DG category of perfect complexes in $\mathsf{Rep}^{\mathsf{nil}}_{\mathbb{F}_q} Q_{\mathsf{Jor}}$ is a unital associative algebra with generators

$$\{Z^{[n]}_{\lambda} \mid n \in \mathbb{Z},\ \lambda \in \mathrm{Par} \setminus \{\emptyset\}\} \sqcup \{Z^{[n]}_{\emptyset} = 1\},$$

and the relations

$$Z^{[n]}_{\lambda} * Z^{[n]}_{\mu} = \sum_{\nu \in \mathrm{Par}} g^{\nu}_{\lambda,\mu} Z^{[n]}_{\nu}, \quad Z^{[n]}_{\lambda} * Z^{[m]}_{\mu} = Z^{[m]}_{\mu} * Z^{[n]}_{\lambda}, \quad (|n-m|>1),$$

$$Z^{[n]}_{\lambda} * Z^{[n+1]}_{\mu} = \sum_{\alpha,\beta \in \mathrm{Par}} \gamma^{\alpha,\beta}_{\lambda,\mu} Z^{[n+1]}_{\alpha} * Z^{[n]}_{\beta}, \tag{$\sharp$}$$

Proposition (Shimoji-Y.)**.** The relation ($\sharp$) is equivalent to the following Heisenberg relation: For $k \in \mathbb{Z}_{>0}$, define $b^{[n]}_{\pm k} \in \mathbf{DH}_{\mathsf{cl}}$ by

$$b^{[n]}_{k} := \sum_{|\lambda|=k} (q;q)_{\ell(\lambda)-1} Z^{[n]}_{\lambda}, \quad b^{[n]}_{-k} := \sum_{|\lambda|=k} (q;q)_{\ell(\lambda)-1} Z^{[n+1]}_{\lambda}.$$

Also set $b^{[n]}_{0} := 1 \in \mathbf{DH}_{\mathsf{cl}}$. Then

$$b^{[n]}_{k} * b^{[n]}_{l} - b^{[n]}_{k} * b^{[n]}_{l} = \delta_{k+l,0} \frac{k}{q^k - 1}.$$

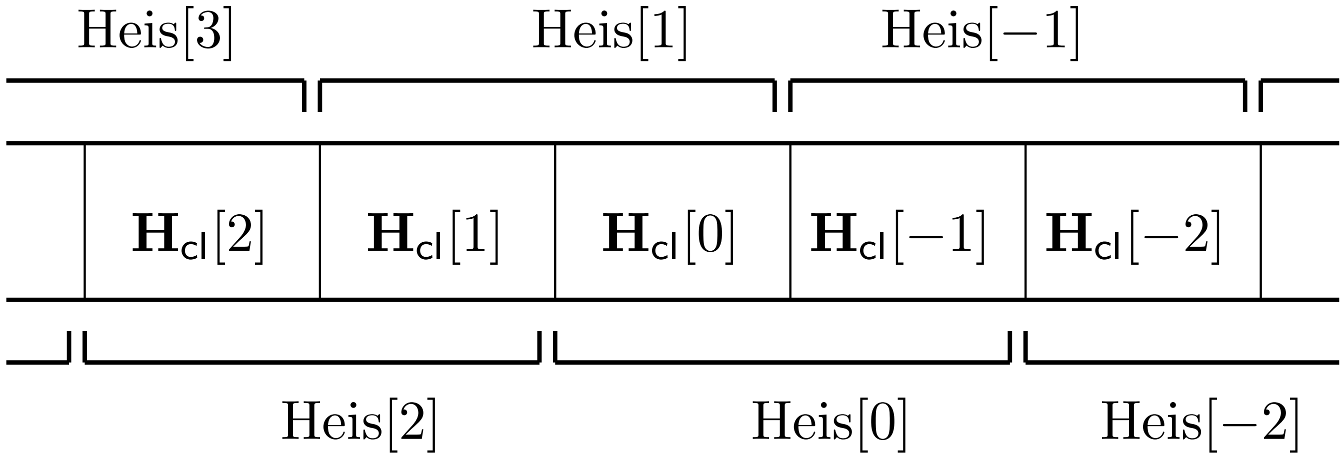


Figure 1 Infinite family of Heisenberg subalgebras in $\mathbf{DH}_{\mathsf{cl}}$

(Lecture, Day 4–5)

3 Outline of geometric construction

D: a locally finite DG category over $\mathbb{F}_q$

Theorem (Toën-Vaquié (2009))**.** The moduli stack $\mathcal{P}(\mathsf{D})$ of perfect DG D^{op}-modules over exists as a derived stack, locally geometric and locally of finite type.

We can also construct the moduli stack $\mathcal{G}(\mathsf{D})$ of cofibrations $X \to Y$ of perfect DG D^{op}-modules, and have the diagram of geometric correspondence

$$\begin{array}{ccc} \mathcal{G}(\mathsf{D}) & \xrightarrow{c} & \mathcal{P}(\mathsf{D}) \\ {\scriptstyle p}\downarrow & & \\ \mathcal{P}(\mathsf{D}) \times \mathcal{P}(\mathsf{D}) & & \end{array} \qquad \begin{array}{ccc} (x \rightarrowtail y) & \longmapsto & y \\ \downarrow & & \\ (y \coprod_x 0, x) & & \end{array}$$

Next, we construct the theory of the derived category $\mathsf{D}^b_{\mathsf{c}}(\mathcal{X}, \overline{\mathbb{Q}}_\ell)$ of constructible lisse-étale $\overline{\mathbb{Q}}_\ell$-sheaves over a locally geometric derived stack $\mathcal{X}$, and Grothendieck's six operations.

Applying the general theory to the present situation, we have

$$\begin{array}{ccc} \mathsf{D}^b_{\mathsf{c}}(\mathcal{G}(\mathsf{D}), \overline{\mathbb{Q}}_\ell) & \xrightarrow{c_!} & \mathsf{D}^b_{\mathsf{c}}(\mathcal{P}(\mathsf{D}), \overline{\mathbb{Q}}_\ell) \\ {\scriptstyle p^*}\uparrow & & \\ \mathsf{D}^b_{\mathsf{c}}(\mathcal{P}(\mathsf{D}) \times \mathcal{P}(\mathsf{D}), \overline{\mathbb{Q}}_\ell) & & \end{array}$$

Now we set

$$\mu\colon \mathsf{D}^b_{\mathsf{c}}(\mathcal{P}(\mathsf{D}) \times \mathcal{P}(\mathsf{D}), \overline{\mathbb{Q}}_\ell) \longrightarrow \mathsf{D}^b_{\mathsf{c}}(\mathcal{P}(\mathsf{D}), \overline{\mathbb{Q}}_\ell), \quad M \longmapsto c_! p^*(M)[\dim p]$$

Main Theorem. The operation $M \star N := \mu(M \boxtimes N)$ is associative.

- We have an associative operation on complexes, but the rest part is yet to be done. In order to recover the derived Hall algebra, we need to determine a small enough subcategory $\mathsf{Q} \subset \mathsf{D}^b_{\mathsf{c}}(\mathcal{P}(\mathsf{D}), \overline{\mathbb{Q}}_\ell)$ on which we have the sheaf-function dictionary.

4 Derived stacks

4.1 Derived schemes and derived stacks

Notations on ∞-categories:

- $\Lambda^n_j \subset \Delta^n$ denotes the j-th horn of the n-simplex Δ^n $(0 \le j \le n)$.
- An ∞-category is a simplicial set K such that for any $n \in \mathbb{N}$ and any $0 < i < n$, any map $f_0 : \Lambda^n_i \to K$ of simplicial sets admits an extension $f : \Delta^n \to K$.

Notations on commutative simplicial algebras:

- k: a commutative ring.
- sCom: the category of commutative simplicial k-algebra.
- sCom_∞: the ∞-category obtained by localizing sCom via the set of weak equivalences in the Kan model category sCom $\subset$ sSet.

Definition. We call $\mathrm{dAff}_\infty := (\mathrm{sCom}_\infty)^{\mathrm{op}}$ the ∞-category of affine derived schemes.

Turn to the definition of derived stacks.

Definition. A morphism $A \to B$ in sCom_∞ is called étale [smooth] if

- the induced $\pi_0(A) \to \pi_0(B)$ is an étale [smooth] map of commutative k-algebras,
- the induced $\pi_i(A) \otimes_{\pi_0(A)} \pi_0(B) \to \pi_i(B)$ is an isomorphism for any i.

Étale morphisms endow $\mathsf{dAff}_\infty = (\mathsf{sCom}_\infty)^{\mathsf{op}}$ with a Grothendieck topology et.
(I will explain Grothendieck topologies on ∞-categories in the next page.)

Definition. The ∞-category of derived stacks is defined to be

$$\mathsf{dSt}_\infty := \mathsf{Sh}_{\infty,\mathsf{et}}(\mathsf{dAff}_\infty) \subset \mathsf{PSh}_\infty(\mathsf{dAff}_\infty) := \mathsf{Fun}_\infty((\mathsf{dAff}_\infty)^{\mathsf{op}}, \mathcal{S}).$$

$\mathcal{S}$: the ∞-category of spaces. (See [Lurie, "Higher Topos Theory"] for the detail.)

- Kan $\subset$ sSet: the full subcategory of Kan complexes, which is a simplicial category.
- $\mathrm{N}_{\mathsf{sp}}(\ )$: simplicial nerve construction,
  a functor mapping a simplicial category to a simplicial set.
- $\mathcal{S} := \mathrm{N}_{\mathsf{sp}}(\mathsf{Kan})$.
- The homotopy category of the ∞-category $\mathcal{S}$ is equivalent to $\mathcal{H} := \mathrm{Ho}(\mathsf{sSet})$, the homotopy category of spaces (Quillen equivalence).

Grothendieck topology on an ∞-category [Lurie, HTT, §6.2.2], [Toën-Vezzosi].

Definition. 1. A sieve on an ∞-category C is a full sub-∞-category $\mathsf{C}^{(0)} \subset \mathsf{C}$ s.t. $X \in \mathsf{C}^{(0)}$ holds for any $Y \in \mathsf{C}^{(0)}$ and any morphism $f : X \to Y$ in C.

2. A sieve on $X \in \mathsf{C}$ is a sieve on the over-∞-category $\mathsf{C}_{/X}$.

- For a functor $F : \mathsf{C} \to \mathsf{D}$ of ∞-categories and a sieve $\mathsf{D}^{(0)} \subset \mathsf{D}$, the homotopy fiber product gives a sieve $F^{-1}\mathsf{D}^{(0)} := \mathsf{D}^{(0)} \times_{\mathsf{D}} \mathsf{C} \subset \mathsf{C}$ on C.
- For a morphism $f : X \to Y$ in C and a sieve $\mathsf{C}^{(0)}_{/Y}$ on Y, we have a sieve $f^*\mathsf{C}^{(0)}_{/Y} := (f_*)^{-1}\mathsf{C}^{(0)}_{/Y}$ on X. ($f_* : \mathsf{C}_{/X} \to \mathsf{C}_{/Y}$: the natural functor of over-∞-categories.)

Definition. A Grothendieck topology τ on an ∞-category C is a choice of a collection $\mathrm{Cov}(X)$ of sieves on each $X \in \mathsf{C}$ (covering sieves on X) s.t.

- For any $X \in \mathsf{C}$, $\mathsf{C}_{/X} \in \mathrm{Cov}(X)$.
- For any $f : X \to Y$ in C and any $\mathsf{C}^{(0)}_{/Y} \in \mathrm{Cov}(Y)$, $f^*\mathsf{C}^{(0)}_{/Y} \in \mathrm{Cov}(X)$.
- For $Y \in \mathsf{C}$ and $\mathsf{C}^{(0)}_{/Y} \in \mathrm{Cov}(Y)$, if $\mathsf{C}^{(1)}_{/Y}$ is a sieve on Y s.t. $f^*\mathsf{C}^{(1)}_{/Y} \in \mathrm{Cov}(X)$ holds for any $(f : X \to Y) \in \mathsf{C}^{(0)}_{/Y}$, then $\mathsf{C}^{(1)}_{/Y} \in \mathrm{Cov}(Y)$.

If C is a nerve of a category C, then a Grothendieck topology on C is equiv. to that on C.

Back to the definition of derived stacks:

$$\mathsf{dSt}_\infty := \mathsf{Sh}_{\infty,\mathsf{et}}(\mathsf{dAff}_\infty) \subset \mathsf{PSh}_\infty(\mathsf{dAff}_\infty) := \mathsf{Fun}_\infty((\mathsf{dAff}_\infty)^{\mathsf{op}}, \mathcal{S}),$$

where $\mathsf{Sh}_{\infty,\mathsf{et}}(\mathsf{dAff}_\infty)$ denotes the ∞-category of sheaves with respect to the Grothendieck topology et.

A derived stack corresponds to a stack in the ordinary algebraic geometry. In the next subsection, I explain geometric derived stacks in the sense of Toën-Vezzosi, which corresponds to an algebraic/Artin stack.

Remark. I use the terminology "geometric derived stacks" following [Toën-Vezzosi, Homotopical Algebraic Geometry II, Mem. AMS, 2008]. It is equivalent to "derived Artin stacks" in [Toën, Derived algebraic geometry, EMS Surv. Math. Sci., 2014].

4.2 Geometric derived stacks

For $n \in \mathbb{Z}_{\geq -1}$, one defines an n-geometric derived stack inductively on n.
At the same time one also defines an n-atlas, a n-representable morphism and a n-smooth morphism of derived stacks.

- Let $n = -1$.
  1. A (-1)-geometric derived stack is defined to be an affine derived scheme.
  2. A morphism $f : \mathcal{X} \to \mathcal{Y}$ of derived stacks is called (-1)-representable if for any affine derived scheme U and any morphism $U \to \mathcal{Y}$ of derived stacks, the pullback $\mathcal{X} \times_{\mathcal{Y}} U$ is an affine derived scheme.
  3. A morphism $f : \mathcal{X} \to \mathcal{Y}$ of derived stacks is called (-1)-smooth if it is (-1)-representable, and if for any affine derived scheme U and any morphism $U \to \mathcal{Y}$ of derived stacks, the induced morphism $\mathcal{X} \times_{\mathcal{Y}} U \to U$ is a smooth morphism of affine derived schemes.
  4. A (-1)-atlas of a stack $\mathcal{X}$ is defined to be the one-member family $\{\mathcal{X}\}$.

Recall: A morphism $A \to B$ in sCom$_\infty$ is called étale [smooth] if

- the induced $\pi_0(A) \to \pi_0(B)$ is an étale [smooth] map of commutative k-algebras,
- the induced $\pi_i(A) \otimes_{\pi_0(A)} \pi_0(B) \to \pi_i(B)$ is an isomorphism for any i.

- Let $n \in \mathbb{N}$.
  1. Let $\mathcal{X}$ be a derived stack. An n-atlas of $\mathcal{X}$ is a small family $\{U_i \to \mathcal{X}\}_{i \in I}$ of morphisms of derived stacks satisfying the following three conditions.
     – Each U_i is an affine derived scheme.
     – Each morphism $U_i \to \mathcal{X}$ is $(n-1)$-smooth.
     – The morphism $\coprod_{i \in I} U_i \to \mathcal{X}$ is an epimorphism.
  2. A derived stack $\mathcal{X}$ is called n-geometric if the following two conditions are satisfied.
     – The diagonal morphism $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is $(n-1)$-representable.
     – There exists an n-atlas of $\mathcal{X}$.
  3. A morphism $f : \mathcal{X} \to \mathcal{Y}$ of derived stacks is called n-representable if for any affine derived scheme U and for any morphism $U \to \mathcal{Y}$ of derived stacks, the derived stack $\mathcal{X} \times_{\mathcal{Y}} U$ is n-geometric.
  4. A morphism $f : \mathcal{X} \to \mathcal{Y}$ of derived stacks is called n-smooth if for any affine derived scheme U and any morphism $U \to \mathcal{Y}$ of derived stacks, there exists an n-atlas $\{U_i\}_{i \in I}$ of $\mathcal{X} \times_{\mathcal{Y}} U$ such that for each $i \in I$ the composition $U_i \to \mathcal{X} \times_{\mathcal{Y}} U \to U$ is a smooth morphism of affine derived schemes.

To an algebraic stack $\mathcal{X}$ in the ordinary sense, one can attach a derived stack $j(\mathcal{X})$ functorially.

Fact (Toën-Vezzosi (2008))**.** For an algebraic stack $\mathcal{X}$, the derived stack $j(\mathcal{X})$ is 1-geometric.

Remark. To schemes and algebraic spaces X, we can also attach derived stacks $j(X)$.
For affine schemes X, the derived stack $j(X)$ is (-1)-geometric.
For schemes and algebraic spaces X, the derived stacks $j(X)$ are 1-geometric.

5 Moduli spaces of complexes

In this section we review the theory of moduli stacks of modules over DG categories via derived stacks [Toen-Vaquié].

5.1 Moduli functor of perfect objects

- $A \in \mathsf{sCom}$: a commutative simplicial k-algebra.
- $N(A)$ the normalized chain complex with the structure of a comm. DG k-algebra.
- Regarding $N(A)$ as a DG category, we have the DG category of DG $N(A)$-modules:

$$\mathsf{M}(A) := \mathsf{M}(N(A))$$

- The full sub-DG category of cofibrant and perfect objects in $\mathsf{M}(A)$:

$$\mathrm{P}(A) := \mathrm{P}(N(A)) \subset \mathsf{M}(A).$$

Definition. For a DG category D over k and $A \in \mathsf{sCom}$, we set

$$\mathcal{M}_{\mathsf{D}}(A) := \mathrm{Map}_{\mathrm{dgCat}}(\mathsf{D}^{\mathsf{op}}, \mathrm{P}(A)),$$

where $\mathrm{Map}_{\mathrm{dgCat}}$ denotes the mapping space in the model category dgCat of DG categories, which is regarded as a simplicial set.

$$\mathcal{M}_{\mathsf{D}}(A) := \mathrm{Map}_{\mathrm{dgCat}}(\mathsf{D}^{\mathsf{op}}, \mathrm{P}(A)),$$

Here the model structure is the one introduced by [Tabuada, 2005]:
A DG functor $f : \mathsf{D} \to \mathsf{D}'$ is

- a weak equivalence if f is a quasi-isomorphism, and
- a fibration if
  (i) for any $M, N \in \mathsf{D}$, the morphism $f_{MN} : \mathrm{Hom}_{\mathsf{D}}(M, N) \to \mathrm{Hom}_{\mathsf{D}'}(f(M), f(N))$ is an epimorphism of DG k-modules, and
  (ii) for any $M \in \mathsf{D}$ and any isomorphism $v : N \to f(M)$ in $H^0(\mathsf{D}')$, there is an isomorphism $u : M \to M'$ in $H^0(\mathsf{D})$ such that $H^0(f_{M,N})(u) = v$.

For a morphism $A \to B$ in sCom, we obtain a morphism $\mathcal{M}_{\mathsf{D}}(A) \to \mathcal{M}_{\mathsf{D}}(B)$ in sSet by composition with $N(B) \otimes_{N(A)} - : \mathrm{P}(A) \to \mathrm{P}(B)$. Thus we obtain a functor

$$\mathcal{M}_{\mathsf{D}} : \mathsf{sCom} \longrightarrow \mathsf{sSet}, \quad \mathcal{M}_{\mathsf{D}}(A) := \mathrm{Map}_{\mathrm{dgCat}}(\mathsf{D}^{\mathsf{op}}, \mathrm{P}(A)).$$

This construction gives rise to a functor of ∞-categories

$$\mathcal{M}_{\mathsf{D}} \in \mathsf{PSh}_{\infty}(\mathsf{dAff}_{\infty}) \;=\; \mathsf{Fun}_{\infty}((\mathsf{dAff}_{\infty})^{\mathsf{op}}, \mathcal{S}).$$

Fact ([Toën-Vaquié, Lemma 3.1]). The presheaf $\mathcal{M}_{\mathsf{D}} \in \mathsf{PSh}_{\infty}(\mathsf{dAff}_{\infty})$ is a derived stack over k. We call it the moduli stack of perfect DG D^{op}-modules

Remark. • The 0-th homotopy $\pi_0(\mathcal{M}_{\mathsf{D}}(k))$ is bijective to the set of isomorphism classes of compact DG D-modules in $\mathrm{Ho}(\mathsf{M}(\mathsf{D}))$.

• For each $x \in \mathrm{Ho}(\mathsf{M}(\mathsf{D}))$, we have

$$\pi_1(\mathcal{M}_{\mathsf{D}}, x) \simeq \mathrm{Aut}_{\mathrm{Ho}(\mathsf{M}(\mathsf{D}))}(x, x), \quad \pi_i(\mathcal{M}_{\mathsf{D}}, x) \simeq \mathrm{Ext}^{-i}_{\mathrm{Ho}(\mathsf{M}(\mathsf{D}))}(x, x) \ (i \in \mathbb{Z}_{\geq 2}),$$

where $\mathrm{Ho}(\mathsf{M}(\mathsf{D}))$ is regarded as a triangulated category.

5.2 Geometricity of moduli stacks of perfect objects

We explain the main result in [Toën-Vaquié, 2009].

Definition. A DG category D over k is of finite type if there exists a DG k-algebra B which is homotopically finitely presented in the model category dgAlg_k of DG algebras s.t. P(D) is quasi-equivalent to $\mathsf{Mod}_{\mathsf{dg}}(B)$.

Fact (Toën-Vaquié)**.** If D is a DG category over k of finite type, then the derived stack $\mathcal{M}_{\mathsf{D}}$ is locally geometric and locally of finite presentation.

Here I used

Definition. A derived stack $\mathcal{X}$ is called locally geometric if $\mathcal{X}$ is equivalent to a filtered colimit $\varinjlim_{i \in I} \mathcal{X}_i$ of derived stacks $\{\mathcal{X}_i\}_{i \in I}$ s.t.

- each derived stack $\mathcal{X}_i$ is n_i-geometric for some $n_i \in \mathbb{Z}_{\geq -1}$,
- each morphism $\mathcal{X}_i \to \mathcal{X}_i \times_{\mathcal{X}} \mathcal{X}_i$ of derived stacks induced by $\mathcal{X}_i \to \mathcal{X}$ is an equivalence in the ∞-category dSt_∞ of derived stacks.

Definition. 1. An n-geometric derived stack $\mathcal{X}$ is called locally of finite presentation if it has an n-atlas $\{U_i\}_{i \in I}$ such that for each representable derived stack $U_i \simeq \operatorname{Spec} A_i$ the simplicial k-algebra A_i is finitely presented (see below).

2. A locally geometric derived stack $\mathcal{X}$ is locally of finite presentation if each geometric derived stack $\mathcal{X}_i$ in $\mathcal{X} \simeq \varinjlim_i \mathcal{X}_i$ can be chosen to be locally of finite presentation in the sense of 1.

Definition. 1. A morphism $f : A \to B$ in sCom_∞ is called finitely presented if for any filtered system $\{C_i\}_{i \in I}$ of objects in $(\mathsf{sCom}_\infty)_{A/}$ the natural morphism

$$\varinjlim_{i \in I} \operatorname{Map}_{(\mathsf{sCom}_\infty)_{A/}}(B, C_i) \longrightarrow \operatorname{Map}_{(\mathsf{sCom}_\infty)_{A/}}(B, \varinjlim_{i \in I} C_i)$$

is an isomorphism in $\mathcal{H}$.

2. $A \in \mathsf{sCom}_\infty$ is called finitely presented or of finite presentation if the morphism $k \to A$ is finitely presented in the sense of 1.

5.3 Moduli stack of complexes of quiver representations

- kQ: the path algebra of a quiver Q over k.
- Regard kQ as a DG algebra over k, and as a DG category over k.
- $\mathrm{Mod}_{\mathrm{dg}}(kQ)$ is the DG category of complexes of representations of Q over k.

Definition. We call the derived stack $\mathcal{M}_{kQ}$ the derived stack of perfect complexes of representations of Q and denote it by

$$\mathcal{P}(Q) := \mathcal{M}_{kQ}.$$

Fact 1. Let Q be a finite quiver with no loops. Then the derived stack $\mathcal{P}(Q)$ is locally geometric and locally of finite presentation over k.

$\pi_0(\mathcal{P}(Q)(k))$ is the set of isom. classes of perfect complexes of reps. of Q over k.

6 Constructible sheaves on derived stacks

6.1 Lisse-étale ∞-site

We will introduce the lisse-étale ∞-site for a geometric derived stack, an analogue of the lisse-étale site for an algebraic stack [Laumon, Moret-Bailly, 2000].

- $(\mathsf{dSt}_\infty)_{/\mathcal{X}}$: the over-$\infty$-category of derived stacks over a derived stack $\mathcal{X}$.
- $\mathsf{dAff}_\infty/\mathcal{X} \subset (\mathsf{dSt}_\infty)_{/\mathcal{X}}$: the full sub-$\infty$-category spanned by affine derived schemes

Definition. Let $n \in \mathbb{Z}_{\geq -1}$ and $\mathcal{X}$ be an n-geometric derived stack.
The lisse-étale ∞-site

$$\mathsf{Lis\text{-}Et}^n_\infty(\mathcal{X}) = (\mathsf{Lis}^n_\infty(\mathcal{X}), \mathsf{lis\text{-}et})$$

on $\mathcal{X}$ is the ∞-site given by the following description.

- $\mathsf{Lis}^n_\infty(\mathcal{X})$ is the full sub-∞-category of $\mathsf{dAff}_\infty/\mathcal{X}$ spanned by (U, u) where the morphism $u : U \to \mathcal{X}$ is n-smooth.
- The set $\mathrm{Cov}_{\mathsf{lis\text{-}et}}(U, u)$ of covering sieves on (U, u) consists of $\{(U_i, u_i) \to (U, u)\}_{i \in I}$ in $\mathsf{Lis}^n_\infty(\mathcal{X})$ s.t. $\{U_i \to U\}_{i \in I}$ is an étale covering.

Recall: A morphism $A \to B$ in sCom_∞ is called étale [smooth] if

- the induced $\pi_0(A) \to \pi_0(B)$ is an étale [smooth] map of commutative k-algebras,
- the induced $\pi_i(A) \otimes_{\pi_0(A)} \pi_0(B) \to \pi_i(B)$ is an isomorphism for any i.

6.2 Constructible lisse-étale sheaves

Recall the notion of a constructible sheaf on an ordinary scheme:

A sheaf $\mathcal{F}$ on a scheme X is called constructible if for any affine Zariski open $U \subset X$ there is a finite decomposition $U = \cup_i U_i$ into constructible locally closed subschemes U_i such that $\mathcal{F}|_{U_i}$ is a locally constant sheaf with value in a finite set.

We introduce an analogue of this notion for derived stacks.

Definition. Let $\mathcal{X}$ be a geometric derived stack. An object of the ∞-category $\mathrm{Sh}_{\infty,\text{lis-et}}(\mathrm{Lis}_{\infty}(\mathcal{X}))$ is called a lisse-étale sheaf.

For an affine derived scheme U, we denote by $\pi_0(U)$ the associated affine scheme.

Definition. A lisse-étale sheaf $\mathcal{F}$ on $\mathcal{X}$ is called constructible if

(i) it is cartesian, i.e., for any morphism $f : T \to T'$ in $\mathcal{X}_{\text{lis-et}}$, the natural morphism $f^{-1}\mathcal{F}_T \to \mathcal{F}_{T'}$ is an equivalence, and

(ii) for any $U \in \mathrm{Lis}_{\infty}(\mathcal{X})$ the restriction $\pi_0(\mathcal{F})|_{\pi_0(U)}$ is a constructible sheaf on $\pi_0(U)$.

Definition. Λ: a commutative ring.
A lisse-étale sheaf of Λ-modules is an object of the ∞-category

$$\mathsf{Sh}_{\infty,\text{lis-et}}(\mathsf{Lis}_{\infty}(\mathcal{X})\,, \mathrm{N}(\mathsf{Mod}(\Lambda))).$$

We then have the DG category of complexes consisting of lisse-étale sheaves of Λ-modules. By the dg nerve construction, we obtain an ∞-category.

Definition. We denote the obtained ∞-category of complexes of lisse-étale sheaves by

$$\mathsf{Mod}_{\infty}(\mathcal{X}_{\text{lis-et}}, \Lambda)\,.$$

For $* \in \{+, -, b\}$ we denote by

$$\mathsf{Mod}^{*}_{\infty}(\mathcal{X}_{\text{lis-et}}, \Lambda) \subset \mathsf{Mod}_{\infty}(\mathcal{X}_{\text{lis-et}}, \Lambda)$$

the full sub-∞-category spanned by complexes whose homologies are bounded below (resp. bounded above, resp. bounded).
The full sub-∞-categories with constructible homologies are denoted by

$$\mathsf{Mod}^{\mathsf{c}}_{\infty}(\mathcal{X}_{\text{lis-et}}, \Lambda)\,, \quad \mathsf{Mod}^{\mathsf{c},*}_{\infty}(\mathcal{X}_{\text{lis-et}}, \Lambda) := \mathsf{Mod}^{\mathsf{c}}_{\infty}(\mathcal{X}_{\text{lis-et}}, \Lambda) \cap \mathsf{Mod}^{*}_{\infty}(\mathcal{X}_{\text{lis-et}}, \Lambda)\,.$$

7 Derived category and derived functors

7.1 Derived ∞-category of constructible lisse-étale sheaves

Proposition. $\mathcal{X}$: a locally geometric derived stack. Λ: a commutative ring.
The ∞-category of complexes of constructible lisse-étale Λ-sheaves

$$\mathsf{Mod}_{\infty}^{\mathsf{c},*}(\mathcal{X}_{\mathsf{lis\text{-}et}}, \Lambda)$$

is stable in the sense of [Lurie, Higher Algebra].
In particular, the homotopy category $\mathrm{Ho}\,\mathsf{Mod}_{\infty}^{\mathsf{c},*}(\mathcal{X}_{\mathsf{lis\text{-}et}}, \Lambda)$ has a structure of a triangulated category (explained below).

Definition. The (left bounded, resp. right bounded, resp. bounded) derived category of constructible sheaves of Λ-modules on $\mathcal{X}$ is defined to be

$$\mathsf{D}_{\mathsf{c}}^{*}(\mathcal{X}, \Lambda) := \mathrm{Ho}\,\mathsf{Mod}_{\infty}^{\mathsf{c},*}(\mathcal{X}_{\mathsf{lis\text{-}et}}, \Lambda) \quad (* \in \{\emptyset, +, -, b\}).$$

Below we give a brief recollection on stable ∞-categories.

Definition (Lurie, HA, §1.1.1)**.** An ∞-category is stable if

(i) it has a zero object $0 \in \mathsf{C}$,

(ii) any morphism has a fiber and cofiber, and

(iii) a triangle in C is a pullback square iff it is a pushout square.

A triangle in C is a square of the following form:

$$\begin{array}{ccc} X & \to & Y \\ \downarrow & & \downarrow \\ 0 & \to & Z \end{array}$$

For a stable ∞-category C, we can define a suspension functor $\Sigma : \mathsf{C} \to \mathsf{C}$ and a loop functor $\Omega : \mathsf{C} \to \mathsf{C}$ [Lurie, HA, §1.1.2].

Fact (Lurie, HA, §1.1.2)**.** For a stable ∞-category C, the homotopy category $\mathrm{Ho}\,\mathsf{C}$ has a structure of a triangulated category with $[1] = \Sigma : \mathrm{Ho}\,\mathsf{C} \to \mathrm{Ho}\,\mathsf{C}$ and the distinguished triangles in the next page.

A distinguished triangle in $\mathrm{Ho}\,\mathsf{C}$ is a diagram of the form

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

such that there is a diagram in C of the form

$$\begin{array}{ccccc} X & \xrightarrow[\widetilde{f}]{} & Y & \longrightarrow & 0 \\ \downarrow & & \downarrow \widetilde{g} & & \downarrow \\ 0' & \longrightarrow & Z & \xrightarrow{\widetilde{h}} & W \end{array}$$

satisfying the following 4 conditions.

(i) $0, 0' \in \mathsf{C}$ are zero objects.

(ii) The two squares are pushout square in C.

(iii) Morphisms $\widetilde{f}, \widetilde{g}$ in C represent f, g in $\mathrm{Ho}\,\mathsf{C}$ respectively.

(iv) h is equal to the composition of the homotopy class of $\widetilde{h}$ and the equivalence $W \simeq X[1]$ given by the outer rectangle.

Using this fact, we can lift notions on triangulated categories to those on stable ∞-categories. For example:

Definition. A t-structure of a stable ∞-category C is a t-structure on the homotopy category $\mathrm{Ho}\,\mathsf{C}$.

Below we explain derived ∞-categories [Lurie, HA, §1.3.2].

- A: an abelian category with enough injectives.
- C(A): the DG category of complexes in A (with injective model structure).
- $\mathsf{C}^+(\mathsf{A}_{\mathsf{inj}}) \subset \mathsf{C}(\mathsf{A})$: the full subcat. of complexes bounded below of injectives.

The dg nerve construction gives an ∞-category

$$\mathsf{D}^+_\infty(\mathsf{A}) := \mathrm{N}_{\mathsf{dg}}(\mathsf{C}^+(A_{\mathsf{inj}})),$$

which is known to be stable. It is called the derived ∞-category of A.

$\mathsf{D}^+_\infty(\mathsf{A})$ has a t-structure determined by $(\mathsf{D}^+_\infty(\mathsf{A})_{\le 0}, \mathsf{D}^+_\infty(\mathsf{A})_{\ge 0})$ with
$\mathsf{D}^+_\infty(\mathsf{A})_{\ge 0}$: the full sub-$\infty$-cat. of $H_n(M) := \pi_0(M[n]) \simeq 0$ in $\mathrm{N}(\mathsf{A})$ for $n < 0$,
$\mathsf{D}^+_\infty(\mathsf{A})_{\le 0}$: similarly defined.
This t-structure enjoys the following properties.

1. The core $\mathsf{D}^+_\infty(\mathsf{A})^\heartsuit := \mathsf{D}^+_\infty(\mathsf{A})_{\le 0} \cap \mathsf{D}^+_\infty(\mathsf{A})_{\ge 0}$ is equivalent to $\mathrm{N}(\mathsf{A})$.
2. $\mathrm{Ho}\,\mathsf{D}^+_\infty(\mathsf{A}) \simeq \mathsf{D}^+(\mathsf{A})$ as triangulated categories, and the t-structure on $\mathrm{Ho}\,\mathsf{D}^+_\infty(\mathsf{A})$ is equivalent to the standard t-structure on $\mathsf{D}^+(\mathsf{A})$.

7.2 Derived functors

On the derived category of constructible lisse-étale sheaves

$$\mathsf{D}^{*}_{\mathsf{c}}(\mathcal{X}, \overline{\mathbb{Q}}_\ell) := \mathrm{Ho}\,\mathsf{Mod}^{\mathsf{c},*}_{\infty}\big(\mathcal{X}_{\text{lis-et}}, \overline{\mathbb{Q}}_\ell\big) \quad (* \in \{\emptyset, +, -, b\}),$$

we can construct analogue of Grothendieck's six derived functors.

Precisely speaking, for

- $\mathcal{X}, \mathcal{Y}$: locally geometric derived stacks locally of finite presentation,
- $f : \mathcal{X} \to \mathcal{Y}$: a morphism locally of finite presentation,

we can define triangulated functors

$$\begin{aligned} \mathrm{R}f_{*} &: \mathsf{D}^{+}_{\mathsf{c}}(\mathcal{X}, \overline{\mathbb{Q}}_\ell) \longrightarrow \mathsf{D}^{+}_{\mathsf{c}}(\mathcal{Y}, \overline{\mathbb{Q}}_\ell), & \mathrm{R}f_{!} &: \mathsf{D}^{-}_{\mathsf{c}}(\mathcal{X}, \overline{\mathbb{Q}}_\ell) \longrightarrow \mathsf{D}^{-}_{\mathsf{c}}(\mathcal{Y}, \overline{\mathbb{Q}}_\ell), \\ \mathrm{L}f^{*} &: \mathsf{D}_{\mathsf{c}}(\mathcal{Y}, \overline{\mathbb{Q}}_\ell) \longrightarrow \mathsf{D}_{\mathsf{c}}(\mathcal{X}, \overline{\mathbb{Q}}_\ell), & \mathrm{R}f^{!} &: \mathsf{D}_{\mathsf{c}}(\mathcal{Y}, \overline{\mathbb{Q}}_\ell) \longrightarrow \mathsf{D}_{\mathsf{c}}(\mathcal{X}, \overline{\mathbb{Q}}_\ell) \end{aligned}$$

and $\mathrm{R}\,\mathcal{H}om$, $\otimes^{\mathrm{L}}$. These functors are compatible with those for algebraic stacks developed by Laszlo and Olsson (2008).

7.3 Base-change theorem

The constructed derived functors satisfy the standard properties. Today I only explain the base-change theorem, which will be used to show the associativity of Hall algebra.

Assume that we have the following cartesian diagram in the ∞-category of locally geometric derived stacks, and that f is locally of finite presentation.

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow[\pi]{} & \mathcal{X} \\ {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{Y}' & \xrightarrow{p} & \mathcal{Y} \end{array}$$

We have a morphism $p^* f_! \to \varphi_! \pi^*$ in $\mathsf{Fun}_\infty(\mathsf{Mod}_\infty^{\mathsf{c},-}(\mathcal{X}_{\text{lis-et}}, \overline{\mathbb{Q}}_\ell), \mathsf{Mod}_\infty^{\mathsf{c},-}(\mathcal{Y}'_{\text{lis-et}}, \overline{\mathbb{Q}}_\ell))$, and $p^! f_* \to \phi_* \pi^!$ in $\mathsf{Fun}_\infty(\mathsf{Mod}_\infty^{\mathsf{c},+}(\mathcal{X}_{\text{lis-et}}, \overline{\mathbb{Q}}_\ell), \mathsf{Mod}_\infty^{\mathsf{c},+}(\mathcal{Y}'_{\text{lis-et}}, \overline{\mathbb{Q}}_\ell))$.

Proposition (Y., §6.6)**.** If p is smooth, then

$$(p^* f_! \to \varphi_! \pi^*) \simeq (p^! f_* \to \phi_* \pi^!) \quad \text{in } \mathsf{Fun}_\infty(\mathsf{Mod}_\infty^{\mathsf{c},b}(\mathcal{X}_{\text{lis-et}}, \overline{\mathbb{Q}}_\ell), \mathsf{Mod}_\infty^{\mathsf{c},b}(\mathcal{Y}'_{\text{lis-et}}, \overline{\mathbb{Q}}_\ell)).$$

As a consequence, we have

$$(\mathrm{L}p^* \mathrm{R}f_! \to \mathrm{R}\varphi_! \mathrm{L}\pi^*) \simeq (\mathrm{L}p^! \mathrm{R}f_* \to \mathrm{R}\phi_* \mathrm{L}\pi^!) \quad \text{in } \mathsf{Fun}(\mathsf{D}_{\mathsf{c}}^b(\mathcal{X}, \overline{\mathbb{Q}}_\ell), \mathsf{D}_{\mathsf{c}}^b(\mathcal{Y}', \overline{\mathbb{Q}}_\ell)).$$

8 Geometric construction of derived Hall algebras

D: a DG category of finite type (in the sense of Toën-Vaquié) over $k = \overline{\mathbb{F}}_q$.
(E.g. the DG category $\mathsf{Mod}_{\mathsf{dg}}(kQ)$ of complexes of reps. of a quiver Q without loops.)
$\mathcal{P}(\mathsf{D})$: the moduli space of perfect DG D^{op}-modules.
: a locally geometric derived stack locally of finite presentation.
Decomposition of $\mathcal{P}(\mathsf{D})$:

$$\mathcal{P}(\mathsf{D}) = \bigcup_{a \le b} \mathcal{P}(\mathsf{D})^{[a,b]}, \quad \mathcal{P}(\mathsf{D})^{[a,b]} = \coprod_{\alpha \in K_0(\mathrm{Ho}\,\mathsf{P}(\mathsf{D}))} \mathcal{P}(\mathsf{D})^{[a,b],\alpha}.$$

The component $\mathcal{P}(\mathsf{D})^{[a,b],\alpha}$ parametrizes DG modules M whose cohomologies concentrate in $[a,b]$ and $\overline{M} = \alpha$.
Decomposition of the moduli space $\mathcal{G}(\mathsf{D})$ of cofibrations:

$$\mathcal{G}(\mathsf{D}) = \bigcup_{a \le b} \mathcal{G}(\mathsf{D})^{[a,b]}, \quad \mathcal{G}(\mathsf{D})^{[a,b]} = \coprod_{\alpha,\beta \in K_0(\mathrm{Ho}\,\mathsf{P}(\mathsf{D}))} \mathcal{G}(\mathsf{D})^{[a,b],\alpha,\beta}$$

$\mathcal{G}(\mathsf{D})^{[a,b],\alpha,\beta}$ parametrizes cofibrations $X \hookrightarrow Y$ such that cohomologies of Y concentrate in $[a,b]$ and $\alpha = \overline{X}$, $\beta = \overline{Y \coprod^X 0}$.

Diagram of correspondence:

$$\begin{array}{ccc} \mathcal{G}(\mathsf{D})^{[a,b],\alpha,\beta} & \xrightarrow{c} & \mathcal{P}(\mathsf{D})^{[a,b],\alpha+\beta} \\ \downarrow{\scriptstyle p} & & \\ \mathcal{P}(\mathsf{D})^{[a,b],\alpha} \times \mathcal{P}(\mathsf{D})^{[a,b],\beta} & & \end{array} \qquad \begin{array}{ccc} (X \hookrightarrow Y) & \longmapsto & Y \\ \downarrow & & \\ (X, Y \coprod^{X} 0) & & \end{array}$$

The multiplication μ of derived Hall algebra:

$$\begin{aligned} \mu_{\alpha,\beta} : \mathsf{D}^b_{\mathsf{c}}(\mathcal{P}(\mathsf{D})^{\alpha}, \overline{\mathbb{Q}}_\ell) \times \mathsf{D}^b_{\mathsf{c}}(\mathcal{P}(\mathsf{D})^{\beta}, \overline{\mathbb{Q}}_\ell) &\longrightarrow \mathsf{D}^b_{\mathsf{c}}(\mathcal{P}(\mathsf{D})^{\alpha+\beta}, \overline{\mathbb{Q}}_\ell) \\ M &\longmapsto \mathrm{R}c_!\, \mathrm{L}p^*(M)[\dim p]. \end{aligned}$$

(ℓ is invertible in $\mathbb{F}_q$.)
Associativity:

$$\mu_{\alpha,\beta+\gamma} \circ (\mathrm{id} \times \mu_{\beta,\gamma}) \simeq \mu_{\alpha+\beta,\gamma} \circ (\mu_{\alpha,\beta} \times \mathrm{id}).$$

Outline of the proof of associativity.
The LHS $\mu_{\alpha,\beta+\gamma} \circ (\mathrm{id} \times \mu_{\beta,\gamma})$ corresponds to the rigid arrows in

$$
\begin{array}{ccccc}
\mathcal{G}^{\alpha,(\beta,\gamma)} & \overset{p_2''}{\dashrightarrow} & \mathcal{G}^{\alpha,\beta+\gamma} & \xrightarrow{p_2'} & \mathcal{P}^{\alpha+\beta+\gamma} \\
{\scriptstyle p_1''}\,\vdots & & \downarrow{\scriptstyle p_1'} & & \\
\mathcal{P}^{\alpha} \times \mathcal{G}^{\beta,\gamma} & \xrightarrow[p_2]{} & \mathcal{P}^{\alpha} \times \mathcal{P}^{\beta+\gamma} & & \\
{\scriptstyle p_1}\downarrow & & & & \\
\mathcal{P}^{\alpha} \times \mathcal{P}^{\beta} \times \mathcal{P}^{\gamma} & & & &
\end{array}
$$

The dotted arrows are determined by

$$\mathcal{G}^{\alpha,(\beta,\gamma)} := (\mathcal{P}^{\alpha} \times \mathcal{G}^{\beta,\gamma}) \times_{\mathcal{P}^{\alpha} \times \mathcal{P}^{\beta+\gamma}} \mathcal{G}^{\alpha,\beta+\gamma},$$

which parametrizes $(N \hookrightarrow M, M \hookrightarrow L)$ such that $\overline{N} = \gamma$, $\overline{M} = \beta + \gamma$, $\overline{L} = \alpha + \beta + \gamma$.
By the smoothness of p_1'', the base-change theorem implies

$$\mu_{\alpha,\beta+\gamma} \circ (\mathrm{id} \times \mu_{\beta,\gamma}) \simeq \mathrm{R}(p_2' p_2'')_! \, \mathrm{L}(p_1 p_1'')^* [\dim(p_1 p_1'')].$$

The RHS $\mu_{\alpha+\beta,\gamma} \circ (\mu_{\alpha,\beta} \times \mathrm{id})$ corresponds to

$$\begin{array}{ccccc}
\mathcal{G}^{(\alpha,\beta),\gamma} & \overset{q_2''}{\dashrightarrow} & \mathcal{G}^{\alpha+\beta,\gamma} & \xrightarrow{q_2'} & \mathcal{P}^{\alpha+\beta+\gamma} \\
{\scriptstyle q_1''}\Big\downarrow & & \Big\downarrow{\scriptstyle q_1'} & & \\
\mathcal{G}^{\alpha,\beta} \times \mathcal{P}^{\gamma} & \xrightarrow[q_2]{} & \mathcal{P}^{\alpha+\beta} \times \mathcal{P}^{\gamma} & & \\
{\scriptstyle q_1}\Big\downarrow & & & & \\
\mathcal{P}^{\alpha} \times \mathcal{P}^{\beta} \times \mathcal{P}^{\gamma} & & & &
\end{array}$$

The dotted arrows are determined by

$$\mathcal{G}^{(\alpha,\beta),\gamma} := (\mathcal{G}^{\alpha,\beta} \times \mathcal{P}^{\gamma}) \times_{\mathcal{P}^{\alpha+\beta} \times \mathcal{P}^{\gamma}} \mathcal{G}^{\alpha+\beta,\gamma}$$

which parametrizes $(R \to L \coprod^M 0, M \to L)$ such that $\overline{M} = \gamma$, $\overline{R} = \beta$, $\overline{L} = \alpha + \beta + \gamma$. By the smoothness of q_1'', the base-change theorem implies

$$\mu_{\alpha+\beta,\gamma} \circ (\mu_{\alpha,\beta} \times \mathrm{id}) \simeq \mathrm{R}(q_2' q_2'')_! \, \mathrm{L}(q_1 q_1'')^* [\dim(q_1 q_1'')].$$

Thus LHS and RHS are given by

$$\mu_{\alpha,\beta+\gamma} \circ (\mathrm{id} \times \mu_{\beta,\gamma}) \simeq \mathrm{R}p_! \, \mathrm{L}(p')^*[\dim p'], \quad \mu_{\alpha+\beta,\gamma} \circ (\mu_{\alpha,\beta} \times \mathrm{id}) \simeq \mathrm{R}q_! \, \mathrm{L}(q')^*[\dim q']$$

with

$$\begin{array}{ccc} \mathcal{G}^{\alpha,(\beta,\gamma)} & \xrightarrow{p} & \mathcal{P}^{\alpha+\beta+\gamma} \\ \downarrow{\scriptstyle p'} & & \\ \mathcal{P}^{\alpha} \times \mathcal{P}^{\beta} \times \mathcal{P}^{\gamma} & & \end{array} \qquad \begin{array}{ccc} \mathcal{G}^{(\alpha,\beta),\gamma} & \xrightarrow{q} & \mathcal{P}^{\alpha+\beta+\gamma} \\ \downarrow{\scriptstyle q'} & & \\ \mathcal{P}^{\alpha} \times \mathcal{P}^{\beta} \times \mathcal{P}^{\gamma} & & \end{array}$$

Then the associativity follows from the isomorphism of the derived stacks

$$\mathcal{G}^{\alpha,(\beta,\gamma)} \simeq \mathcal{G}^{(\alpha,\beta),\gamma} .$$

This isomorphism is shown by reduction to the values on the closed points.

Thank you for the listening.