Derived gluing construction of chiral algebras

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Contents

Based on

Y. "Derived gluing construction of chiral algebras",
Lett. Math. Phys. 2021; arXiv:2004.10055.

1.1. Moore-Tachikawa 2-dim Topological Quantum Field Theory

[G. Moore, Y. Tachikawa, "On 2d TQFTs whose values are holomorphic symplectic varieties", String-Math 2011, Proc. Sympos. Pure Math. 85 (2012); arXiv:1106.5698]

Moore and Tachikawa conjectured the existence of a functor

$$\eta_G : \mathrm{Bo}_2 \longrightarrow \mathrm{HS}$$

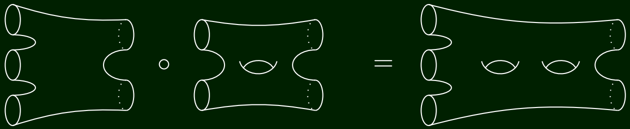
between symmetric monoidal categories with duality.

Bo_2: the 2-bordism category.

Objects: $(S^1)^n$ for $n \in \mathbb{N} := \mathbb{Z}_{\geq 0}$, identified with n.

Morphisms: $\Sigma_{g,n_1+n_2} : n_1 \to n_2$, 2-dim. oriented manifolds with genus g and boundary $(S^1)^{n_1} \sqcup -(S^1)^{n_2}$.

Composition := gluing.



$$(\Sigma_{0,2+3} : 2 \to 3) \circ (\Sigma_{1,2+2} : 2 \to 2) = (\Sigma_{2,2+3} : 2 \to 3)$$

1.1. Moore-Tachikawa 2d TQFT $\eta_G : \mathrm{Bo}_2 \to \mathrm{HS}$ [MT, arXiv:1106.5698]

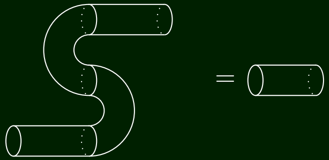
Bo_2 is a symmetric monoidal category with duality.

$\otimes := \sqcup$, disjoint union of manifolds.

Duality:

- In general, for each object A, there is a dual object A^*, and there are $p_A \in \mathrm{Hom}(A \times A^*, 1)$, $q_A \in \mathrm{Hom}(1, A^* \times A)$ such that $(p_A \times \mathrm{id}_A) \circ (\mathrm{id}_A \times q_A) = \mathrm{id}_A$, $(\mathrm{id}_{A^*} \times p_A) \circ (q_A \times \mathrm{id}_{A^*}) = \mathrm{id}_{A^*}$.

In case of Bo_2, $n^* := n$ and $p_n := (\Sigma_{0,2+0})^{\sqcup n}$, $q_n := (\Sigma_{0,0+2})^{\sqcup n}$, which satisfy the relation "S-bordism is equal to the tube".



$(p_1 \times \mathrm{id}_1) \circ (\mathrm{id}_1 \times q_1) = \mathrm{id}_1$

1.1. Moore-Tachikawa 2d TQFT $\eta_G : \mathrm{Bo}_2 \to \mathrm{HS}$ [MT, arXiv:1106.5698]

HS: category "of holomorphic symplectic varieties"

- Objects: semisimple algebraic groups over $\mathbb{C}$ (including the trivial group).
- Morphisms: $X : G_1 \to G_2$, holomorphic symplectic variety X with Hamiltonian $G_1 \times G_2$-action.

$G \curvearrowright (Y, \omega)$ is Hamiltonian if $\exists \mu \colon Y \to \mathfrak{g}^* := \mathrm{Lie}(G)^*$, the moment map, s.t.

$\langle d\mu(\cdot), \xi \rangle = -\iota_{\xi_Y} \omega$ **with** $\xi_Y(y) := \frac{d}{dt} e^{t\xi} . y \Big|_{t=0}$ **for** $\xi \in \mathfrak{g}$, **and** $\mu(g.y) = \mathrm{ad}^*_{g^{-1}}\, \mu(y)$ **for** $g \in G$.

- Composition: For $X_{12} \in \mathrm{Hom}_{\mathrm{HS}}(G_1, G_2)$ and $X_{23} \in \mathrm{Hom}_{\mathrm{HS}}(G_2, G_3)$,

$$X_{23} \circ X_{12} := (X_{12}^{\mathrm{op}} \times X_{23}) /\!\!/_{\mu} \Delta(G_2) = \mu^{-1}(0)/\Delta(G_2).$$

$/\!\!/_{\mu}$: Hamiltonian reduction (symplectic quotient) for the moment map

$$\mu \colon X_{12} \times X_{23} \to \mathfrak{g}_2^* := \mathrm{Lie}(G_2)^*, \quad \mu(x, y) := -\mu_{12}(x) + \mu_{23}(y)$$

with μ_{12} the $\mathfrak{g}_2^*$-component of momentum map $X_{12} \to \mathfrak{g}_1^* \times \mathfrak{g}_2^*$.

- $\otimes$: given by Cartesian product.

1.1. Moore-Tachikawa 2d TQFT $\eta_G\colon \mathrm{Bo}_2 \to \mathrm{HS}$ [MT, arXiv:1106.5698]

- The identity morphism for $G \in \mathrm{HS}$:

$$(\mathrm{id}_G\colon G \to G) := T^*G = G \times \mathfrak{g}^*$$

with standard symplectic structure and two commuting G-actions

$$g.(h,x) := (gh,x), \quad g.(h,x) := (hg^{-1}, g.x).$$

These are Hamiltonian with moment maps

$$(\mu_L,\mu_R)\colon G \times \mathfrak{g}^* \to \mathfrak{g}^* \times \mathfrak{g}^*, \quad \mu_L(g,x) = x, \quad \mu_R(g,x) = g.x.$$

- T^*G is indeed the identity morphism in HS. For $Y \in \mathrm{Hom}_{\mathrm{HS}}(G', G)$,

$$\begin{aligned} T^*G \circ Y &= Y^{\mathrm{op}} \times (G \times \mathfrak{g}^*) /\!\!/ \Delta(G) \\ &= \{(y,g,x) \in Y \times G \times \mathfrak{g}^* \mid \mu_Y(y) = \mu_L(g,x)\}/\Delta(G) \\ &= \{(y,g,x) \mid x = \mu_Y(y)\}/\Delta(G) = G \times Y/G = Y. \end{aligned}$$

1.1. Moore-Tachikawa 2d TQFT $\eta_G\colon \mathrm{Bo}_2 \to \mathrm{HS}$ [MT, arXiv:1106.5698]

They conjectured that, for each simply-connected semisimple G,

(SL_n, Spin_n (univ. cover of SO_n), Sp_n, exceptional groups)

there exists a functor $\eta_G\colon \mathrm{Bo}_2 \to \mathrm{HS}$ with $\eta_G(n) = G^n$ and

$$\eta_G(\Sigma_{g,n_1+n_2}) : \text{holo. symplectic variety with Ham. } G^{n_1+n_2}\text{-action}$$
$$\text{(Moore-Tachikawa symplectic variety).}$$

The functoriality of η_G means that taking symplectic quotients of $\eta_G(\Sigma)$'s is compatible with gluing bordisms Σ's.

$$\begin{array}{ccc}
\eta_G(\Sigma'_{n_2+n_3} \circ \Sigma_{n_1+n_2}) & \overset{\text{gluing}}{=\!=\!=\!=} & \eta_G(\Sigma''_{n_1+n_3}) \\
\Big\| \ \text{functoriality} & & \\
\eta_G(\Sigma'_{n_2+n_3}) \circ \eta_G(\Sigma_{n_1+n_2}) & = & \big(\eta_G(\Sigma_{n_1+n_2})^{\mathrm{op}} \times \eta_G(\Sigma'_{n_2+n_3})\big) /\!\!/ \Delta(G^{n_2})
\end{array}$$

1.2. Braverman-Finkelberg-Nakajima construction of η_G

["Ring objects in the equivariant derived Satake category arising from Coulomb branches", Adv. Theor. Math. Phys. (2019); arXiv:1706.02112]

Theorem (Braverman-Finkelberg-Nakajima)

The Moore-Tachikawa 2d TQFT η_G exists.

- They introduced, in some equivariant derived constructible category $D_{G_\mathcal{O}}(\mathrm{Gr}_G)$ on the affine Grassmannian

$$\mathrm{Gr}_G = G_\mathcal{K}/G_\mathcal{O}, \quad G_\mathcal{O} := G\big(\mathbb{C}[[z]]\big),\ G_\mathcal{K} := G\big(\mathbb{C}((z))\big),$$

two distinguished objects $\mathcal{A}, \mathcal{B} \in D_{G_\mathcal{O}}(\mathrm{Gr}_G)$ which are ring objects with respect to the convolution product $\star$.

- Using these ring objects for the Langlands dual G^L, they showed that

$$\eta_G(\Sigma_{g,n}) := \mathrm{Spec}\big(H^*_{G^L_\mathcal{O}}(\mathrm{Gr}_{G^L}, i^!_\Delta(\mathcal{A}^{\boxtimes n} \boxtimes \mathcal{B}^{\boxtimes g})), \star\big)$$

has a symplectic structure, and satisfies the gluing condition $\eta_G(\Sigma \circ \Sigma') \simeq \eta_G(\Sigma) \circ \eta_G(\Sigma')$.

1.2. Braverman-Finkelberg-Nakajima construction of η_G [BFN, arXiv:1706.02112]

A few varieties in genus zero part can be described explicitly.

Denoting $W_G^n := \eta_G(\Sigma_{g=0,n})$, the gluing condition gives

$$W_G^n \circ W_G^m \simeq W_G^{n+m-2}.$$

- The case $n = 2$ is already explained:

$W_G^2 = \eta_G\Big($  $\Big) = \mathrm{id}_G = T^*G = G \times \mathfrak{g}^*.$

- The case $n = 1$ is a bit non-trivial.

$W_G^1 = \eta_G\Big($  $\Big) = \eta_G\Big($  $\Big) = G \times \mathcal{S}_{\mathrm{reg}}$

with $\mathcal{S}_{\mathrm{reg}} \subset \mathfrak{g}^*$ the Slodowy slice of regular nilpotent $f_{\mathrm{reg}} \in \mathfrak{g}$.
$\mathcal{S}_{\mathrm{reg}} := f_{\mathrm{reg}} + \mathfrak{g}^e \subset \mathfrak{g} \simeq \mathfrak{g}^*$ via Killing form $(\cdot|\cdot)$

$\{e, f_{\mathrm{reg}}, h\} \subset \mathfrak{g}$ an $\mathfrak{sl}_2$-triple assoc. to f_{reg}, $\mathfrak{g}^e \subset \mathfrak{g}$ the centralizer of e.

- The case $n = 3$ for $G = \mathrm{SL}_2$ and SL_3 is

$$W_{\mathrm{SL}_2}^3 = (\mathbb{C}^2)^{\times 3}, \quad W_{\mathrm{SL}_3}^3 = \overline{O_{\mathrm{min}}} \text{ in } E_6.$$

$\overline{O_{\mathrm{min}}}$: closure of coadjoint orbit of minimal nilpotent element

Remark for Slodowy slice

- $\mathcal{N} := \{x \in \mathfrak{g} \mid \mathrm{ad}(x) \in \mathrm{End}(\mathfrak{g}) \text{ is nilpotent}\}$: the nilpotent cone.
  $\mathcal{N} \ni f$: a nilpotent element. $\{e, f, h\} \subset \mathfrak{g}$: an $\mathfrak{sl}_2$-triple assoc. to f.
  $$\mathcal{S}_f := f + \mathfrak{g}^e \subset \mathfrak{g} \simeq \mathfrak{g}^* \quad \text{via Killing form } (\cdot|\cdot)$$
  with $\mathfrak{g}^e$ the centralizer of e in $\mathfrak{g}$.
- G acts on $\mathfrak{g}^* = \operatorname{Spec} \mathbb{C}[\mathfrak{g}]$ by coadjoint action.
  $\chi\colon \mathfrak{g}^* \to \operatorname{Spec} \mathbb{C}[\mathfrak{g}]^G \simeq \mathfrak{h}^*/W \simeq \operatorname{Spec} \mathbb{C}[p_1, \ldots, p_{\operatorname{rank} \mathfrak{g}}]$ by Chevalley.
  $\mathcal{N} = \chi^{-1}(0)$, $\dim \mathcal{N} = \dim \mathfrak{g} - \operatorname{rank} \mathfrak{g}$.
  $\mathcal{N} = \bigcup_f O_f$ orbit stratification for coadjoint action.
  $T_f O_f = f + [\mathfrak{g}, f]$: tangent space, $\mathfrak{g} = [\mathfrak{g}, f] \perp \mathfrak{g}^e$.
  Thus $\mathcal{S}_f$ is a transversal slice of O_f at f.
- The regular orbit O_{reg} is the unique orbit of max. dim.
  $\mathcal{S}_{\mathrm{reg}} := \mathcal{S}_{f_{\mathrm{reg}}}$ with $f_{\mathrm{reg}} \in O_{\mathrm{reg}}$.
- $\mathfrak{g} = \mathfrak{sl}_2 = \{\left[\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right]\}$, $\mathcal{N} = \{X = \left[\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right] \mid \det X = -a^2 - bc = 0\}$.
  $f = f_{\mathrm{reg}} = \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right] \in O_{\mathrm{reg}} = \{\left[\begin{smallmatrix} zw & -z^2 \\ w^2 & -zw \end{smallmatrix}\right] \mid zw \neq 0\}$, $e = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$, $\mathfrak{g}^e = \mathbb{C}e$.
  $$\mathcal{S}_{\mathrm{reg}} = f_{\mathrm{reg}} + \mathfrak{g}^e = \begin{bmatrix} 0 & * \\ 1 & 0 \end{bmatrix}.$$

1.3. Arakawa's chiral quantization $\eta^{\mathrm{ch}}_{G,g=0}$

[T. Arakawa, "**Chiral algebras of class $\mathcal{S}$ and Moore-Tachikawa symplectic varieties**", arXiv:1811.01577]

- Arakawa considered chiralization of Moore-Tachikawa TQFT η_G:

$$\eta_G^{\mathrm{ch}} : \mathsf{Bo}_2 \longrightarrow \mathsf{HS}^{\mathrm{ch}}.$$

  - Target category $\mathsf{HS}^{\mathrm{ch}}$:
    - Objects: semisimple algebraic groups (same as HS).
    - Morphisms $V \colon G_1 \to G_2$: vertex algebras V equipped with
      $$V_{-h_1^\vee}(\mathfrak{g}_1) \otimes V_{-h_2^\vee}(\mathfrak{g}_2) \to V \text{ (+ some cond.).}$$
    - Composition of $V_{12} \colon G_1 \to G_2$ and $V_{23} \colon G_2 \to G_3$:
      $$V_{23} \circ V_{12} := H^{\frac{\infty}{2}+0}(\widehat{\mathfrak{g}}_{-2h_2^\vee}, \mathfrak{g}_2, V_{12}^{\mathrm{op}} \otimes V_{23}),$$
      $H^{\frac{\infty}{2}+*}(\cdot,\cdot,\cdot)$: relative BRST (semi-infinite) cohomology
      (vertex algebra analogue of Hamiltonian reduction)
  - The functor η_G^{ch} should sit in a commutative diagram

$$\begin{array}{ccc}
\mathsf{Bo}_2 & \xrightarrow{\ \eta_G^{\mathrm{ch}}\ } & \mathsf{HS}^{\mathrm{ch}} \\
\Big\| & & \Big\downarrow \text{Spec } R_{(-)} \text{ taking associated scheme} \\
\mathsf{Bo}_2 & \xrightarrow{\ \eta_G\ } & \mathsf{HS}
\end{array}$$

1.3. Arakawa's chiral quantization $\eta^{\mathrm{ch}}_{G,g=0}$ [Arakawa, arXiv:1811.01577]

- Arakawa built genus 0 part $\eta^{\mathrm{ch}}_{G,g=0}$: $\mathsf{Bo}_2|_{g=0} \to \mathsf{HS}^{\mathrm{ch}}$.

Theorem (Arakawa)

There is a family $\{\mathsf{V}^{\mathcal{S}}_{G,n} = \eta^{\mathrm{ch}}_{G,g=0}(\Sigma_{g=0,n}) \mid n \in \mathbb{N}\}$ *of vertex algebras such that*

$$\mathsf{V}^{\mathcal{S}}_{G,1} \simeq H^0_{\mathrm{DS}}(\mathcal{D}^{\mathrm{ch}}_G), \quad \mathsf{V}^{\mathcal{S}}_{G,2} \simeq \mathcal{D}^{\mathrm{ch}}_G, \quad \mathsf{V}^{\mathcal{S}}_{G,m} \circ \mathsf{V}^{\mathcal{S}}_{G,n} \simeq \mathsf{V}^{\mathcal{S}}_{G,m+n-2},$$

and their associated schemes are Moore-Tachikawa symplectic varieties:

$$W^n_G \simeq \mathrm{Spec}\, R_{\mathsf{V}^{\mathcal{S}}_{G,n}}.$$

- $\rightsquigarrow$ Beem-Rastelli conjecture

[“Vertex operator algebras, Higgs branches, and modular differential equations”, arXiv:1707.07679]

$$\mathcal{M}_{\mathrm{Higgs}}(\mathcal{T}) \overset{?}{\simeq} \mathrm{Specm}(R_{V_{\mathcal{T}}}) \quad \forall\, \mathcal{T} : \mathcal{N} = 2 \text{ 4d SCFT}$$

is affirmatively solved for genus 0 class $\mathcal{S}$ theories $\mathcal{T} = \mathcal{T}^{\mathcal{S}}_{\Sigma_{0,n}}$.

1.4. Motivation and main result

- I learned these theories in Arakawa's intensive lectures at Nagoya Univ., November 2019, with many comments and problems. Today's talk stems from one of them.
- There is a subtlety on the construction of $\eta_G\colon \mathrm{Bo}_2 \to \mathrm{HS}$ for higher genus cases due to the non-flatness of the moment map.
  Composition of morphisms in HS

$$X_{23} \circ X_{12} := (X_{12}^{\mathrm{op}} \times X_{23})/\!\!/_{\mu}\Delta(G_2) = \mu^{-1}(0)/\Delta(G_2).$$

- To construct $\eta_G^{\mathrm{ch}} : \mathrm{Bo}_2 \to \mathrm{HS}^{\mathrm{ch}}$ for higher genus cases, it would be necessary to modify $\mathrm{HS}^{\mathrm{ch}}$ and HS, since BRST reduction for a non-flat moment map does NOT yield a stalk complex.
  Composition of morphisms in $\mathrm{HS}^{\mathrm{ch}}$:

$$V_{23} \circ V_{12} := H^{\frac{\infty}{2}+0}(\widehat{\mathfrak{g}}_{-2h_2^\vee}, \mathfrak{g}_2, V_{12}^{\mathrm{op}} \otimes V_{23}).$$

- In the intensive lectures, Arakawa commented:

derived symplectic geometry を使うとできるかもしれない.

1.4. Motivation and main result

- What is derived symplectic geometry ?
  In this talk, it means the study of shifted symplectic derived schemes/stacks in the realm of derived algebraic geometry (DAG).
- Very naively speaking, one can transfer objects in classical algebraic geometry (scheme theory) to DAG by the next replacement.

classical	derived
set	∞-category (simplicial set)
comm. ring A	simplicial/dg comm. ring $A^\bullet$
scheme $(X, \mathcal{O}_X)$	derived scheme $(X, \mathcal{O}_X^\bullet)$
$\mathrm{Hom}_{\mathsf{Sch}}(X, Y)$: morphism set	$\mathrm{Map}_{\mathsf{dSch}}(X, Y)$: morphism space

- As we should replace algebras by dg algebras, the notion of symplectic/Poisson structure in DAG should admit shift (as complex)... $\rightsquigarrow$ shifted symplectic/Poisson structure.

1.4. Motivation and main result

- The idea of using derived symplectic geometry to construct η_G in full genera is originally due to Calaque, who introduced the ∞-category MT of derived Moore-Tachikawa varieties:

  ["Lagrangian structures on mapping stacks and semi-classical TFTs", arXiv:1306.3235]

  - Objects: semisimple algebraic groups (same as HS)
  - Morphisms $R\colon G_1 \to G_2$: dg Poisson commutative algebras R with Hamiltonian $(\mathfrak{g}_1 \oplus \mathfrak{g}_2)$-action.
  - Composition of $R_{12} \in \mathrm{Map}_{\mathsf{MT}}(G_1, G_2)$ and $R_{23} \in \mathrm{Map}_{\mathsf{MT}}(G_2, G_3)$:

$$R_{23} \,\widetilde{\circ}\, R_{12} := \left(R_{12}^{\mathrm{op}} \otimes R_{23}\right) /\!\!/_{\mu}^{\mathbb{L}} \mathrm{Sym}(\mathfrak{g}_2).$$

  $/\!\!/_{\mu}^{\mathbb{L}}$: derived Hamiltonian reduction of dg Poisson algebras $\mu := -\mu_{12}^2 \otimes 1 + 1 \otimes \mu_{23}^1$. We call $R_{23} \,\widetilde{\circ}\, R_{12}$ derived gluing.

1.4. Motivation and main result

The main statement of this talk: I define the ∞-category $\mathsf{MT}^{\mathrm{ch}}$ by

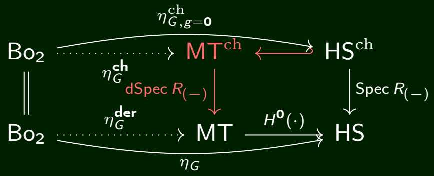
- Objects: semisimple algebraic groups (same as $\mathsf{HS}, \mathsf{HS}^{\mathrm{ch}}$).
- Morphisms: dg vertex algebras V with $\mu_V : V_k(\mathfrak{g}_1) \otimes V_l(\mathfrak{g}_2) \to V$.
- Compos. of $V_{12} : G_1 \to G_2$ and $V_{23} : G_2 \to G_3$ is given by BRST reduction: $V_{23} \mathbin{\widetilde{\circ}} V_{12} := \mathrm{BRST}(\widehat{\mathfrak{g}}_{l+m}, V_{12}^{\mathrm{op}} \otimes V_{23}, \mu)$ (chiral derived gluing).

Theorem ([Y. "Derived gluing construction of chiral algebras", Lett. Math. Phys. 2021, arXiv:2004.10055])

Taking associated derived scheme gives a functor

$$\mathrm{dSpec}\, R_{(-)} : \mathsf{MT}^{\mathrm{ch}} \longrightarrow \mathsf{MT},$$

i.e., $R_{V \widetilde{\circ} W} \simeq R_V \mathbin{\widetilde{\circ}} R_W$ *in* MT.



2. Vertex algebras in derived setting

2.1. Recollection on vertex algebras

c.f. [Frenkel, Ben-Zvi, "Vertex Algebras and Algebraic Curves", AMS (2001)]

- A vertex algebra $(V, |0\rangle, T, Y)$ consists of
  - a linear space V, called state space,
  - an element $|0\rangle \in V$, called vacuum,
  - an endomorphism $T \in \mathrm{End}\, V$, called translation,
  - a linear map $Y(\cdot, z) : V \to (\mathrm{End}\, V)[[z^{\pm 1}]]$, called state-field corresp., denoted as $Y(a, z) = a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$ for each $a \in V$,

  satisfying

  (i) $a(z)b \in V((z))$ for any $a, b \in V$,

  (ii) $Y(|0\rangle, z) = \mathrm{id}_V$, $a(z)|0\rangle = a + O(z)$ for any $a \in V$,

  (iii) $T|0\rangle = 0$, $[T, a(z)] = \partial_z a(z)$ for any $a \in V$,

  (iv) $\forall a, b \in V$, $\exists N_{a,b} \in \mathbb{N}$ s.t. $(z-w)^{N_{a,b}}[a(z), b(w)] = 0$.
- A vertex algebra can be regarded as a linear space V equipped with infinitely-many binary operations $(a, b) \mapsto a_{(n)}b \ \ (n \in \mathbb{Z})$.

2.1. Recollection on vertex algebras

- A vertex algebra $(V, |0\rangle, T, Y)$ consists of
  - state space V,
  - vacuum $|0\rangle \in V$,
  - translation $T \in \operatorname{End} V$,
  - state-field correspondence $Y(a, z) = a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$,

  satisfying (i), (ii), (iii) and

  (iv) [locality] $\forall a, b \in V$, $\exists N_{a,b} \in \mathbb{N}$ s.t. $(z - w)^{N_{a,b}}[a(z), b(w)] = 0$.
- $V = (V, |0\rangle, T, Y)$ is called commutative if $N_{a,b} = 0$ $\forall a, b \in V$. Such V is equivalent to a commutative algebra $(V, \cdot)$ with unit $|0\rangle$ and derivation T under the correspondence $a(z)b = e^{zT} a \cdot b$, i.e.,

$$a_{(n)} b = \frac{1}{(-n-1)!}(T^{-n-1} a) \cdot b \quad (n \leq -1).$$

  In particular, $a_{(-1)} b = a \cdot b$.

2.2. Li filtration, Zhu's C_2-algebra, and associated scheme

- Li filtration of a vertex algebra $V = (V, |0\rangle, T, Y)$.
  [H. Li, "Abelianizing vertex algebras", 2005]

$$V = F^0 V \supset F^1 V \supset F^2 V \supset \cdots$$
$$F^p V := \left\langle (a_1)_{(-n_1)} \cdots (a_r)_{(-n_r)} v \mid a_i, v \in V, n_i \in \mathbb{Z}_{>0}, \textstyle\sum_i n_i \geq p \right\rangle_{\mathrm{lin}}.$$

- The 0-th graded part

$$R_V := F^0 V / F^1 V = V / C_2(V), \quad C_2(V) := \left\langle a_{(-2)} b \mid a, b \in V \right\rangle_{\mathrm{lin}}.$$

is a Poisson (commutative) algebra, called Zhu's C_2-algebra.

[Y. Zhu, "Modular invariance of characters of vertex operator algebras", 1996]

Multiplication $\cdot$ and Poisson bracket $\{-,-\}$ are

$$\overline{a} \cdot \overline{b} := \overline{a_{(-1)} b}, \quad \{\overline{a}, \overline{b}\} := \overline{a_{(0)} b} \qquad (\overline{a} \in R_V \text{ for } a \in V).$$

$\operatorname{Spec} R_V$ is a Poisson scheme, called the associated scheme of V.

2.3. dg version [Y. arXiv:2004.10055]

I introduced dg (:= differential graded) version of these notions.

- A complex (V, d) means $V = \bigoplus_{i \in \mathbb{Z}} V^i$ with a sequence $\{d_i : V^i \to V^{i+1} \mid i \in \mathbb{Z}\}$ of linear maps satisfying $d_{i+1} d_i = 0$.
- A dg vertex algebra is a complex (V, d) equipped with a vertex superalgebra structure $(|0\rangle, T, Y)$ on $V^{\mathrm{even}} \oplus V^{\mathrm{odd}}$ such that
  - $|0\rangle \in V^0$ and $T \in \underline{\mathrm{End}}(V)^0 = \mathrm{Hom}_{\mathsf{dgVec}}(V, V)$.
  - d is an odd derivation (a la Kac) of $(V^{\mathrm{even}} \oplus V^{\mathrm{odd}}, |0\rangle, T, Y)$.
  - $a_{(n)} V^j \subset V^{i+j}$ for any $a \in V^i$ and $n \in \mathbb{Z}$.
- Li filtration $F^{\bullet} V$ is defined by the same formula as non-dg.

Lemma [Y]

For a dg vertex algebra V, $F^{\bullet} V$ is a decreasing filtration of complexes.

- Zhu's C_2-algebra $R_V := F^0 V / F^1 V = V / C_2(V)$ is a dg Poisson alg. $\mathrm{dSpec}(R_V)$ is a derived Poisson scheme, which I call the associated derived scheme of V.

3. dg Poisson algebras and derived Hamiltonian reduction

3.1. dg n-Poisson algebras (shifted Poisson structures)

[Pantev, Toën, Vaquié, Vezzosi, "Shifted symplectic structures", PIHES (2013)]

[Calaque, Pantev, Toën, Vaquié, Vezzosi,
"Shifted Poisson structures and deformation quantization", J. Top. (2017)]

- For $n \in \mathbb{Z}$, a dg n-Poisson algebra $(R, \cdot, \{\ ,\ \})$ consist of
  - dg commutative algebra $(R, \cdot)$
  - dg morphism $\{\ ,\ \} : R \otimes R \longrightarrow R[1-n]$ (n-Poisson bracket)

  satisfying
  - $\{\ ,\ \}$ is a Lie bracket on $R[n-1]$.
  - $\{f, g \cdot h\} = \{f, g\} \cdot h + (-1)^{|g||h|}\{f, h\} \cdot g$ for homog. $f, g, h \in R$.

  In the case $n = 1$, we call it a dg Poisson algebra.
- Examples. $\mathfrak{l}$: dg Lie algebra.
  - Kirillov-Kostant dg Poisson algebra $(\mathrm{Sym}(\mathfrak{l}), \{\ ,\ \}_{\mathrm{KK}})$
  - Chevalley-Eilenberg complex $\mathrm{CE}(\mathfrak{l}, \mathrm{Sym}(\mathfrak{l})) = \mathrm{Sym}(\mathfrak{l}^*[-1]) \otimes \mathrm{Sym}(\mathfrak{l})$ is a dg 2-Poisson algebra with $\cup$ product and Schouten bracket.

3.2. Safronov's derived Hamiltonian reduction

[Safronov, "Poisson reduction as a coisotropic intersection", High. Struct. (2017)]

- R: dg Poisson algebra, $\mathfrak{l}$: dg Lie algebra.
  A morphism $\mu\colon \mathfrak{l} \to R$ of dg Lie algebras is called momentum map.
  It induces $\mathsf{CE}(\mu) : \mathsf{CE}(\mathfrak{l}, \mathsf{Sym}(\mathfrak{l})) \to \mathsf{CE}(\mathfrak{l}, R)$.
  - Taking $R = \mathbb{C}$, trivial dg Poisson algebra, we also have $\mathsf{CE}(0) : \mathsf{CE}(\mathfrak{l}, \mathsf{Sym}(\mathfrak{l})) \to \mathsf{CE}(\mathfrak{l}, \mathbb{C})$
- $\mathsf{CE}(0)$ and $\mathsf{CE}(\mu)$ are coisotropic, and the derived tensor product

$$R /\!/_{\mu}^{\mathbb{L}} \mathsf{Sym}(\mathfrak{l}) := \mathsf{CE}(\mathfrak{l}, R) \otimes^{\mathbb{L}}_{\mathsf{CE}(\mathfrak{l}, \mathsf{Sym}(\mathfrak{l}))} \mathsf{CE}(\mathfrak{l}, \mathbb{C})$$

is a (homotopy) dg Poisson commutative algebra.

3.3. Relation to non-derived Hamiltonian reduction [Safronov, 2017]

- $\mathfrak{l}$: dg Lie algebra $\rightsquigarrow$ a dg Poisson commutative algebra

$$\overline{\mathrm{Cl}}(\mathfrak{l}) = (\mathrm{Sym}(\mathfrak{l}[1] \oplus \mathfrak{l}^*[-1]), d_{\overline{\mathrm{Cl}}(\mathfrak{l})}),$$

  called the classical Clifford algebra.

- R: dg Poisson commutative algebra, $\mu \colon \mathfrak{l} \to R$: momentum map $\rightsquigarrow$ classical BRST complex, a dg Poisson commutative algebra

$$\mathrm{BRST}_{\mathrm{cl}}(\mathfrak{l}, R, \mu) = (\overline{\mathrm{Cl}}(\mathfrak{l}) \otimes R, d_{\overline{\mathrm{Cl}}(\mathfrak{l}) \otimes R} + \{\overline{Q}, -\}),$$

  tensor product as graded Poisson algebras and BRST differential.

Theorem [Safronov]

For $\mathfrak{l} = \mathfrak{g}$, a finite dimensional Lie algebra,

$$R /\!\!/_{\mu}^{\mathbb{L}} \mathrm{Sym}(\mathfrak{g}) \simeq \mathrm{BRST}_{\mathrm{cl}}(\mathfrak{g}, R, \mu)$$

as (homotopy) dg Poisson algebras

3.3. Relation to non-derived Hamiltonian reduction [Safronov, 2017]

- Non-derived Hamiltonian reduction $X /\!\!/ G$, appearing in the composition in HS $X_{23} \circ X_{12} = \left(X_{12}^{\mathrm{op}} \otimes X_{23}\right) /\!\!/_{\mu} \Delta(G_2)$ is a special case of classical BRST:

$$R_{23} \circ R_{12} = H^0 \, \mathrm{BRST}_{\mathrm{cl}}(\mathfrak{g}_2, R_{12} \otimes R_{23}, \mu),$$

  where R_{ij} is a non-dg Poisson algebra with $X_{ij} = \operatorname{Spec} R_{ij}$.

- By Safronov's Theorem, composition in MT can be regarded as classical BRST reduction:

$$R_{23} \,\widetilde{\circ}\, R_{12} = \left(R_{12}^{\mathrm{op}} \otimes R_{23}\right) /\!\!/_{\mu}^{\mathbb{L}} \mathrm{Sym}(\mathfrak{g}_2) \simeq \mathrm{BRST}_{\mathrm{cl}}(\mathfrak{g}_2, R_{12} \otimes R_{23}, \mu).$$

  As a result, we have a Quillen adjunction

$$\mathrm{MT} \underset{H^0}{\overset{i}{\leftrightarrows}} \mathrm{HS}$$

4. dg vertex Poisson algebras and derived arc spaces

4.1. Vertex Poisson algebras and arc spaces

- For a vertex algebra V,

$$V \twoheadrightarrow \mathrm{gr}^F V = \bigoplus_n F^n V / F^{n+1} V \twoheadrightarrow R_V = F^0 V / F^1 V.$$

  $\mathrm{gr}^F V$ has a structure of vertex Poisson algebra.
- vertex Poisson algebra $(P, |0\rangle, T, Y_+, Y_-) :=$
  comm. vertex algebra $(P, |0\rangle, T, Y_+)$ + vertex Lie algebra (P, d, T, Y_-)
  [Frenkel, Ben-Zvi, "Vertex Algebras and Algebraic Curves", AMS (2001)]
- Example of vertex Poisson algebra from arc space (∞-jet scheme).
  [Arakawa, "A remark on the C_2 cofiniteness condition on vertex algebras", 2012]
  - For a commutative algebra A, there is a commutative algebra $J_\infty(A)$ with derivation T s.t.

$$\mathrm{Hom}_{\mathrm{ComAlg}}(J_\infty(A), -) = \mathrm{Hom}_{\mathrm{ComAlg}}(A, - \otimes \mathbb{C}[[z]]).$$

  - Proposition [Arakawa] (level 0 vertex Poisson structure)
    For a Poisson algebra R, $J_\infty(R)$ is a vertex Poisson algebra with

$$u_{(n)}(T^l v) = \begin{cases} \frac{l!}{(l-n)!} T^{l-n} \{u, v\}_R & (l \geq n) \\ 0 & (l < n) \end{cases} \qquad (u, v \in R \subset J_\infty(R) \quad l, n \in \mathbb{N}).$$

4.2. dg vertex Poisson algebra from derived arc space [Y. arXiv:2004.10055]

- For a dg commutative algebra A, $\exists J_\infty(A)$ with derivation T s.t.

$$\mathrm{Map}_{\mathsf{dSch}}(-, \mathrm{dSpec}\, J_\infty(A)) \simeq \mathrm{Map}_{\mathsf{dSch}}(- \times^{\mathbb{L}} \mathrm{dSpec}\, \mathbb{C}[[z]], \mathrm{dSpec}\, A).$$

- dg vertex Poisson algebra $(P, d, |0\rangle, T, Y_+, Y_-)$
  := dg commutative vertex algebra + dg vertex Lie algebra
  - dgVA: ∞-category of dg vertex algebras
  - dgVP: ∞-category of dg vertex Poisson algebras
  - dgPA: ∞-category of dg Poisson algebras
- Sequence of functors

$$R_{(-)} = \left(\mathrm{dgVA} \xrightarrow{\mathrm{gr}^F} \mathrm{dgVP} \xrightarrow{R^{\mathrm{co}}_{(-)}} \mathrm{dgPA}\right)$$
$$V \longmapsto \bigoplus_n F^n V / F^{n+1} V \longmapsto R_V := F^0 V / F^1 V$$
$$\simeq R^{\mathrm{co}}_{\mathrm{gr}^F V} := (\mathrm{gr}^F V)/(\mathrm{Im}\, T).$$

- Lemma [Y] (level 0 dg vertex Poisson structure):
  For a dg Poisson algebra P, $J_\infty(P)$ is a dg vertex Poisson algebra satisfying $R^{\mathrm{co}}_{J_\infty(P)} = P$.

4.3. Coisson BRST reduction for dg vertex Poisson algebra [Y. arXiv:2004.10055]

As a "vertex Poisson lift" of MT, I introduced the ∞-category $\mathrm{MT}^{\mathrm{co}}$ with

- Objects: semisimple algebraic groups (same as MT)
- Morphisms $P\colon G_1 \to G_2$: dg vertex Poisson algebras P with morphism $J_\infty(\mathrm{Sym}(\mathfrak{g}_1 \oplus \mathfrak{g}_2)) \to P$ in dgVP.
  - $J_\infty(\mathrm{Sym}(\mathfrak{g})) = \mathrm{Sym}(\mathfrak{g}[[t]])$ with level 0 dg vertex Poisson structure.
- Composition of $P_{12}\colon G_1 \to G_2$ and $P_{23}\colon G_2 \to G_3$:

$$R_{23} \mathbin{\widetilde{\circ}} R_{12} := \mathrm{BRST}_{\mathrm{co}}(\mathfrak{g}_2, P_{12} \otimes P_{23}, \mu_{\mathrm{co}}).$$

coisson BRST reduction (vertex Poisson analogue of Hamiltonian reduction).

- $\mathfrak{l}$: dg Lie algebra $\rightsquigarrow$ Clifford vertex Poisson algebra $\mathrm{Cl}_{\mathrm{co}}(J_\infty(\mathfrak{l})) = \mathrm{Sym}(J_\infty(\mathfrak{l})[1] \oplus J_\infty(\mathfrak{l})^*[-1]) \in \mathrm{dgVP}$.
- $P \in \mathrm{dgVP}$ with $\mu_{\mathrm{co}} : J_\infty(\mathrm{Sym}(\mathfrak{l})) \to P$ (coisson momentum map) $\rightsquigarrow \mathrm{BRST}_{\mathrm{co}}(\mathfrak{l}, P, \mu_{\mathrm{co}}) = (\mathrm{Cl}_{\mathrm{co}}(J_\infty(\mathfrak{l})) \otimes P, d_{\mathrm{co}})$: coisson BRST cplx.

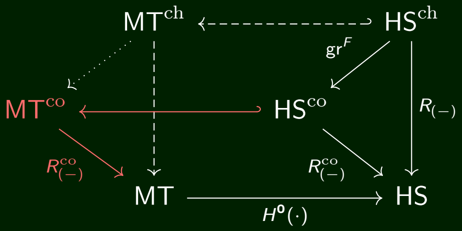
4.3. Coisson BRST reduction for dg vertex Poisson algebra [Y. arXiv:2004.10055]

Proposition [Y]

Given a momentum map $\mu\colon \mathfrak{l} \to R$ (morphism in dgPA), we have a coisson momentum map $\mu_{\mathrm{co}} = J_\infty(\mu) : J_\infty(\mathrm{Sym}(\mathfrak{l})) \to J_\infty(R)$ (morphism in dgVP) and

$$R^{\mathrm{co}}_{\mathrm{BRST}_{\mathrm{co}}(J_\infty(\mathfrak{l}), J_\infty(R), J_\infty(\mu))} \simeq \mathrm{BRST}_{\mathrm{cl}}(\mathfrak{l}, R, \mu).$$

- At this stage, we have a commutative diagram



5. Chiral derived gluing

5.1. BRST reduction

- Back to the setting in part 1.
  G: semisimple group. $\mathfrak{g} = \mathrm{Lie}(G)$.
  $V_k(\mathfrak{g})$: universal affine vertex algebra at level $k \in \mathbb{C}$
  - As a linear space, $V_k(\mathfrak{g}) = U(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[[t]] \oplus \mathbb{C}K)} \mathbb{C}_k$ with $\widehat{\mathfrak{g}} = \mathfrak{g}((t)) \oplus \mathbb{C}K$ affine Lie alg., $\mathbb{C}_k = \mathbb{C}\,|0\rangle$, $K\,|0\rangle = k\,|0\rangle$.
  - For $a = (x \otimes t^n)\,|0\rangle \in V_k(\mathfrak{g})$ ($x \in \mathfrak{g}$, $n \leq -1$), $Y(a,z) := \frac{1}{(-n-1)!}\partial_z^{-n-1} \sum_m (x \otimes t^m) z^{-m-1}$.
- $V \in$ dgVA with $\mu\colon V_k(\mathfrak{g}) \to V$ (chiral momentum map), $\mathrm{BRST}(\widehat{\mathfrak{g}}_k, V, \mu) = (V \otimes \bigwedge^{\frac{\infty}{2}}(\mathfrak{g}), d_{\mathrm{ch}}) \in$ dgVA: BRST complex.
  c.f. [Frenkel, Ben-Zvi, "Vertex Algebras and Algebraic Curves", AMS (2001)]

Proposition [Y], compatibility of reductions

$\mathrm{gr}^F \mathrm{BRST}(\widehat{\mathfrak{g}}_k, V, \mu) \simeq \mathrm{BRST}_{\mathrm{co}}(J_\infty(\mathfrak{g}), \mathrm{gr}^F V, \mathrm{gr}^F \mu)$ in dgVP,

$R_{\mathrm{BRST}(\widehat{\mathfrak{g}}_k, V, \mu)} \simeq \mathrm{BRST}_{\mathrm{cl}}(\mathfrak{g}, R_V, R_\mu)$ in dgPA.

5.2. Definition of $\mathrm{MT}^{\mathrm{ch}}$ [Y. arXiv:2004.10055]

Definition of $\mathrm{MT}^{\mathrm{ch}}$ [Y]

- Objects: simply connected semi-simple groups G.
- Morphism $(V, \mu_V) : G_1 \to G_2$: $V \in$ dgVA with chiral momentum map $\mu_V\colon V_k(\mathfrak{g}_1) \otimes V_l(\mathfrak{g}_2) \to V$.
- Composition of $(V, \mu_V)\colon G_1 \to G_2$ and $(W, \mu_W)\colon G_2 \to G_3$:

$$W \,\tilde{\circ}\, V := \mathrm{BRST}(\widehat{\mathfrak{g}}_{2\,l+m}, V^{\mathrm{op}} \otimes W, \mu),$$

  where V^{op} with $Y_{V^{\mathrm{op}}}(a, z) := Y_V(a, -z)$ and $\mu := -\mu_V^2 + \mu_W^1$.
  chiral derived gluing (dg vertex algebra analogue of Hamiltonian reduction).
- $\otimes$ is the tensor product of dg vertex algebras.
- There is a natural duality structure.

5.3. Main statement [Y. arXiv:2004.10055]

Theorem ([Y])

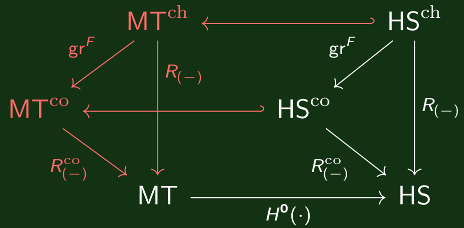
The functors

$\mathrm{gr}^F : \mathrm{dgVA} \to \mathrm{dgVP}$, $R^{\mathrm{co}} : \mathrm{dgVP} \to \mathrm{dgPA}$ *and* $R : \mathrm{dgVA} \to \mathrm{dgPA}$

yield functors

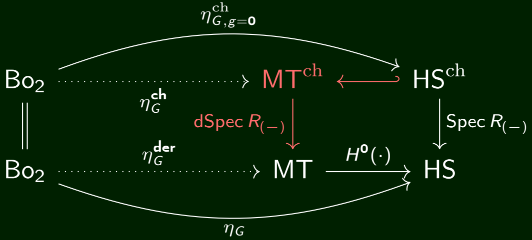
$\mathrm{MT}^{\mathrm{ch}} \to \mathrm{MT}^{\mathrm{co}}$, $\mathrm{MT}^{\mathrm{co}} \to \mathrm{MT}$, $\mathrm{MT}^{\mathrm{ch}} \to \mathrm{MT}$ *respectively,*

which sit in a commutative diagram of ∞*-categories*



5.4. Concluding remark

- We may expect to have the following commutative diagram:



- There seems no explicit description of higher genus Moore-Tachikawa varieties $\eta_G(\Sigma_{g,n}) \in \mathrm{HS}$ $(g \geq 1)$.

Thank you.