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Osaka AG Seminar

Geometric derived Hall algebra II

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0 Introduction

In the first talk, I explained Toën's derived Hall algebra (§1) and what will be needed to construct it geometrically (§2).

- D: a dg-category of finite type over $\mathbb{F}_q$
- $\mathcal{P}(\mathsf{D})$: the moduli stack of perfect dg-modules over D^{op}.
- $\mathcal{G}(\mathsf{D})$: the moduli stack of cofibrations $X \hookrightarrow Y$ of perfect dg-modules over D^{op}.
  Diagram of correspondence

$$\begin{array}{ccc} \mathcal{G}(\mathsf{D}) & \xrightarrow{\ c\ } & \mathcal{P}(\mathsf{D}) \\ {\scriptstyle p}\downarrow & & \\ \mathcal{P}(\mathsf{D}) \times \mathcal{P}(\mathsf{D}) & & \end{array} \qquad \begin{array}{ccc} (X \hookrightarrow Y) & \longmapsto & Y \\ \big\downarrow & & \\ (X, Y \coprod^X 0) & & \end{array}$$

- $\mathsf{D}_\mathsf{c}^b(\mathcal{X}, \overline{\mathbb{Q}}_\ell)$: the bounded derived category of constructible lisse-étale ℓ-adic sheaves over a locally geometric derived stack $\mathcal{X}$, and Grothendieck's six operations $\mathrm{R}f_*, \mathrm{L}f^*, \mathrm{R}f_!, \mathrm{L}f^!, \otimes^{\mathrm{L}}, \mathrm{R}\,\mathcal{H}om$.

$$\begin{array}{ccc} \mathsf{D}_\mathsf{c}^b(\mathcal{G}(\mathsf{D}), \overline{\mathbb{Q}}_\ell) & \xrightarrow{\ \mathrm{R}c_!\ } & \mathsf{D}_\mathsf{c}^b(\mathcal{P}(\mathsf{D}), \overline{\mathbb{Q}}_\ell) \\ {\scriptstyle \mathrm{L}p^*}\uparrow & & \\ \mathsf{D}_\mathsf{c}^b(\mathcal{P}(\mathsf{D}) \times \mathcal{P}(\mathsf{D}), \overline{\mathbb{Q}}_\ell) & & \end{array}$$

- Geometric Hall-algebra multiplication μ will be associative.

$$\mu : \mathsf{D}_\mathsf{c}^b(\mathcal{P}(\mathsf{D}), \overline{\mathbb{Q}}_\ell)^{\times 2} \longrightarrow \mathsf{D}_\mathsf{c}^b(\mathcal{P}(\mathsf{D}), \overline{\mathbb{Q}}_\ell), \quad M \longmapsto \mathrm{R}c_!\, \mathrm{L}p^*(M)[\dim p]$$

Today I will explain

- Moduli spaces of complexes via geometric derived stacks [Toën-Vaquié] (§4).
- Lisse-étale constructible ℓ-adic sheaves over derived stacks (§5).
- Derived category and derived functors (§6).
- Geometric construction of derived Hall algebras (§7).
- Toward canonical bases for derived Hall algebras (§8).

The constructions in §5 and §6 are natural analogue of those for algebraic stacks developed by [Laszlo-Olsson, 2008].

Based on my preprint
S. Yanagida, "Geometric derived Hall algebra", arXiv:1912.05442.

See also
柳田伸太郎, 「幾何学的導来Hall代数」代数学シンポジウム講演集 (2020).

4 Moduli spaces of complexes

Everything is defined over a commutative ring k.

4.1 Recollection: Moduli functor of perfect objects

For a dg-category D over k and $A \in \mathsf{sCom}$, we consider

$$\mathcal{M}_{\mathsf{D}}(A) := \mathrm{Map}_{\mathrm{dgCat}}(\mathsf{D}^{\mathsf{op}}, \mathsf{P}(A)), \quad \mathsf{P}(A) := \mathsf{Mod}_{\mathsf{dg}}(\mathrm{N}_{\mathsf{dg}}(A))^{\circ}_{\mathsf{perf}},$$

which gives rise to a functor of ∞-categories

$$\mathcal{M}_{\mathsf{D}} \in \mathsf{PSh}_{\infty}(\mathsf{dAff}_{\infty}) := \mathsf{Fun}_{\infty}((\mathsf{dAff}_{\infty})^{\mathsf{op}}, \mathcal{S}) = \mathsf{Fun}_{\infty}(\mathsf{sCom}_{\infty}, \mathcal{S}).$$

Fact ([Toën-Vaquié). $\mathcal{M}_{\mathsf{D}} \in \mathsf{PSh}_{\infty}(\mathsf{dAff}_{\infty})$ is a derived stack over k. We call it the moduli stack of perfect D^{op}-dg-modules

Remark. • The 0-th homotopy $\pi_0(\mathcal{M}_{\mathsf{D}}(k))$ is bijective to the set of isomorphism classes of perfect D^{op}-dg-modules in $\mathrm{Ho}(\mathsf{M}(\mathsf{D}))$.

• For each $x \in \mathsf{M}(\mathsf{D})$, we have

$$\pi_1(\mathcal{M}_{\mathsf{D}}, x) \simeq \mathrm{Aut}_{\mathrm{Ho}(\mathsf{M}(\mathsf{D}))}(x, x), \quad \pi_i(\mathcal{M}_{\mathsf{D}}, x) \simeq \mathrm{Ext}^{-i}_{\mathrm{Ho}(\mathsf{M}(\mathsf{D}))}(x, x) \ (i \in \mathbb{Z}_{\geq 2}).$$

4.2 Geometricity of moduli stacks of perfect objects

We explain the main result in [Toën-Vaquié, 2009].

Definition. A dg-category D over k is of finite type if there exists a k-dg-algebra B which is homotopically finitely presented in the model category dgAlg_k of dg-algebras s.t. $\mathsf{P}(\mathsf{D}) = \mathsf{Mod}_{\mathsf{dg}}(\mathsf{D})^{\circ}_{\mathsf{perf}}$ is quasi-equivalent to $\mathsf{Mod}_{\mathsf{dg}}(B)^{\circ}$.

Fact (Toën-Vaquié)**.** If D is a dg-category over k of finite type, then the derived stack $\mathcal{M}_{\mathsf{D}}$ is locally geometric and locally of finite presentation.

Here I used

Definition. A derived stack $\mathcal{X}$ is called locally geometric if $\mathcal{X}$ is equivalent to a filtered colimit $\varinjlim_{i \in I} \mathcal{X}_i$ of derived stacks $\{\mathcal{X}_i\}_{i \in I}$ s.t.

- each derived stack $\mathcal{X}_i$ is n_i-geometric for some $n_i \in \mathbb{Z}_{\geq -1}$,
- each morphism $\mathcal{X}_i \to \mathcal{X}_i \times_{\mathcal{X}} \mathcal{X}_i$ of derived stacks induced by $\mathcal{X}_i \to \mathcal{X}$ is an equivalence in the ∞-category dSt_{∞} of derived stacks.

Definition. 1. An n-geometric derived stack $\mathcal{X}$ is called locally of finite presentation if it has an n-atlas $\{U_i\}_{i\in I}$ such that for each representable derived stack $U_i \simeq \operatorname{Spec} A_i$ the simplicial k-algebra A_i is finitely presented (see below).

2. A locally geometric derived stack $\mathcal{X}$ is locally of finite presentation if each geometric derived stack $\mathcal{X}_i$ in $\mathcal{X} \simeq \varinjlim_i \mathcal{X}_i$ can be chosen to be locally of finite presentation in the sense of 1.

Definition. 1. A morphism $f : A \to B$ in sCom_∞ is called finitely presented if for any filtered system $\{C_i\}_{i\in I}$ of objects in $(\mathsf{sCom}_\infty)_{A/}$ the natural morphism

$$\varinjlim_{i\in I} \operatorname{Map}_{(\mathsf{sCom}_\infty)_{A/}}(B, C_i) \longrightarrow \operatorname{Map}_{(\mathsf{sCom}_\infty)_{A/}}(B, \varinjlim_{i\in I} C_i)$$

is an isomorphism in $\mathcal{H}$.

2. $A \in \mathsf{sCom}_\infty$ is called finitely presented or of finite presentation if the morphism $k \to A$ is finitely presented in the sense of 1.

4.3 Moduli stack of complexes of quiver representations

- kQ: the path algebra of a quiver Q over k.
- Regard kQ as a dg-algebra over k, and as a dg-category over k.
- $\mathrm{Mod}_{\mathrm{dg}}(kQ)$ is the dg-category of complexes of representations of Q over k.

Definition. We call the derived stack $\mathcal{M}_{kQ}$ the derived stack of perfect complexes of representations of Q and denote it by

$$\mathcal{P}(Q) := \mathcal{M}_{kQ}.$$

Fact 1. Let Q be a finite quiver with no loops. Then the derived stack $\mathcal{P}(Q)$ is locally geometric and locally of finite presentation over k.

$\pi_0(\mathcal{P}(Q)(k))$ is the set of isom. classes of perfect complexes of reps. of Q over k.

5 Constructible sheaves on derived stacks

5.1 Lisse-étale ∞-site

We will introduce the lisse-étale ∞-site for a geometric derived stack, an analogue of the lisse-étale site for an algebraic stack [Laumon, Moret-Bailly, 2000].

- $(\mathsf{dSt}_\infty)_{/\mathfrak{X}}$: the over-$\infty$-category of derived stacks over a derived stack $\mathfrak{X}$.
- $\mathsf{dAff}_\infty/\mathfrak{X} \subset (\mathsf{dSt}_\infty)_{/\mathfrak{X}}$: the full sub-$\infty$-category spanned by affine derived schemes

Definition. Let $n \in \mathbb{Z}_{\geq -1}$ and $\mathfrak{X}$ be an n-geometric derived stack.
The lisse-étale ∞-site

$$\mathsf{Lis\text{-}Et}^n_\infty(\mathfrak{X}) = (\mathsf{Lis}^n_\infty(\mathfrak{X})\,, \mathsf{lis\text{-}et})$$

on $\mathfrak{X}$ is the ∞-site given by the following description.

- $\mathsf{Lis}^n_\infty(\mathfrak{X})$ is the full sub-∞-category of $\mathsf{dAff}_\infty/\mathfrak{X}$ spanned by (U, u) where the morphism $u : U \to \mathfrak{X}$ is n-smooth.
- The set $\mathrm{Cov}_{\mathsf{lis\text{-}et}}(U, u)$ of covering sieves on (U, u) consists of $\{(U_i, u_i) \to (U, u)\}_{i \in I}$ in $\mathsf{Lis}^n_\infty(\mathfrak{X})$ s.t. $\{U_i \to U\}_{i \in I}$ is an étale covering.

Recall: A morphism $A \to B$ in sCom_∞ is called étale [smooth] if

- the induced $\pi_0(A) \to \pi_0(B)$ is an étale [smooth] map of commutative k-algebras,
- the induced $\pi_i(A) \otimes_{\pi_0(A)} \pi_0(B) \to \pi_i(B)$ is an isomorphism for any i.

5.2 Constructible lisse-étale sheaves

Recall the notion of a constructible sheaf on an ordinary scheme:

A sheaf $\mathcal{F}$ on a scheme X is called constructible if for any affine Zariski open $U \subset X$ there is a finite decomposition $U = \cup_i U_i$ into constructible locally closed subschemes U_i such that $\mathcal{F}|_{U_i}$ is a locally constant sheaf with value in a finite set.

We introduce an analogue of this notion for derived stacks.

Definition. Let $\mathcal{X}$ be a geometric derived stack. An object of the ∞-category $\mathrm{Sh}_{\infty,\text{lis-et}}(\mathrm{Lis}_\infty(\mathcal{X}))$ is called a lisse-étale sheaf.

For an affine derived scheme U, we denote by $\pi_0(U)$ the associated affine scheme.

Definition. A lisse-étale sheaf $\mathcal{F}$ on $\mathcal{X}$ is called constructible if

(i) it is cartesian, i.e., for any morphism $f : T \to T'$ in $\mathcal{X}_{\text{lis-et}}$, the natural morphism $f^{-1}\mathcal{F}_T \to \mathcal{F}_{T'}$ is an equivalence, and

(ii) for any $U \in \mathrm{Lis}_\infty(\mathcal{X})$ the restriction $\pi_0(\mathcal{F})|_{\pi_0(U)}$ is a constructible sheaf on $\pi_0(U)$.

Definition. Λ: a commutative ring.
A lisse-étale sheaf of Λ-modules is an object of the ∞-category

$$\mathsf{Sh}_{\infty,\text{lis-et}}(\mathsf{Lis}_{\infty}(\mathcal{X})\,,\mathrm{N}(\mathsf{Mod}(\Lambda))).$$

We then have the dg-category of complexes consisting of lisse-étale sheaves of Λ-modules. By the dg nerve construction, we obtain an ∞-category.

Definition. We denote the obtained ∞-category of complexes of lisse-étale sheaves by

$$\mathsf{Mod}_{\infty}(\mathcal{X}_{\text{lis-et}},\Lambda)\,.$$

For $* \in \{+,-,b\}$ we denote by

$$\mathsf{Mod}^{*}_{\infty}(\mathcal{X}_{\text{lis-et}},\Lambda) \subset \mathsf{Mod}_{\infty}(\mathcal{X}_{\text{lis-et}},\Lambda)$$

the full sub-∞-category spanned by complexes whose homologies are bounded below (resp. bounded above, resp. bounded).
The full sub-∞-categories with constructible homologies are denoted by

$$\mathsf{Mod}^{\mathsf{c}}_{\infty}(\mathcal{X}_{\text{lis-et}},\Lambda)\,,\quad \mathsf{Mod}^{\mathsf{c},*}_{\infty}(\mathcal{X}_{\text{lis-et}},\Lambda) := \mathsf{Mod}^{\mathsf{c}}_{\infty}(\mathcal{X}_{\text{lis-et}},\Lambda) \cap \mathsf{Mod}^{*}_{\infty}(\mathcal{X}_{\text{lis-et}},\Lambda)\,.$$

6 Derived category and derived functors

6.1 Derived ∞-category of constructible lisse-étale sheaves

Proposition. $\mathcal{X}$: a locally geometric derived stack. Λ: a commutative ring. The ∞-category of complexes of constructible lisse-étale Λ-sheaves

$$\mathsf{Mod}^{\mathsf{c},*}_{\infty}(\mathcal{X}_{\mathsf{lis\text{-}et}}, \Lambda)$$

is stable in the sense of [Lurie, Higher Algebra].
In particular, the homotopy category $\mathrm{Ho}\,\mathsf{Mod}^{\mathsf{c},*}_{\infty}(\mathcal{X}_{\mathsf{lis\text{-}et}}, \Lambda)$ has a structure of a triangulated category (explained below).

Definition. The (left bounded, resp. right bounded, resp. bounded) derived category of constructible sheaves of Λ-modules on $\mathcal{X}$ is defined to be

$$\mathsf{D}^{*}_{\mathsf{c}}(\mathcal{X}, \Lambda) := \mathrm{Ho}\,\mathsf{Mod}^{\mathsf{c},*}_{\infty}(\mathcal{X}_{\mathsf{lis\text{-}et}}, \Lambda) \quad (* \in \{\emptyset, +, -, b\}).$$

Below we give a brief recollection on stable ∞-categories.

Definition (Lurie, HA, §1.1.1)**.** An ∞-category is stable if

(i) it has a zero object $0 \in \mathsf{C}$,

(ii) any morphism has a fiber and cofiber, and

(iii) a triangle in C is a pullback square iff it is a pushout square.

A triangle in C is a square of the following form:

$$\begin{array}{ccc} X & \rightarrow & Y \\ \downarrow & & \downarrow \\ 0 & \rightarrow & Z \end{array}$$

For a stable ∞-category C, we can define a suspension functor $\Sigma : \mathsf{C} \to \mathsf{C}$ and a loop functor $\Omega : \mathsf{C} \to \mathsf{C}$ [Lurie, HA, §1.1.2].

Fact (Lurie, HA, §1.1.2)**.** For a stable ∞-category C, the homotopy category $\mathrm{Ho}\,\mathsf{C}$ has a structure of a triangulated category with $[1] = \Sigma : \mathrm{Ho}\,\mathsf{C} \to \mathrm{Ho}\,\mathsf{C}$ and the distinguished triangles in the next page.

A distinguished triangle in $\mathrm{Ho}\,\mathsf{C}$ is a diagram of the form

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

such that there is a diagram in C of the form

$$\begin{array}{ccccc} X & \xrightarrow[\widetilde{f}]{} & Y & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle \widetilde{g}} & & \downarrow \\ 0' & \longrightarrow & Z & \xrightarrow{\widetilde{h}} & W \end{array}$$

satisfying the following 4 conditions.

(i) $0, 0' \in \mathsf{C}$ are zero objects.

(ii) The two squares are pushout square in C.

(iii) Morphisms $\widetilde{f}, \widetilde{g}$ in C represent f, g in $\mathrm{Ho}\,\mathsf{C}$ respectively.

(iv) h is equal to the composition of the homotopy class of $\widetilde{h}$ and the equivalence $W \simeq X[1]$ given by the outer rectangle.

Using this fact, we can lift notions on triangulated categories to those on stable ∞-categories. For example:

Definition. A t-structure of a stable ∞-category C is a t-structure on the homotopy category $\mathrm{Ho}\,\mathsf{C}$.

Below we explain derived ∞-categories [Lurie, HA, §1.3.2].

- A: an abelian category with enough injectives.
- C(A): the dg-category of complexes in A (with injective model structure).
- $\mathsf{C}^+(\mathsf{A}_{\mathsf{inj}}) \subset \mathsf{C}(\mathsf{A})$: the full subcat. of complexes bounded below of injectives.

The dg nerve construction gives an ∞-category

$$\mathsf{D}^+_\infty(\mathsf{A}) := \mathrm{N}_{\mathsf{dg}}(\mathsf{C}^+(A_{\mathsf{inj}})),$$

which is known to be stable. It is called the derived ∞-category of A.

$\mathsf{D}^+_\infty(\mathsf{A})$ has a t-structure determined by $(\mathsf{D}^+_\infty(\mathsf{A})_{\le 0}, \mathsf{D}^+_\infty(\mathsf{A})_{\ge 0})$ with
$\mathsf{D}^+_\infty(\mathsf{A})_{\ge 0}$: the full sub-$\infty$-cat. of $H_n(M) := \pi_0(M[n]) \simeq 0$ in $\mathrm{N}(\mathsf{A})$ for $n < 0$,
$\mathsf{D}^+_\infty(\mathsf{A})_{\le 0}$: similarly defined.
This t-structure enjoys the following properties.

1. The core $\mathsf{D}^+_\infty(\mathsf{A})^\heartsuit := \mathsf{D}^+_\infty(\mathsf{A})_{\le 0} \cap \mathsf{D}^+_\infty(\mathsf{A})_{\ge 0}$ is equivalent to $\mathrm{N}(\mathsf{A})$.
2. $\mathrm{Ho}\,\mathsf{D}^+_\infty(\mathsf{A}) \simeq \mathsf{D}^+(\mathsf{A})$ as triangulated categories, and the t-structure on $\mathrm{Ho}\,\mathsf{D}^+_\infty(\mathsf{A})$ is equivalent to the standard t-structure on $\mathsf{D}^+(\mathsf{A})$.

6.2 Derived functors — finite coefficient case

On the derived category of constructible lisse-étale sheaves

$$\mathsf{D}^*_{\mathsf{c}}(\mathcal{X},\Lambda) := \mathrm{Ho}\,\mathsf{Mod}^{\mathsf{c},*}_{\infty}(\mathcal{X}_{\text{lis-et}},\Lambda) \quad (* \in \{\emptyset,+,-,b\}),$$

we can construct analogue of Grothendieck's six derived functors in the case Λ is a Gorenstein local ring of dimension 0 whose residual characteristic ℓ is invertible in the base ring k.

Precisely speaking, for

- $\mathcal{X},\mathcal{Y}$: locally geometric derived stacks locally of finite presentation,
- $f:\mathcal{X}\to\mathcal{Y}$: a morphism locally of finite presentation,

we can define triangulated functors

$$\mathrm{R}f_* : \mathsf{D}^+_{\mathsf{c}}(\mathcal{X},\Lambda) \longrightarrow \mathsf{D}^+_{\mathsf{c}}(\mathcal{Y},\Lambda), \quad \mathrm{R}f_! : \mathsf{D}^-_{\mathsf{c}}(\mathcal{X},\Lambda) \longrightarrow \mathsf{D}^-_{\mathsf{c}}(\mathcal{Y},\Lambda),$$
$$\mathrm{L}f^* : \mathsf{D}_{\mathsf{c}}(\mathcal{Y},\Lambda) \longrightarrow \mathsf{D}_{\mathsf{c}}(\mathcal{X},\Lambda), \quad \mathrm{R}f^! : \mathsf{D}_{\mathsf{c}}(\mathcal{Y},\Lambda) \longrightarrow \mathsf{D}_{\mathsf{c}}(\mathcal{X},\Lambda)$$

and $\mathrm{R}\,\mathcal{H}om$, $\otimes^{\mathrm{L}}$. These functors are compatible with those for algebraic stacks developed by Laszlo and Olsson (2008).

Today I only explain the functors $\mathrm{R}f_!$ and $\mathrm{L}f^*$ appearing in the construction of derived Hall algebras.

6.2.1 $\mathrm{R}f_*$ and $\mathrm{R}f_!$

For $f : \mathcal{X} \to \mathcal{Y}$, we define the direct image functor f_* of ∞-category to be

$$f_* : \mathsf{Mod}_\infty(\mathcal{X}_{\text{lis-et}}, \Lambda) \longrightarrow \mathsf{Mod}_\infty(\mathcal{Y}_{\text{lis-et}}, \Lambda)\,, \quad (f_*\mathcal{F})(U) \;:=\; \mathcal{F}(U \times_{\mathcal{Y}} \mathcal{X}).$$

It induces a triangulated functor $\mathrm{R}f_* : \mathsf{D}(\mathcal{X}, \Lambda) := \mathrm{Ho}\,\mathsf{Mod}_\infty(\mathcal{X}_{\text{lis-et}}, \Lambda) \to \mathsf{D}(\mathcal{Y}, \Lambda)$.
If f is moreover locally of finite presentation, then we have

$$\mathrm{R}f_* : \mathsf{D}_{\mathsf{c}}^{+}(\mathcal{X}, \Lambda) \longrightarrow \mathsf{D}_{\mathsf{c}}^{+}(\mathcal{Y}, \Lambda).$$

For f locally of finite presentation, we define the shrink direct image functor $f_!$ to be

$$f_! \;:=\; D_{\mathcal{Y}} \circ f_* \circ D_{\mathcal{X}} : \mathsf{Mod}^{\mathsf{c}}_\infty(\mathcal{X}_{\text{lis-et}}, \Lambda) \longrightarrow \mathsf{Mod}^{\mathsf{c}}_\infty(\mathcal{Y}_{\text{lis-et}}, \Lambda)\,.$$

where $D_{\mathcal{X}}, D_{\mathcal{Y}}$ are the dualizing functors (introduced in the next page).
It induces a triangulated functor

$$\mathrm{R}f_! : \mathsf{D}_{\mathsf{c}}^{-}(\mathcal{X}, \Lambda) \longrightarrow \mathsf{D}_{\mathsf{c}}^{-}(\mathcal{Y}, \Lambda).$$

The dualizing functor $D_{\mathcal{X}}$ is given by the dualizing object $\Omega_{\mathcal{X}} \in \mathsf{Mod}^{\mathsf{c}}_{\infty}(\mathcal{X}_{\mathsf{lis\text{-}et}}, \Lambda)$ as

$$D_{\mathcal{X}} := \mathcal{H}om(\ , \Omega_{\mathcal{X}}) : \mathsf{Mod}^{\mathsf{c}}_{\infty}(\mathcal{X}_{\mathsf{lis\text{-}et}}, \Lambda) \longrightarrow \mathsf{Mod}^{\mathsf{c}}_{\infty}(\mathcal{X}_{\mathsf{lis\text{-}et}}, \Lambda)^{\mathsf{op}} .$$

For the existence of $\Omega_{\mathcal{X}}$, we need $\mathcal{X}$ to be locally of finite presentation and Λ to satisfy the assumption.

The dualizing functor satisfies the following property [Y, §5.5]:

1. The natural morphism $\mathrm{id} \to D_{\mathcal{X}} \circ D_{\mathcal{X}}$ is an equivalence.
2. For $M, N \in \mathsf{Mod}^{\mathsf{c}}_{\infty}(\mathcal{X}_{\mathsf{lis\text{-}et}}, \Lambda)$, we have $\mathcal{H}om(M, N) \simeq \mathcal{H}om(D_{\mathcal{X}}(N), D_{\mathcal{X}}(M))$.

6.2.2 $\mathrm{L}f^*$

We need a special care on the construction of the inverse image $\mathrm{L}f^*$ as in the case of algebraic stacks:

- In [Laumon, Moret-Bailly, "Champs algébriques", 2000], there is an error on the definition of the functor $\mathrm{L}f^*$.
- A correct definition of $\mathrm{L}f^*$ is given in [Olsson, "Sheaves on Artin stacks", 2007].
- The work [Laszlo, Olsson, 2008] is based on the corrected definition.

Our definition of $\mathrm{L}f^*$ is a simple analogue of Olsson's definition for algebraic stacks.

$\mathcal{X}$: geometric derived stack.
Take an (n-)atlas $\{U_i\}_{i \in I}$, and set $X_0 := \coprod_{i \in I} U_i$, $X_k := X_0 \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} X_0$ (k-times fiber prod.).
We have smooth epimorphisms $X_k \to \mathcal{X}$, and denote them as $e_X : X_\bullet \to \mathcal{X}$, $X_\bullet := \{X_k\}_{k \in \mathbb{N}}$ (coskeleton of $\mathcal{X}$).
For a morphism $f : \mathcal{X} \to \mathcal{Y}$ of geometric derived stacks, the coskeletons $e_X : X_\bullet \to \mathcal{X}$ and $e_Y : Y_\bullet \to \mathcal{Y}$ gives a family of morphisms $f_\bullet : X_\bullet \to Y_\bullet$ with a commutative diagram

$$\begin{array}{ccc} X_\bullet & \xrightarrow{f_\bullet} & Y_\bullet \\ {\scriptstyle e_X}\downarrow & & \downarrow{\scriptstyle e_Y} \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \end{array}$$

Recall the étale topology et defined by étale morphisms in the ∞-category sCom_∞ of simplicial commutative rings. We have ∞-topoi $X_{\bullet,\mathsf{et}}$ and $Y_{\bullet,\mathsf{et}}$.
[As an ∞-category, an object of $X_{\bullet,\mathsf{et}}$ is a data $F_\bullet = (F_n, F(\delta))$ consisting of sheaves F_n on $X_{n,\mathsf{et}}$ for each n and morphisms $F(\delta) : \delta^{-1}F_n \to F_m$ for each $\delta : [n] \to [m]$, satisfying a compatibility with composition, where we denoted by $\delta : X_m \to X_n$ the map coming from the simplicial structure.]
Then $f_\bullet$ induces a morphism of ∞-topoi

$$f_{\bullet,\mathsf{et}} : X_{\bullet,\mathsf{et}} \longrightarrow Y_{\bullet,\mathsf{et}}.$$

Denote by $\mathsf{Mod}_\infty(X_{\bullet,\mathsf{et}}, \Lambda)$ the ∞-category of sheaves of Λ-modules on $X_{\bullet,\mathsf{et}}$. Then the standard adjunction (f^{-1}, f_*) on étale sheaves yields a functor

$$f_\bullet^* : \mathsf{Mod}_\infty^{\mathsf{cart}}(Y_{\bullet,\mathsf{et}}, \Lambda) \longrightarrow \mathsf{Mod}_\infty^{\mathsf{cart}}(X_{\bullet,\mathsf{et}}, \Lambda),$$

where cart denotes the cartesian sheaves.
On the other hand, the descent argument gives

$$r_{\mathcal{X}} : \mathsf{Mod}_\infty^{\mathsf{cart}}(\mathcal{X}_{\mathsf{lis\text{-}et}}, \Lambda) \xrightarrow{\sim} \mathsf{Mod}_\infty^{\mathsf{cart}}(X_{\bullet,\mathsf{et}}, \Lambda)\,, \quad r_{\mathcal{Y}} : \mathsf{Mod}_\infty^{\mathsf{cart}}(\mathcal{Y}_{\mathsf{lis\text{-}et}}, \Lambda) \xrightarrow{\sim} \mathsf{Mod}_\infty^{\mathsf{cart}}(Y_{\bullet,\mathsf{et}}, \Lambda)\,.$$

Using these stuffs, we define

$$f^* := r_{\mathcal{X}}^{-1} \circ f_\bullet^* \circ r_{\mathcal{Y}} : \mathsf{Mod}_\infty^{\mathsf{cart}}(\mathcal{Y}_{\mathsf{lis\text{-}et}}, \Lambda) \longrightarrow \mathsf{Mod}_\infty^{\mathsf{cart}}(\mathcal{X}_{\mathsf{lis\text{-}et}}, \Lambda)\,.$$

If f is locally of finite presentation, then we have a functor for constructible sheaves

$$f^* : \mathsf{Mod}_\infty^{\mathsf{c}}(\mathcal{Y}_{\mathsf{lis\text{-}et}}, \Lambda) \longrightarrow \mathsf{Mod}_\infty^{\mathsf{c}}(\mathcal{X}_{\mathsf{lis\text{-}et}}, \Lambda)\,,$$

which induces a triangulated functor

$$\mathrm{L}f^* : \mathsf{D}_{\mathsf{c}}^+(\mathcal{Y}, \Lambda) \longrightarrow \mathsf{D}_{\mathsf{c}}^+(\mathcal{X}, \Lambda).$$

6.3 Base-change theorem

The constructed derived functors satisfy the standard properties. Today I only explain the base-change theorem, which will be used to show the associativity of Hall algebra.

Assume that we have the following cartesian diagram in the ∞-category of locally geometric derived stacks, and that f is locally of finite presentation.

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{\pi} & \mathcal{X} \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{Y}' & \xrightarrow{p} & \mathcal{Y} \end{array}$$

We have a morphism $p^* f_! \to \varphi_! \pi^*$ in $\mathsf{Fun}_\infty(\mathsf{Mod}_\infty^{\mathsf{c},-}(\mathcal{X}_{\mathsf{lis\text{-}et}}, \Lambda), \mathsf{Mod}_\infty^{\mathsf{c},-}(\mathcal{Y}'_{\mathsf{lis\text{-}et}}, \Lambda))$, and $p^! f_* \to \phi_* \pi^!$ in $\mathsf{Fun}_\infty(\mathsf{Mod}_\infty^{\mathsf{c},+}(\mathcal{X}_{\mathsf{lis\text{-}et}}, \Lambda), \mathsf{Mod}_\infty^{\mathsf{c},+}(\mathcal{Y}'_{\mathsf{lis\text{-}et}}, \Lambda))$.

Proposition (Y., §6.6)**.** If p is smooth, then

$$(p^* f_! \to \varphi_! \pi^*) \simeq (p^! f_* \to \phi_* \pi^!) \quad \text{in } \mathsf{Fun}_\infty(\mathsf{Mod}_\infty^{\mathsf{c},b}(\mathcal{X}_{\mathsf{lis\text{-}et}}, \Lambda), \mathsf{Mod}_\infty^{\mathsf{c},b}(\mathcal{Y}'_{\mathsf{lis\text{-}et}}, \Lambda)).$$

As a consequence, we have

$$(\mathrm{L}p^* \mathrm{R}f_! \to \mathrm{R}\varphi_! \mathrm{L}\pi^*) \simeq (\mathrm{L}p^! \mathrm{R}f_* \to \mathrm{R}\phi_* \mathrm{L}\pi^!) \quad \text{in } \mathsf{Fun}(\mathsf{D}_{\mathsf{c}}^b(\mathcal{X}, \Lambda), \mathsf{D}_{\mathsf{c}}^b(\mathcal{Y}', \Lambda)).$$

6.4 The case of ℓ-adic coefficients

- So far I explained the case when Λ satisfies a certain condition.
  But for the construction of derived Hall algebras we need the case $\Lambda = \overline{\mathbb{Q}}_\ell$, which does not satisfy the condition.
- For a complete DVR Λ of char. $\ell > 0$, regarding $\Lambda = \varprojlim_n (\Lambda/\mathfrak{m}^n)$, we can construct derived categories and functors as limits on n [Y., §7].
  This construction is a simple analogue of that for algebraic stacks developed by [Laszlo-Olsson, 2008].

7 Geometric construction of derived Hall algebras

D: a dg-category of finite type (in the sense of Toën-Vaquié) over $k = \mathbb{F}_q$.
(E.g. the dg-category $\mathsf{Mod}_{\mathsf{dg}}(kQ)$ of reps. of a quiver Q without loops.)

$\mathcal{P}(\mathsf{D})$: the moduli space of perfect D^{op}-dg-modules.
: a locally geometric derived stack locally of finite presentation.

Decomposition of $\mathcal{P}(\mathsf{D})$:

$$\mathcal{P}(\mathsf{D}) = \bigcup_{a \leq b} \mathcal{P}(\mathsf{D})^{[a,b]}, \quad \mathcal{P}(\mathsf{D})^{[a,b]} = \coprod_{\alpha \in K_0(\mathrm{Ho}\,\mathsf{P}(\mathsf{D}))} \mathcal{P}(\mathsf{D})^{[a,b],\alpha}.$$

The component $\mathcal{P}(\mathsf{D})^{[a,b],\alpha}$ parametrizes dg-modules M whose cohomologies concentrate in $[a, b]$ and $\overline{M} = \alpha$.

Decomposition of the moduli space $\mathcal{G}(\mathsf{D})$ of cofibrations:

$$\mathcal{G}(\mathsf{D}) = \bigcup_{a \leq b} \mathcal{G}(\mathsf{D})^{[a,b]}, \quad \mathcal{G}(\mathsf{D})^{[a,b]} = \coprod_{\alpha,\beta \in K_0(\mathrm{Ho}\,\mathsf{P}(\mathsf{D}))} \mathcal{G}(\mathsf{D})^{[a,b],\alpha,\beta}$$

$\mathcal{G}(\mathsf{D})^{[a,b],\alpha,\beta}$ parametrizes cofibrations $X \hookrightarrow Y$ such that cohomologies of Y concentrate in $[a, b]$ and $\alpha = \overline{X}$, $\beta = \overline{Y \coprod^X 0}$.

Diagram of correspondence:

$$
\begin{array}{ccc}
\mathcal{G}(\mathsf{D})^{[a,b],\alpha,\beta} & \xrightarrow{\ c\ } & \mathcal{P}(\mathsf{D})^{[a,b],\alpha+\beta} \\
\Big\downarrow{\scriptstyle p} & & \\
\mathcal{P}(\mathsf{D})^{[a,b],\alpha} \times \mathcal{P}(\mathsf{D})^{[a,b],\beta} & &
\end{array}
\qquad
\begin{array}{ccc}
(X \hookrightarrow Y) & \longmapsto & Y \\
\Big\downarrow & & \\
(X, Y \coprod^X 0) & &
\end{array}
$$

The multiplication μ of derived Hall algebra:

$$
\begin{aligned}
\mu_{\alpha,\beta} : \mathsf{D}^b_{\mathsf{c}}(\mathcal{P}(\mathsf{D})^{\alpha}, \overline{\mathbb{Q}}_\ell) \times \mathsf{D}^b_{\mathsf{c}}(\mathcal{P}(\mathsf{D})^{\beta}, \overline{\mathbb{Q}}_\ell) &\longrightarrow \mathsf{D}^b_{\mathsf{c}}(\mathcal{P}(\mathsf{D})^{\alpha+\beta}, \overline{\mathbb{Q}}_\ell) \\
M &\longmapsto \mathrm{R}c_! \, \mathrm{L}p^*(M)[\dim p].
\end{aligned}
$$

(ℓ is invertible in $\mathbb{F}_q$.)
Associativity:

$$\mu_{\alpha,\beta+\gamma} \circ (\mathrm{id} \times \mu_{\beta,\gamma}) \simeq \mu_{\alpha+\beta,\gamma} \circ (\mu_{\alpha,\beta} \times \mathrm{id}).$$

Outline of the proof of associativity.

The LHS $\mu_{\alpha,\beta+\gamma} \circ (\mathrm{id} \times \mu_{\beta,\gamma})$ corresponds to the rigid arrows in

$$\begin{array}{ccccc}
\mathcal{G}^{\alpha,(\beta,\gamma)} & \overset{p_2''}{\dashrightarrow} & \mathcal{G}^{\alpha,\beta+\gamma} & \xrightarrow{p_2'} & \mathcal{P}^{\alpha+\beta+\gamma} \\
{\scriptstyle p_1''}\vdots & & \downarrow{\scriptstyle p_1'} & & \\
\mathcal{P}^{\alpha} \times \mathcal{G}^{\beta,\gamma} & \xrightarrow[p_2]{} & \mathcal{P}^{\alpha} \times \mathcal{P}^{\beta+\gamma} & & \\
{\scriptstyle p_1}\downarrow & & & & \\
\mathcal{P}^{\alpha} \times \mathcal{P}^{\beta} \times \mathcal{P}^{\gamma} & & & &
\end{array}$$

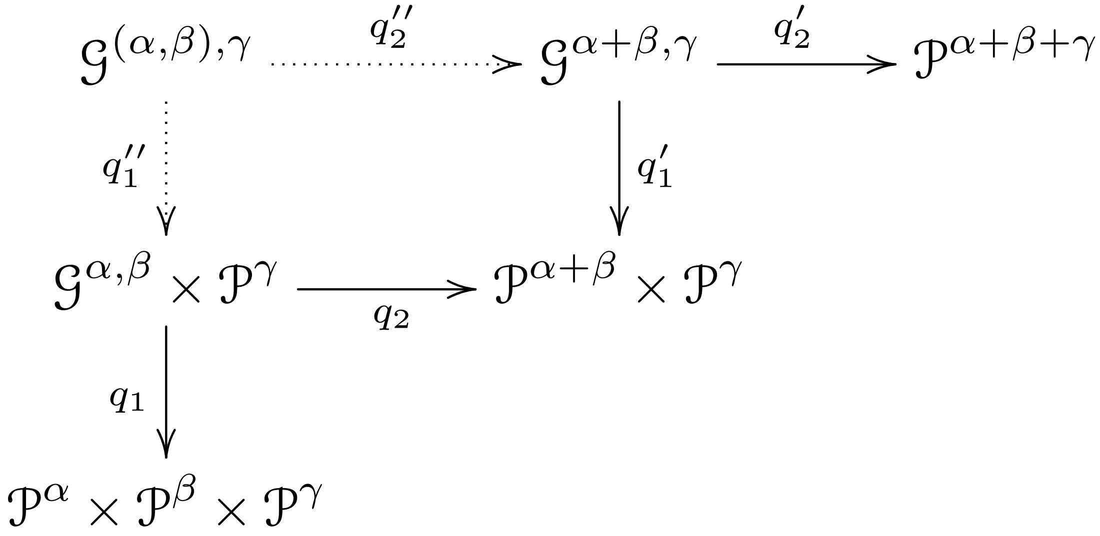
The dotted arrows are determined by

$$\mathcal{G}^{\alpha,(\beta,\gamma)} := (\mathcal{P}^{\alpha} \times \mathcal{G}^{\beta,\gamma}) \times_{\mathcal{P}^{\alpha} \times \mathcal{P}^{\beta+\gamma}} \mathcal{G}^{\alpha,\beta+\gamma},$$

which parametrizes $(N \hookrightarrow M, M \hookrightarrow L)$ such that $\overline{N} = \gamma$, $\overline{M} = \beta + \gamma$, $\overline{L} = \alpha + \beta + \gamma$.
By the smoothness of p_1'', the base-change theorem implies

$$\mu_{\alpha,\beta+\gamma} \circ (\mathrm{id} \times \mu_{\beta,\gamma}) \simeq \mathrm{R}(p_2' p_2'')_! \, \mathrm{L}(p_1 p_1'')^* [\dim(p_1 p_1'')].$$

The RHS $\mu_{\alpha+\beta,\gamma} \circ (\mu_{\alpha,\beta} \times \mathrm{id})$ corresponds to



The dotted arrows are determined by

$$\mathcal{G}^{(\alpha,\beta),\gamma} := (\mathcal{G}^{\alpha,\beta} \times \mathcal{P}^{\gamma}) \times_{\mathcal{P}^{\alpha+\beta} \times \mathcal{P}^{\gamma}} \mathcal{G}^{\alpha+\beta,\gamma}$$

which parametrizes $(R \to L \coprod^{M} 0, M \to L)$ such that $\overline{M} = \gamma$, $\overline{R} = \beta$, $\overline{L} = \alpha + \beta + \gamma$. By the smoothness of q_1'', the base-change theorem implies

$$\mu_{\alpha+\beta,\gamma} \circ (\mu_{\alpha,\beta} \times \mathrm{id}) \simeq \mathrm{R}(q_2' q_2'')_! \, \mathrm{L}(q_1 q_1'')^* [\dim(q_1 q_1'')].$$

Thus LHS and RHS are given by

$$\mu_{\alpha,\beta+\gamma} \circ (\mathrm{id} \times \mu_{\beta,\gamma}) \simeq \mathrm{R}p_! \, \mathrm{L}(p')^*[\dim p'], \quad \mu_{\alpha+\beta,\gamma} \circ (\mu_{\alpha,\beta} \times \mathrm{id}) \simeq \mathrm{R}q_! \, \mathrm{L}(q')^*[\dim q']$$

with

$$\begin{array}{ccc} \mathcal{G}^{\alpha,(\beta,\gamma)} & \xrightarrow{\ p\ } & \mathcal{P}^{\alpha+\beta+\gamma} \\ {\scriptstyle p'}\big\downarrow & & \\ \mathcal{P}^{\alpha} \times \mathcal{P}^{\beta} \times \mathcal{P}^{\gamma} & & \end{array} \qquad \begin{array}{ccc} \mathcal{G}^{(\alpha,\beta),\gamma} & \xrightarrow{\ q\ } & \mathcal{P}^{\alpha+\beta+\gamma} \\ {\scriptstyle q'}\big\downarrow & & \\ \mathcal{P}^{\alpha} \times \mathcal{P}^{\beta} \times \mathcal{P}^{\gamma} & & \end{array}$$

Then the associativity follows from the isomorphism of the derived stacks

$$\mathcal{G}^{\alpha,(\beta,\gamma)} \simeq \mathcal{G}^{(\alpha,\beta),\gamma} .$$

This isomorphism is shown by reduction to the values on the closed points.

8 Toward canonical bases for derived Hall algebras

Q: a quiver without loops with the vertex set I

$\mathsf{Mod}_{\mathsf{dg}}(kQ)$: dg-category of reps. of Q, whose Grothendieck group is $K_0 = \mathbb{Z}^I$

$\mathcal{P}(Q) := \mathcal{M}_{\mathsf{Mod}_{\mathsf{dg}}(kQ)}$: the moduli stack of complexes of representations of Q

with the decomposition $\mathcal{P}(Q) = \bigcup_{a \le b} \mathcal{P}(Q)^{[a,b]}$, $\mathcal{P}(Q)^{[a,b]} = \bigsqcup_{\alpha,\beta \in \mathbb{Z}^I} \mathcal{P}(Q)^{[a,b],\alpha,\beta}$

Let us consider

$$\mathbf{1}_\alpha^n := \overline{\mathbb{Q}}_\ell\big|_{\mathcal{P}(Q)^{[n,n],\alpha}} [\dim \mathcal{P}(Q)^{[n,n],\alpha}] \in \mathsf{Mod}_\infty^{\mathsf{c},b}\big(\mathcal{P}(Q), \overline{\mathbb{Q}}_\ell\big) \quad (n \in \mathbb{Z},\ \alpha \in \mathbb{N}^I).$$

These belong to the ∞-category of perverse sheaves, which gives a t-structure of the stable ∞-category $\mathsf{Mod}_\infty^{\mathsf{c},b}\big(\mathcal{P}(Q), \overline{\mathbb{Q}}_\ell\big)$ in the sense of Lurie.

The multiplication $\star := \mu$ preserves the ∞-category of perverse sheaves, and the product

$$L_{\alpha_1,\ldots,\alpha_l}^{n_1,\ldots,n_l} := \mathbf{1}_{\alpha_1}^{n_1} \star \cdots \star \mathbf{1}_{\alpha_l}^{n_l}$$

can be regarded as an analogue of the Lusztig sheaf appearing in the construction of the canonical basis for quantum groups.

We are now studying a derived analogue of Lusztig's construction of canonical bases.

Thank you for the listening.