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Geometric derived Hall algebra I

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0 Introduction

The derived Hall algebra introduced by Toën (2006) is a version of Ringel-Hall algebra. Roughly it is a "Hall algebra for complexes".

In the case of ordinary Ringel-Hall algebra, we know Lusztig's geometric formulation using the theory of derived categories of constructible sheaves on moduli spaces of Quiver representations, which are realized as Artin stacks.

I will explain a geometric formulation of derived Hall algebras using the theory of derived categories of constructible sheaves on moduli spaces of complexes of Quiver representations, which are realized as geometric derived stacks.

Based on my preprint
S. Yanagida, "Geometric derived Hall algebra", arXiv:1912.05442.

See also
柳田伸太郎, 「幾何学的導来Hall代数」代数学シンポジウム講演集 (2020).

Today I will explain

- Ringel-Hall algebras and Toën's derived Hall algebras (§1).
- Outline of geometric construction of derived Hall algebras (§2).
- Derived stacks and geometric derived stacks [Toën-Vezzossi, 2008] (§3).
- Moduli spaces of complexes via geometric derived stacks [Toën-Vaquié, 2009] (§4).

In the second talk I will explain

- Moduli spaces of complexes via geometric derived stacks (§4).
- Lisse-étale constructible ℓ-adic sheaves over derived stacks (§5).
- Derived category and derived functors (§6).
- Geometric construction of derived Hall algebras (§7).

1 Ringel-Hall algebra and derived Hall algebra

1.1 Ringel-Hall algebra

A: an $\mathbb{F}_q$-linear abelian category of finite global dimension,

$\mathrm{Iso}(\mathsf{A})$: the set of isomorphism classes of objects in A,

$\mathbb{Q}_c(\mathsf{A})$: the linear space of $\mathbb{Q}$-valued functions on $\mathrm{Iso}(\mathsf{A})$ with finite supports,

$1_{[M]}$: the characteristic function of $[M] \in \mathrm{Iso}(\mathsf{A})$, forming a basis of $\mathbb{Q}_c(\mathsf{A})$.

Theorem (Ringel, 1990). $H(\mathsf{A}) := \left(\mathbb{Q}_c(\mathsf{A}), *, 1_{[0]}\right)$ is a unital associative $\mathbb{Q}$-algebra, where

$$1_{[M]} * 1_{[N]} := \sum_{[L] \in \mathrm{Iso}(\mathsf{A})} g^L_{M,N} 1_{[L]},$$

$$g^L_{M,N} := a_M^{-1} a_N^{-1} e^L_{M,N}, \quad a_M := |\mathrm{Aut}(M)|,$$

$$e^L_{M,N} := |\{0 \to N \to L \to M \to 0 \mid \text{exact in } \mathsf{A}\}|.$$

Another definition of $g^L_{M,N}$:

$$g^L_{M,N} = \left|\mathcal{G}^L_{M,N}\right|, \quad \mathcal{G}^L_{M,N} := \{N' \subset L \mid N' \simeq N,\ L/N \simeq M\}.$$

$g^L_{M,N}$ counts the number of pairs $N \subset L$.

The extended Hall algebra $\widetilde{H}(\mathsf{A}) := \mathbb{C}K_0(\mathsf{A}) \otimes_{\mathbb{C}} H(\mathsf{A})$

- Fix $\nu = q^{1/2} \in \mathbb{C}$. Denote $k_\alpha \in \mathbb{C}K_0(\mathsf{A})$ for $\alpha \in K_0(\mathsf{A})$.
- $k_\alpha * [M] = \nu^{\chi_S(\alpha,M)}[M] * k_\alpha$,
  $\chi(\cdot,\cdot) := \sum_{i\geq 0}(-1)^i \dim_{\mathbb{F}_q} \mathrm{Ext}^i_{\mathsf{A}}(\cdot,\cdot)$: Euler form, $\chi_S(M,N) := \chi(M,N) + \chi(N,M)$.

Ringel's realization of quantum groups via Hall algebras

- Q: a quiver without loop.
  $\mathrm{Rep}^{\mathrm{nilp}}_{\mathbb{F}_q} Q = \text{mod-}\mathbb{F}_q Q$ **:** the category of representations of Q.
  $\rightsquigarrow$ the extended Hall algebra $\widetilde{H}(\mathrm{Rep}^{\mathrm{nilp}}_{\mathbb{F}_q} Q)$.
- $U_\nu(\mathfrak{g}_Q)$ **:** the quantum group associated to the Kac-Moody Lie algebra $\mathfrak{g}_Q$.
  $U_\nu(\mathfrak{b}_Q) \subset U_\nu(\mathfrak{g}_Q)$ **:** Borel subalgebra.

Theorem (Ringel, 1990)**.** There is an algebra embedding

$$U_\nu(\mathfrak{b}_Q) \hookrightarrow \widetilde{H}(\mathrm{Rep}^{\mathrm{nilp}}_{\mathbb{F}_q} Q), \quad E_i \longmapsto [S_i],\ K_i \longmapsto k_{\overline{S_i}}.$$

If Q is of type ADE, then it is an isomorphism.

Theorem (Green, 1995; Xiao, 1997)**.** If A is a hereditary finitary abelian category, then $\widetilde{H}(\mathsf{A})$ has a structure of bialgebra.
If moreover the number of subobjects for any object is finite, then $\widetilde{H}(\mathsf{A})$ has a structure of Hopf algebra.

Green's coproduct and Hopf inner product:

$$\Delta([L]) := \sum_{[M],[N]} \nu^{\chi(M,N)} \frac{|\mathrm{Ext}_{\mathsf{A}}(M,N)_L|}{|\mathrm{Aut}_{\mathsf{A}}(L)|}[M] \otimes [N],$$

$$([M] * k_\alpha, [N] * k_\beta) = \frac{\delta_{M,N}\, \chi_S(\alpha,\beta)}{|\,\mathrm{Aut}_{\mathsf{A}}(M)|}.$$

Theorem (Green, 1995; Xiao, 1997)**.** Ringel's algebra embedding $U_\nu(\mathfrak{b}_Q) \hookrightarrow \widetilde{H}(\mathsf{Rep}^{\mathsf{nilp}}_{\mathbb{F}_q} Q)$ is an embedding of Hopf algebras.

Hall algebras for smooth projective curves

- C: a smooth projective curve over $\mathbb{F}_q$.
  The abelian category $\mathrm{Coh}(C)$ is finitary and hereditary.
  $\rightsquigarrow$ A topological bialgebra $\widetilde{H}(\mathrm{Coh}(C))$.
- [Kapranov,1997]: The case $C = \mathbb{P}^1$.
  a central extension $\widetilde{H}'(\mathbb{P}^1)$ of $\widetilde{H}(\mathrm{Coh}(\mathbb{P}^1))$ has a sub-bialgebra isomorphic to the Borel subalgebra $U_\nu(\mathcal{L}\mathfrak{b}_+)$ of the quantum loop algebra $U_\nu(\mathcal{L}\mathfrak{sl}_2)$:

$$U_\nu(\mathcal{L}\mathfrak{b}_+) \hookrightarrow \widetilde{H}'(\mathbb{P}^1).$$

- [Burban-Schiffmann, 2006 (2012)]: The case C is an elliptic curve E.
  a central extension $\widetilde{H}'(E)$ of $\widetilde{H}(\mathrm{Coh}(E))$ has a sub-bialgebra isomorphic to the "Borel subalgebra" of the $\mathfrak{gl}_1$-quantum toroidal algebra (Ding-Iohara-Miki algebra).

1.2 Derived Hall algebra

Toën introduced an analogue of Ringel-Hall algebra for a dg-category.
Rough idea: instead of counting subobjects, count cofibrations up to homotopy.

Notations on dg-categories and of model categories

$\mathsf{C}(\mathbb{F}_q)$: the model dg-category of complexes of $\mathbb{F}_q$-modules.
: A fibration is defined to be an epimorphism.
A weak equivalence is defined to be a quasi-isomorphism.

D: a dg-category over $\mathbb{F}_q$.

M(D): the model dg-category of dg-modules over D^{op}.
: A dg-module means a $\mathsf{C}(\mathbb{F}_q)$-enriched functor $\mathsf{D}^{\mathsf{op}} \to \mathsf{C}(\mathbb{F}_q)$.
: The model structure is induced by that of $\mathsf{C}(\mathbb{F}_q)$.

Notations on the simplicial homotopy theory

$\mathsf{sSet} := \mathsf{Fun}(\boldsymbol{\Delta}^{\mathsf{op}}, \mathsf{Set})$: the category of simplicial sets and simplicial maps

: having Kan model structure where a fibration is a Kan fibration and a weak equivalence is a homotopy equivalence of geom. realizations.

$\mathcal{H} := \mathrm{Ho}\,\mathsf{sSet}$: the homotopy category of the model category sSet, called the homotopy category of spaces.

We have the standard Quillen adjunction

$$|\ | : \mathsf{sSet} \rightleftarrows \mathcal{CG} : \mathrm{Sing}$$

between sSet and the category $\mathcal{CG}$ of compactly generated Hausdorff spaces.
Thus $\mathcal{H} = \mathrm{Ho}\,\mathsf{sSet} \simeq \mathrm{Ho}\,\mathcal{CG}$.

Preliminaries for counting cofibrations

For a dg-category D, the category M(D) is C($\mathbb{F}_q$)-enriched, so one can attach a simplicial set

$$\mathrm{Map}_{\mathsf{M(D)}}(X,Y) := \mathrm{N}(\mathrm{Hom}_{\mathsf{M(D)}}(X,Y)) \in \mathsf{sSet}$$

where $\mathrm{N}(\ ) : \mathsf{C}(\mathbb{F}_q) \to \mathsf{sSet}$ denotes the nerve construction.

Definition. A dg-module $X \in$ M(D) is perfect if for any filtered system $\{Y_i\}_{i\in I}$ in M(D) the natural morphism

$$\varinjlim_{i\in I} \mathrm{Map}_{\mathsf{M(D)}}(X, Y_i) \longrightarrow \mathrm{Map}_{\mathsf{M(D)}}(X, \varinjlim_{i\in I} Y_i)$$

is an isomorphism in $\mathcal{H}$.

The diagram of correspondence

D: dg-category

P(D): the sub-dg-category of M(D) of cofibrant and perfect objects and of weak equivalences

G′(D) := Fun(Δ^1, M(D)), where $I = \Delta^1$ is the 1-simplex

: with the model structure induced levelwise by M(D)

G(D): the sub-dg-cat. of G′(D) of cofibrant and perfect objects

: considered as the category of cofibrations $X \hookrightarrow Y$

For an object $u : X \to Y$ in G(D),

$$s(u) := X, \quad c(u) := Y, \quad t(u) := Y \coprod^{X} 0,$$

which yield a diagram of dg-categories:

$$\begin{array}{ccc} \mathsf{G}(\mathsf{D}) & \xrightarrow{\;c\;} & \mathsf{P}(\mathsf{D}) \\ {\scriptstyle s\times t}\downarrow & & \\ \mathsf{P}(\mathsf{D}) \times \mathsf{P}(\mathsf{D}) & & \end{array} \qquad \begin{array}{ccc} (X \hookrightarrow Y) & \longmapsto & Y \\ \downarrow & & \\ (X, \text{“}Y/X\text{”}) & & \end{array}$$

Define $X^{(0)}(\mathsf{D}), X^{(1)}(\mathsf{D}) \in \mathcal{H}$ by

$$X^{(0)}(\mathsf{D}) := [\mathrm{N}_{\mathsf{dg}}(\mathsf{P}(\mathsf{D}))], \quad X^{(1)}(\mathsf{D}) := [\mathrm{N}_{\mathsf{dg}}(\mathsf{G}(\mathsf{D}))],$$

where N_{dg} denotes the dg nerve construction [Lurie, Higher Algebra] and $[\cdot] : \mathsf{sSet} \to \mathcal{H}$. Then we have the diagram of homotopy types

$$\begin{array}{ccc} X^{(1)}(\mathsf{D}) & \xrightarrow{\;c\;} & X^{0}(\mathsf{D}) \\ \downarrow{\scriptstyle s\times t} & & \\ X^{(0)}(\mathsf{D}) \times X^{(0)}(\mathsf{D}) & & \end{array}$$

Lemma. If the dg-category D is locally finite, then $s \times t$ is proper and the homotopy types $X^{(i)}(\mathsf{D}) \in \mathcal{H}$ are locally finite.

Here we used:

Definition. A dg-category D is called locally finite if the complex $\mathrm{Hom}_{\mathsf{D}}(x, y)$ is homologically bounded with finite-dimensional homology groups for any $x, y \in \mathsf{D}$.

Definition. A homotopy type $X \in \mathcal{H}$ is called locally finite if for any $x \in X$ the group $\pi_i(X, x)$ is finite and there exists an $n \in \mathbb{N}$ such that $\pi_i(X, x)$ is trivial for $i > n$.
$\mathcal{H}^{\mathsf{lf}}$: the full subcategory of $\mathcal{H}$ spanned by locally finite objects

The definition of derived Hall algebra

For $X \in \mathcal{H}^{\mathrm{lf}}$, we denote $\mathbb{Q}_c(X) := \{\alpha : \pi_0(X) \to \mathbb{Q} \mid \text{having compact support}\}$.
For a morphism $f : X \to Y$ in $\mathcal{H}^{\mathrm{lf}}$, define $f^* : \mathbb{Q}_c(Y) \to \mathbb{Q}_c(X)$ by

$$f^*(\alpha)(x) := \alpha(f(x)) \quad (\alpha \in \mathbb{Q}_c(Y),\ x \in \pi_0(X)).$$

Also define a linear map $f_! : \mathbb{Q}_c(X) \to \mathbb{Q}_c(Y)$ by

$$f_!(\alpha)(y) := \sum_{x \in \pi_0(X),\, f(x)=y} \alpha(x) \cdot \prod_{i>0} \Big(|\pi_i(X,x)|^{(-1)^i} \, |\pi_i(Y,y)|^{(-1)^{i+1}} \Big).$$

Theorem (Toën 2006)**.** Let D be a locally finite dg-category over $\mathbb{F}_q$. Then

$$H(\mathsf{D}) = \mathbb{Q}_c(X^{(0)}(\mathsf{D}))$$

has a structure of a unital associative $\mathbb{Q}$-algebra with the multiplication

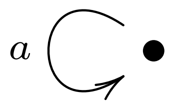
$$\mu := c_! \circ (s \times t)^* : H(\mathsf{D}) \otimes_{\mathbb{Q}} H(\mathsf{D}) \longrightarrow H(\mathsf{D}).$$

We call $H(\mathsf{D})$ the derived Hall algebra of D.

1.3 An example of derived Hall algebra — Jordan quiver

[Shimoji-Y., "A study of symmetric functions via derived Hall algebra"]

- $Q = (\{\bullet\}, \{a\})$: the Jordan quiver



- $\mathsf{A} = \mathrm{Rep}^{\mathsf{nilp}}_{\mathbb{F}_q} Q$: the category of nilpotent representations of Q over $k = \mathbb{F}_q$.

  : a hereditary abelian category

  $\rightsquigarrow$ the Ringel-Hall algebra $\mathrm{H}_{\mathsf{cl}} := H(\mathsf{A})$ is a Hopf algebra,

  called the classical Hall algebra.

- For a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$, $\lambda_i \in \mathbb{N}$, $\lambda_1 \geq \lambda_2 \geq \cdots$, $\lambda_{n \gg 0} = 0$, we define $I_\lambda \in \mathsf{A}$ by

$$I_\lambda := (k^{|\lambda|}, J_\lambda), \quad J_\lambda := J_{\lambda_1} \oplus J_{\lambda_2} \oplus \cdots .$$

  Here we used $|\lambda| := \sum_{i \geq 1} \lambda_i$ and

$$J_n := \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{bmatrix}.$$

  We also consider $\emptyset = ()$ as a partition, and set $I_\emptyset = (0, 0)$.

Lemma. Objects in A are described as follows.

(1) $\mathrm{Iso}(\mathsf{A}) = \{[I_\lambda] \mid \lambda : \text{partitions}\}$.

(2) Simple objects of A are isomorphic to $I_{(1)} = (k, 0)$.

(3) Indecomposable objects are isomorphic to $I_{(n)}$ with some $n \in \mathbb{N}$.

Fact (Hall,Steiniz,Macdonald)**.** The classical Hall algebra $\mathrm{H}_{\mathsf{cl}} = H(\mathsf{A})$ is described as

$$\mathrm{H}_{\mathsf{cl}} = \big(\mathbb{Q}_c(\mathsf{A}), *, [0], \Delta, \epsilon, S\big), \quad \mathbb{Q}_c(\mathsf{A}) = \oplus_{\lambda:\text{partitions}} \mathbb{Q}[I_\lambda],$$

$$[I_\mu] * [I_\nu] = \sum_{\lambda:\text{partitions}} g^\lambda_{\mu,\nu} [I_\lambda], \quad g^\lambda_{\mu,\nu} := \left|\mathcal{G}^\lambda_{\mu,\nu}\right|,$$

$$\mathcal{G}^\lambda_{\mu,\nu} := \mathcal{G}^{I_\lambda}_{I_\mu, I_\nu} = \{N \subset I_\lambda \mid N \simeq I_\nu, I_\lambda/N \simeq I_\mu\},$$

$$\Delta([I_\lambda]) := \sum_{\mu,\nu} a_\lambda^{-1} a_\mu a_\nu g^\lambda_{\mu,\nu} \cdot [I_\mu] \otimes [I_\nu], \quad a_\lambda := a_{I_\lambda} = \left|\mathrm{Aut}_{\mathsf{A}}(I_\lambda)\right|.$$

Lemma. On the structure constant $g^\nu_{\lambda,\mu} = \left|\mathcal{G}^\nu_{\lambda,\mu}\right|$ of $H(\mathsf{A})$, we have

(1) $\mathcal{G}^{(1^n)}_{(1^{n-r}),(1^r)} = \mathrm{Gr}(n, r)$, and $g^{(1^n)}_{(1^{n-r}),(1^r)} = \begin{bmatrix} n \\ r \end{bmatrix}_q$.

(2) $\mathcal{G}^{(n)}_{(n-r),(r)}$ consists of one point, and $g^{(n)}_{(n-r),(r)} = 1$.

Theorem (Shimoji-Y.). The derived Hall algebra $\mathrm{DH}_{\mathsf{cl}}$ for the perfect complexes in $\mathsf{A} = \mathsf{Rep}^{\mathsf{nilp}}_{\mathbb{F}_q} Q$ is a unital associative algebra with generators

$$\{Z^{[n]}_{\lambda} \mid n \in \mathbb{Z},\ \lambda \neq \emptyset : \text{partitions}\}$$

and $Z^{[n]}_{\emptyset} = 1$, and the relations

$$Z^{[n]}_{\lambda} * Z^{[n]}_{\mu} = \sum_{\nu:\text{partitions}} g^{\nu}_{\lambda,\mu} Z^{[n]}_{\nu}, \quad Z^{[n]}_{\lambda} * Z^{[m]}_{\mu} = Z^{[m]}_{\mu} * Z^{[n]}_{\lambda}, \quad (|n-m|>1),$$

$$Z^{[n]}_{\lambda} * Z^{[n+1]}_{\mu} = \sum_{\alpha,\beta:\text{partitions}} \gamma^{\alpha,\beta}_{\lambda,\mu} Z^{[n+1]}_{\alpha} * Z^{[n]}_{\beta}, \tag{$\sharp$}$$

The relation ($\sharp$) is equivalent to the following Heisenberg relation: For $k \in \mathbb{Z}_{>0}$ we define $b^{[n]}_{\pm k} \in \mathrm{DH}_{\mathsf{cl}}$ by

$$b^{[n]}_{k} := \sum_{|\lambda|=k} (q;q)_{\ell(\lambda)-1} Z^{[n]}_{\lambda}, \quad b^{[n]}_{-k} := \sum_{|\lambda|=k} (q;q)_{\ell(\lambda)-1} Z^{[n+1]}_{\lambda}.$$

We also set $b_0 := 1 \in \mathrm{DH}_{\mathsf{cl}}$. Then we have

$$b^{[n]}_{k} * b^{[n]}_{l} - b^{[n]}_{k} * b^{[n]}_{l} = \delta_{k+l,0} \frac{k}{q^k - 1}.$$

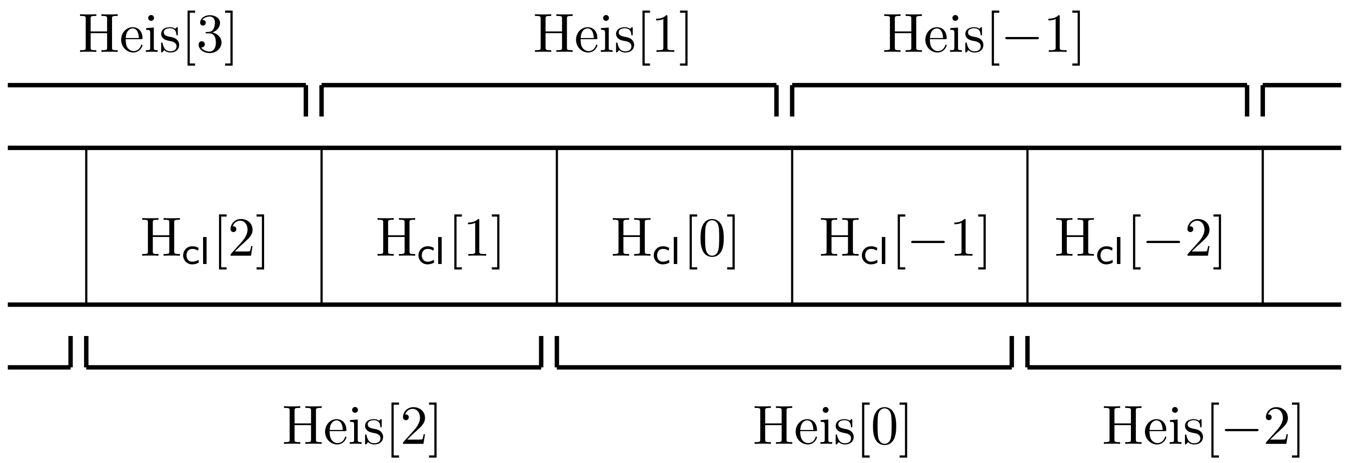


Figure 1 Infinite family of Heisenberg subalgebras in $\mathrm{DH}_{\mathsf{cl}}$

2 Outline of our geometric construction

D: a locally finite dg-category over $\mathbb{F}_q$

Theorem (Toën-Vaquié (2009)). We have the moduli stack $\mathcal{P}(\mathsf{D})$ of perfect dg-modules over D^{op}.
It is a derived stack, locally geometric and locally of finite type.

We can also construct the moduli stack of cofibrations $X \to Y$ of perfect-modules over D^{op}, denoted by $\mathcal{G}(\mathsf{D})$.
There exist morphisms

$$s, c, t : \mathcal{G}(\mathsf{D}) \longrightarrow \mathcal{P}(\mathsf{D})$$

of derived stacks which send $u : X \to Y$ to

$$s(u) = X, \quad c(u) = Y, \quad t(u) = Y \coprod^{X} 0.$$

where s, t are smooth and c is proper.

The diagram of geometric correspondence

Thus we have the diagram

$$\begin{array}{ccc} \mathcal{G}(\mathsf{D}) & \xrightarrow{c} & \mathcal{P}(\mathsf{D}) \\ \downarrow{\scriptstyle p} & & \\ \mathcal{P}(\mathsf{D}) \times \mathcal{P}(\mathsf{D}) & & \end{array}$$

of derived stacks with smooth $p := s \times t$ and proper c.

Next let $\Lambda := \overline{\mathbb{Q}}_\ell$ where ℓ and q are assumed to be coprime.
We have the derived category $\mathsf{D}^b_\mathsf{c}(\mathcal{X}, \Lambda)$ of constructible lisse-étale Λ-sheaves over a locally geometric derived stack $\mathcal{X}$, and Grothedieck's six operations.
Applying the general theory to the present situation, we have

$$\begin{array}{ccc} \mathsf{D}^b_\mathsf{c}(\mathcal{G}(\mathsf{D}), \Lambda) & \xrightarrow{c_!} & \mathsf{D}^b_\mathsf{c}(\mathcal{P}(\mathsf{D}), \Lambda) \\ \uparrow{\scriptstyle p^*} & & \\ \mathsf{D}^b_\mathsf{c}(\mathcal{P}(\mathsf{D}) \times \mathcal{P}(\mathsf{D}), \Lambda) & & \end{array}$$

Main Theorem

$$\begin{array}{ccc} \mathsf{D}^b_{\mathsf{c}}(\mathcal{G}(\mathsf{D}), \Lambda) & \xrightarrow{c_!} & \mathsf{D}^b_{\mathsf{c}}(\mathcal{P}(\mathsf{D}), \Lambda) \\ \uparrow{\scriptstyle p^*} & & \\ \mathsf{D}^b_{\mathsf{c}}(\mathcal{P}(\mathsf{D}) \times \mathcal{P}(\mathsf{D}), \Lambda) & & \end{array}$$

Now we set

$$\mu : \mathsf{D}^b_{\mathsf{c}}(\mathcal{P}(\mathsf{D}) \times \mathcal{P}(\mathsf{D}), \Lambda) \longrightarrow \mathsf{D}^b_{\mathsf{c}}(\mathcal{P}(\mathsf{D}), \Lambda), \quad M \longmapsto c_! p^*(M)[\dim p]$$

Theorem 1. μ is associative.

3 Derived stacks

3.1 Derived schemes and derived stacks

Notations on ∞-categories:

- $\Lambda_j^n \subset \Delta^n$ denotes the j-th horn of the n-simplex Δ^n $(0 \leq j \leq n)$.
- An ∞-category is a simplicial set K such that for any $n \in \mathbb{N}$ and any $0 < i < n$, any map $f_0 : \Lambda_i^n \to K$ of simplicial sets admits an extension $f : \Delta^n \to K$.

Notations on commutative simplicial algebras:

- k: a commutative ring.
- sCom: the category of commutative simplicial k-algebra.
- sCom_∞: the ∞-category obtained by localizing sCom via the set of weak equivalences in the Kan model category sCom $\subset$ sSet.

Definition. We call $\mathrm{dAff}_\infty := (\mathrm{sCom}_\infty)^{\mathrm{op}}$ the ∞-category of affine derived schemes.

Turn to the definition of derived stacks.

Definition. A morphism $A \to B$ in sCom_∞ is called étale [smooth] if

- the induced $\pi_0(A) \to \pi_0(B)$ is an étale [smooth] map of commutative k-algebras,
- the induced $\pi_i(A) \otimes_{\pi_0(A)} \pi_0(B) \to \pi_i(B)$ is an isomorphism for any i.

Étale morphisms endow $\mathsf{dAff}_\infty = (\mathsf{sCom}_\infty)^{\mathsf{op}}$ with a Grothendieck topology et.
(I will explain Grothendieck topologies on ∞-categories in the next page.)

Definition. The ∞-category of derived stacks is defined to be

$$\mathsf{dSt}_\infty := \mathsf{Sh}_{\infty,\mathsf{et}}(\mathsf{dAff}_\infty) \subset \mathsf{PSh}_\infty(\mathsf{dAff}_\infty) := \mathsf{Fun}_\infty((\mathsf{dAff}_\infty)^{\mathsf{op}}, \mathcal{S}).$$

$\mathcal{S}$: the ∞-category of spaces. (See [Lurie, HTT] for the detail.)

- $\mathsf{Kan} \subset \mathsf{sSet}$: the full subcategory of Kan complexes, which is a simplicial category.
- $\mathrm{N}_{\mathsf{sp}}(\ )$: simplicial nerve construction,
  a functor mapping a simplicial category to a simplicial set.
- $\mathcal{S} := \mathrm{N}_{\mathsf{sp}}(\mathsf{Kan})$.
- The homotopy category of the ∞-category $\mathcal{S}$ is equivalent to
  $\mathcal{H} := \mathrm{Ho}\,\mathsf{sSet}$, the homotopy category of spaces (Quillen equivalence).

Grothendieck topology on an ∞-category [Lurie, HTT, §6.2.2], [Toën-Vezzosi 1].

Definition. 1. A sieve on an ∞-category C is a full sub-∞-category $\mathsf{C}^{(0)} \subset \mathsf{C}$ s.t. $X \in \mathsf{C}^{(0)}$ holds for any $Y \in \mathsf{C}^{(0)}$ and any morphism $f : X \to Y$ in C.

2. A sieve on $X \in \mathsf{C}$ is a sieve on the over-∞-category $\mathsf{C}_{/X}$.

- For a functor $F : \mathsf{C} \to \mathsf{D}$ of ∞-categories and a sieve $\mathsf{D}^{(0)} \subset \mathsf{D}$, the homotopy fiber product gives a sieve $F^{-1}\mathsf{D}^{(0)} := \mathsf{D}^{(0)} \times_{\mathsf{D}} \mathsf{C} \subset \mathsf{C}$ on C.
- For a morphism $f : X \to Y$ in C and a sieve $\mathsf{C}^{(0)}_{/Y}$ on Y, we have a sieve $f^*\mathsf{C}^{(0)}_{/Y} := (f_*)^{-1}\mathsf{C}^{(0)}_{/Y}$ on X. ($f_* : \mathsf{C}_{/X} \to \mathsf{C}_{/Y}$: the natural functor of over-∞-categories.)

Definition. A Grothendieck topology τ on an ∞-category C is a choice of a collection $\mathrm{Cov}(X)$ of sieves on each $X \in \mathsf{C}$ (covering sieves on X) s.t.

- For any $X \in \mathsf{C}$, $\mathsf{C}_{/X} \in \mathrm{Cov}(X)$.
- For any $f : X \to Y$ in C and any $\mathsf{C}^{(0)}_{/Y} \in \mathrm{Cov}(Y)$, $f^*\mathsf{C}^{(0)}_{/Y} \in \mathrm{Cov}(X)$.
- For $Y \in \mathsf{C}$ and $\mathsf{C}^{(0)}_{/Y} \in \mathrm{Cov}(Y)$, if $\mathsf{C}^{(1)}_{/Y}$ is a sieve on Y s.t. $f^*\mathsf{C}^{(1)}_{/Y} \in \mathrm{Cov}(X)$ holds for any $(f : X \to Y) \in \mathsf{C}^{(0)}_{/Y}$, then $\mathsf{C}^{(1)}_{/Y} \in \mathrm{Cov}(Y)$.

If C is a nerve of a category C, then a Grothendieck topology on C is equiv. to that on C.

Back to the definition of derived stacks:

$$\mathsf{dSt}_\infty := \mathsf{Sh}_{\infty,\mathsf{et}}(\mathsf{dAff}_\infty) \subset \mathsf{PSh}_\infty(\mathsf{dAff}_\infty) := \mathsf{Fun}_\infty((\mathsf{dAff}_\infty)^{\mathsf{op}}, \mathcal{S}),$$

where $\mathsf{Sh}_{\infty,\mathsf{et}}(\mathsf{dAff}_\infty)$ denotes the ∞-category of sheaves with respect to the Grothendieck topology et.

A derived stack corresponds to a stack in the ordinary algebraic geometry. In the next subsection, I introduce geometric derived stacks in the sense of Toën-Vezzosi, which corresponds to an algebraic/Artin stack.

Remark. I use the terminology "geometric derived stacks" following [Toën-Vezzosi, Homotopical Algebraic Geometry II, Mem. AMS, 2008]. It is equivalent to "derived Artin stacks" in [Toën, Derived algebraic geometry, EMS Surv. Math. Sci., 2014].

3.2 Geometric derived stacks

For $n \in \mathbb{Z}_{\geq -1}$, one defines an n-geometric derived stack inductively on n.
At the same time one also defines an n-atlas, a n-representable morphism and a n-smooth morphism of derived stacks.

- Let $n = -1$.
  1. A (-1)-geometric derived stack is defined to be an affine derived scheme.
  2. A morphism $f : \mathcal{X} \to \mathcal{Y}$ of derived stacks is called (-1)-representable if for any affine derived scheme U and any morphism $U \to \mathcal{Y}$ of derived stacks, the pullback $\mathcal{X} \times_{\mathcal{Y}} U$ is an affine derived scheme.
  3. A morphism $f : \mathcal{X} \to \mathcal{Y}$ of derived stacks is called (-1)-smooth if it is (-1)-representable, and if for any affine derived scheme U and any morphism $U \to \mathcal{Y}$ of derived stacks, the induced morphism $\mathcal{X} \times_{\mathcal{Y}} U \to U$ is a smooth morphism of affine derived schemes.
  4. A (-1)-atlas of a stack $\mathcal{X}$ is defined to be the one-member family $\{\mathcal{X}\}$.

Recall: A morphism $A \to B$ in sCom_∞ is called étale [smooth] if

- the induced $\pi_0(A) \to \pi_0(B)$ is an étale [smooth] map of commutative k-algebras,
- the induced $\pi_i(A) \otimes_{\pi_0(A)} \pi_0(B) \to \pi_i(B)$ is an isomorphism for any i.

- Let $n \in \mathbb{N}$.
  1. Let $\mathcal{X}$ be a derived stack. An n-atlas of $\mathcal{X}$ is a small family $\{U_i \to \mathcal{X}\}_{i \in I}$ of morphisms of derived stacks satisfying the following three conditions.
     - Each U_i is an affine derived scheme.
     - Each morphism $U_i \to \mathcal{X}$ is $(n-1)$-smooth.
     - The morphism $\coprod_{i \in I} U_i \to \mathcal{X}$ is an epimorphism.
  2. A derived stack $\mathcal{X}$ is called n-geometric if the following two conditions are satisfied.
     - The diagonal morphism $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is $(n-1)$-representable.
     - There exists an n-atlas of $\mathcal{X}$.
  3. A morphism $f : \mathcal{X} \to \mathcal{Y}$ of derived stacks is called n-representable if for any affine derived scheme U and for any morphism $U \to \mathcal{Y}$ of derived stacks, the derived stack $\mathcal{X} \times_{\mathcal{Y}} U$ is n-geometric.
  4. A morphism $f : \mathcal{X} \to \mathcal{Y}$ of derived stacks is called n-smooth if for any affine derived scheme U and any morphism $U \to \mathcal{Y}$ of derived stacks, there exists an n-atlas $\{U_i\}_{i \in I}$ of $\mathcal{X} \times_{\mathcal{Y}} U$ such that for each $i \in I$ the composition $U_i \to \mathcal{X} \times_{\mathcal{Y}} U \to U$ is a smooth morphism of affine derived schemes.

To an algebraic stack $\mathcal{X}$ in the ordinary sense, one can attach a derived stack $j(\mathcal{X})$ functorially.

Fact (Toën-Vezzosi (2008))**.** For an algebraic stack $\mathcal{X}$, the derived stack $j(\mathcal{X})$ is 1-geometric.

Remark. To schemes and algebraic spaces X, we can also attach derived stacks $j(X)$.
For affine schemes X, the derived stack $j(X)$ is (-1)-geometric.
For schemes and algebraic spaces X, the derived stacks $j(X)$ are 1-geometric.

4 Moduli spaces of complexes

In this section we review the theory of moduli stacks of modules over dg-categories via derived stacks [Toen-Vaquié].

4.1 Moduli functor of perfect objects

- $A \in \mathsf{sCom}$: a commutative simplicial k-algebra.
- $N(A)$ the normalized chain complex with the structure of a comm. k-dg-algebra.
- Regarding $N(A)$ as a dg-category, we have the dg-category of $N(A)$-dg-modules:

$$\mathsf{M}(A) := \mathsf{M}(N(A))$$

- The full sub-dg-category of cofibrant and perfect objects in $\mathsf{M}(A)$:

$$\mathrm{P}(A) := \mathrm{P}(N(A)) \subset \mathsf{M}(A).$$

Definition. For a dg-category D over k and $A \in \mathsf{sCom}$, we set

$$\mathcal{M}_{\mathsf{D}}(A) := \mathrm{Map}_{\mathrm{dgCat}}(\mathsf{D}^{\mathsf{op}}, \mathrm{P}(A)),$$

where $\mathrm{Map}_{\mathrm{dgCat}}$ denotes the mapping space in the model category dgCat of dg-categories, which is regarded as a simplicial set.

$$\mathcal{M}_{\mathsf{D}}(A) := \mathrm{Map}_{\mathrm{dgCat}}(\mathsf{D}^{\mathsf{op}}, \mathrm{P}(A)),$$

Here the model structure is the one introduced by [Tabuada, 2005]:

A dg-functor $f : \mathsf{D} \to \mathsf{D}'$ is

- a weak equivalence if f is a quasi-isomorphism, and
- a fibration if

(i) for any $M, N \in \mathsf{D}$, the morphism $f_{MN} : \mathrm{Hom}_{\mathsf{D}}(M, N) \to \mathrm{Hom}_{\mathsf{D}'}(f(M), f(N))$ is an epimorphism of k-dg-modules, and

(ii) for any $M \in \mathsf{D}$ and any isomorphism $v : N \to f(M)$ in $H^0(\mathsf{D}')$, there is an isomorphism $u : M \to M'$ in $H^0(\mathsf{D})$ such that $H^0(f_{M,N})(u) = v$.

For a morphism $A \to B$ in sCom, we obtain a morphism $\mathcal{M}_{\mathsf{D}}(A) \to \mathcal{M}_{\mathsf{D}}(B)$ in sSet by composition with $N(B) \otimes_{N(A)} - : \mathrm{P}(A) \to \mathrm{P}(B)$. Thus we obtain a functor

$$\mathcal{M}_{\mathsf{D}} : \mathsf{sCom} \longrightarrow \mathsf{sSet}, \quad \mathcal{M}_{\mathsf{D}}(A) := \mathrm{Map}_{\mathrm{dgCat}}(\mathsf{D}^{\mathsf{op}}, \mathrm{P}(A)).$$

This construction gives rise to a functor of ∞-categories

$$\mathcal{M}_{\mathsf{D}} \in \mathsf{PSh}_{\infty}(\mathsf{dAff}_{\infty}) \;=\; \mathsf{Fun}_{\infty}((\mathsf{dAff}_{\infty})^{\mathsf{op}}, \mathcal{S}).$$

Fact ([Toën-Vaquié, Lemma 3.1]). The presheaf $\mathcal{M}_{\mathsf{D}} \in \mathsf{PSh}_{\infty}(\mathsf{dAff}_{\infty})$ is a derived stack over k. We call it the moduli stack of perfect D^{op}-dg-modules

Remark. • The 0-th homotopy $\pi_0(\mathcal{M}_{\mathsf{D}}(k))$ is bijective to the set of isomorphism classes of perfect D-dg-modules in $\mathrm{Ho}(\mathsf{M}(\mathsf{D}))$.

• For each $x \in \mathrm{Ho}(\mathsf{M}(\mathsf{D}))$, we have

$$\pi_1(\mathcal{M}_{\mathsf{D}}, x) \simeq \mathrm{Aut}_{\mathrm{Ho}(\mathsf{M}(\mathsf{D}))}(x, x), \quad \pi_i(\mathcal{M}_{\mathsf{D}}, x) \simeq \mathrm{Ext}^{-i}_{\mathrm{Ho}(\mathsf{M}(\mathsf{D}))}(x, x) \ (i \in \mathbb{Z}_{\geq 2}),$$

where $\mathrm{Ho}(\mathsf{M}(\mathsf{D}))$ is regarded as a triangulated category.

To be continued.