Derived gluing construction of chiral algebras

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2020/06/11

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- Moore-Tachikawa 2d TFT

[Moore, Tachikawa, "On 2d TQFTs whose values are holomorphic symplectic varieties", arXiv:1106.5698]

$$\eta_G : \mathsf{Bo}_2 \longrightarrow \mathsf{HS}$$

G: simply connected semisimple algebraic group over $\mathbb{C}$.
Bo_2: 2-bordisms category,
objects $(S^1)^n$ $(n \in \mathbb{N})$, morphisms $\Sigma_{g,n_1+n_2} : n_1 \to n_2$

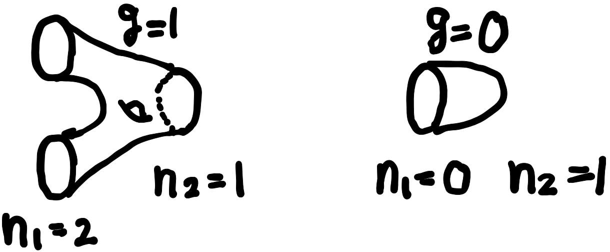

Figure: 2-bordism category Bo_2

- HS: category of "holomorphic symplectic varieties"
  - Objects: simply connected semisimple algebraic groups
  - Morphisms $X : G_1 \to G_2$:
    symplectic variety with Hamiltonian $G_1 \times G_2$-action.
    $\mathrm{id}_G = T^*G$ with the left and right multiplication of G.
  - Composition of $X_{12} \in \mathrm{Hom}_{\mathsf{HS}}(G_1, G_2)$ and $X_{23} \in \mathrm{Hom}_{\mathsf{HS}}(G_2, G_3)$:

$$X_{23} \circ X_{12} := (X_{12}^{\mathrm{op}} \times X_{23}) /\!\!/ \Delta(G_2) = \mu^{-1}(0)/\Delta(G_2).$$

$\mu : X_{12}^{\mathrm{op}} \times X_{23} \to \mathfrak{g}_2^*$, $\mu(x, y) := -\mu_{12}(x) + \mu_{23}(y)$

- Braverman, Finkelberg and Nakajima constructed genus 0 part of η_G.

["Ring objects in the equivariant derived Satake category arising from Coulomb branches", arxiv:1706.02112]

$$\eta_G(n) = G^n, \quad W_G^b := \eta_G(\Sigma_{g=0,b}),$$
$$W_G^1 \simeq G \times \mathcal{S}_{\mathsf{Slodowy}}, \quad W_G^2 \simeq T^*G, \quad W_G^{b'} \circ W_G^b \simeq W_G^{b+b'-2}.$$

We call the 3rd isomorphism gluing condition.

- Arakawa considered chiral quantization of Moore-Tachikawa TFT η_G.

["Chiral algebras of class $\mathcal{S}$ and Moore-Tachikawa symplectic varieties", arXiv:1811.01577]

$$\eta_G^{\mathrm{VA}} : \mathsf{Bo}_2 \longrightarrow \mathsf{VA}$$

Target category VA:

- Objects: semisimple algebraic groups.
- Morphism $V : G_1 \to G_2$: vertex algebra with $V_{-h_1^\vee}(\mathfrak{g}_1) \otimes V_{-h_2^\vee}(\mathfrak{g}_2) \to V$ (+ some cond.).
- Composition of $V_{12} \in \mathrm{Hom}_{\mathsf{VA}}(G_1, G_2)$ and $V_{23} \in \mathrm{Hom}_{\mathsf{VA}}(G_2, G_3)$:

$$V_{23} \circ V_{12} := H^{\frac{\infty}{2}+0}(\widehat{\mathfrak{g}}_{-2h_2^\vee}, \mathfrak{g}_2, V_{12}^{\mathrm{op}} \otimes V_{23}),$$

The functor η_G^{VA} should give commutative diagram

$$\begin{array}{ccc} \mathsf{Bo}_2 & \xrightarrow{\eta_G^{\mathsf{VA}}} & \mathsf{VA} \\ \| & & \Big\downarrow {\scriptstyle \mathrm{Specm}\, R_{(-)} \text{:taking associated variety}} \\ \mathsf{Bo}_2 & \xrightarrow[\eta_G]{} & \mathsf{HS} \end{array}$$

- Arakawa constructed genus 0 part of η_G^{VA}:

$$\eta_G^{\mathsf{VA}}(n) = G^n, \quad \mathbf{V}_{G,b}^{\mathcal{S}} := \eta_G^{\mathsf{VA}}(\Sigma_{0,b}),$$
$$\mathbf{V}_{G,1}^{\mathcal{S}} \simeq H^0_{\mathrm{DS}}(\mathcal{D}_G^{\mathrm{ch}}), \quad \mathbf{V}_{G,2}^{\mathcal{S}} \simeq \mathcal{D}_G^{\mathrm{ch}}, \quad \mathbf{V}_{G,b'}^{\mathcal{S}} \circ \mathbf{V}_{G,b}^{\mathcal{S}} \simeq \mathbf{V}_{G,b+b'-2}^{\mathcal{S}}$$

and showed

$$R_{\mathbf{V}_{G,b}^{\mathcal{S}}} = \mathbb{C}[W_G^b].$$

$\rightarrow$ solve affirmatively Beem-Rastelli conjecture

["Vertex operator algebras, Higgs branches, and modular differential equations", arXiv:1707.07679]

$$\mathcal{M}_{\mathrm{Higgs}}(\mathcal{T}) \overset{?}{\simeq} \mathrm{Specm}(R_{V_{\mathcal{T}}}) \quad \forall \mathcal{T} : \mathcal{N} = 2 \text{ 4d SCFT}$$

for genus 0 class $\mathcal{S}$ theories $\mathcal{T} = \mathcal{T}_{\Sigma_{0,b}}^{\mathcal{S}}$.

- I learned these theories in Arakawa's intensive lectures at Nagoya Univ., November 2019. He gave many comments and problems. This talk stems from one of them.
- As footnoted in Arakawa's paper (p.3), there is a subtlety on the construction of η_G : $\mathrm{Bo}_2 \to$ HS for higher genus cases due to the non-flatness of the moment map.
  Composition of morphisms in HS

$$X_{23} \circ X_{12} := (X_{12}^{\mathrm{op}} \times X_{23}) /\!\!/ \Delta(G_2) = \mu^{-1}(0)/\Delta(G_2).$$

- Thus, in order to construct η_G^{VA} : $\mathrm{Bo}_2 \to$ VA for higher genus cases, it may be necessary to modify the genus 0 construction.
  Composition of morphisms in VA:

$$V_{23} \circ V_{12} := H^{\frac{\infty}{2}+0}(\widehat{\mathfrak{g}}_{-2h_2^\vee}, \mathfrak{g}_2, V_{12}^{\mathrm{op}} \otimes V_{23}),$$

- In the intensive lectures, Arakawa commented:
  derived symplectic geometry を使うとできるかもしれない.
  (Using DSG one may overcome this difficulty.)

- What is derived symplectic geometry?
  In this talk, it means the study of shifted symplectic derived schemes/stacks in the realm of derived algebraic geometry (DAG).
- Very roughly and naively speaking, one can transfer objects in classical algebraic geometry (scheme theory) to DAG by the next replacement.

classical	derived
sets	∞-category (simplicial sets/space)
comm. rings A	comm. simplicial/dg rings $A^\bullet$
scheme $(X, \mathcal{O}_X)$	derived scheme $(X, \mathcal{O}_X^\bullet)$
$\mathrm{Hom}_{\mathsf{Sch}}(X, Y)$: morphism set	$\mathrm{Map}_{\mathbf{dSch}}(X, Y)$: morphism space

- As we should replace algebras by dg algebras, the notion of symplectic/Poisson structure in DAG should admit shift (as complex).
  $\rightarrow$ shifted symplectic/Poisson structure

- The idea of using derived symplectic geometry to realize Moore-Tachikawa varieties in full genera is originally due to Calaque.

  ["Lagrangian structures on mapping stacks and semi-classical TFTs", arXiv:1306.3235]

  He introduced the ∞-category MT of derived Moore-Tachikawa varieties:
  - Objects: G (same as in HS)
  - Morphisms $R : G_1 \to G_2$: Poisson dg algebras with Hamiltonian $(\mathfrak{g}_1 \oplus \mathfrak{g}_2)$-action.
  - Composition of $R_{12} \in \mathrm{Map}_{\mathsf{MT}}(G_1, G_2)$ and $R_{23} \in \mathrm{Map}_{\mathsf{MT}}(G_2, G_3)$:

    $$R_{23} \,\widetilde{\circ}\, R_{12} := \left(R_{12}^{\mathrm{op}} \otimes R_{23}\right) /\!\!/_{\mu}^{\mathbb{L}} \mathrm{Sym}(\mathfrak{g}_2).$$

    $/\!\!/_{\mu}^{\mathbb{L}}$: derived Hamiltonian reduction of Poisson dg algebras (explained in part 3)
    $\mu := -\mu_{12}^2 \otimes 1 + 1 \otimes \mu_{23}^1$. We call $R_{23} \,\widetilde{\circ}\, R_{12}$ derived gluing.

- Finally, I can tell the main statement of this talk.
  I define the ∞-category $\mathsf{MT}^{\mathrm{ch}}$ by
  - Objects: G (same as in HS, VA).
  - Morphisms $V : G_1 \to G_2$: dg vertex algebras with $\mu_V : V_k(\mathfrak{g}_1) \otimes V_l(\mathfrak{g}_2) \to V$ (+ some cond.).
  - Composition of $V_{12} \in \mathrm{Map}_{\mathsf{MT}^{\mathrm{ch}}}(G_1, G_2)$ and $V_{23} \in \mathrm{Map}_{\mathsf{MT}^{\mathrm{ch}}}(G_2, G_3)$ is given by the chiral derived gluing $V_{23} \mathbin{\widetilde{\circ}} V_{12}$ (explained in part 4).

Theorem

Taking the associated derived scheme $R_{(-)}$ gives a functor $\mathsf{MT}^{\mathrm{ch}} \to \mathsf{MT}$.
I.e., $R_{V \widetilde{\circ} W} \simeq R_V \mathbin{\widetilde{\circ}} R_W$ in MT.

Thus we may expect the following commutative diagram

$$\begin{array}{ccccc}
\mathsf{Bo}_2 & \xrightarrow{\eta_G^{\mathsf{ch}}} & \mathsf{MT}^{\mathrm{ch}} & & \supset \mathsf{VA} \\
\| & & \downarrow R_{(-)} & & \\
\mathsf{Bo}_2 & \xrightarrow[\eta_G]{} & \mathsf{MT} & & \supset \mathsf{HS}
\end{array}$$

1. Introduction
2. Vertex algebras in derived setting
    - Recollection on vertex algebras, their singular supports and associated schemes in dg settings.
3. Dg Poisson algebras and derived Hamiltonian reduction
4. Dg vertex Poisson algebras and derived arc spaces
5. Chiral derived gluing

- A complex $(V^\bullet, d)$ means a sequence $\{d_i : V^i \to V^{i+1} \mid i \in \mathbb{Z}\}$ of linear maps with $d_{i+1} d_i = 0$.
  The grading $V^\bullet$ is called the cohomological degree.
  We denote the associated linear superspace by $V^{\mathrm{even}} \oplus V^{\mathrm{odd}}$.
- A dg vertex algebra is a complex $(V^\bullet, d)$ equipped with a vertex superalgebra structure $(|0\rangle, T, Y)$ on $V^{\mathrm{even}} \oplus V^{\mathrm{odd}}$ such that
  - $|0\rangle \in V^0$ and $T \in \underline{\mathrm{End}}(V)^0 = \mathrm{Hom}_{\mathsf{dgVec}}(V, V)$.
  - d is an odd derivation (a la Kac) of the vertex superalgebra $(V^{\mathrm{even}} \oplus V^{\mathrm{odd}}, |0\rangle, T, Y)$.
  - State-field correspondence Y respects the cohomological degree: $a_{(n)} V^j \subset V^{i+j}$ for any $a \in V^i$ and $n \in \mathbb{Z}$.

- Li filtration: For a dg vertex algebra (V, d_V),

$$F^pV := \langle (a_1)_{(-n_1)} \cdots (a_r)_{(-n_r)} v \mid a_i, v \in V, n_i \in \mathbb{Z}_{>0}, \sum_i n_i \geq p \rangle_{\mathsf{lin}}$$

(same as non-dg case) gives a decreasing filtration of complexes

$$V = F^0V \supset F^1V \supset F^2V \supset \cdots .$$

- The associated graded space

$$\operatorname{gr}^F V := \bigoplus_{p \in \mathbb{N}} F^pV / F^{p+1}V$$

is a dg vertex Poisson algebra. [H. Li, "Abelianizing vertex algebras", 2005] for non-dg case

  - Structure of commutative dg algebra with 0-derivation δ:

$$\sigma_p(a) \cdot \sigma_q(b) := \sigma_{p+q}(a_{(-1)}b), \quad \delta\sigma_p(a) := \sigma_{p+1}(a_{(-2)}\,|0\rangle) = \sigma_{p+1}(T(a))$$

$\sigma_p : F^pV \twoheadrightarrow F^pV/F^{p+1}V$ denotes the projection.

  - Structure of dg vertex Lie algebra structure:

$$\sigma_p(a)_{(n)}\sigma_q(b) := \sigma_{p+q-n}(a_{(n)}b) \quad (n \in \mathbb{N}).$$

$SS(V) := \operatorname{Spec}(\operatorname{gr}^F V)$: the singular support of V.

- For a dg vertex algebra V, the complex

$$R_V := F^0 V / F^1 V = V / C_2(V), \quad C_2(V) := \left\langle a_{(-2)} b \mid a, b \in V \right\rangle_{\mathrm{lin}}.$$

  is a dg Poisson algebra, called Zhu's C_2-algebra.
  Multiplication $\cdot$ and Poisson bracket $\{-,-\}$ are

$$\overline{a} \cdot \overline{b} := \overline{a_{(-1)} b}, \quad \{\overline{a}, \overline{b}\} := \overline{a_{(0)} b} \qquad (\overline{a} := \sigma_0(a) \in R_V \text{ for } a \in V).$$

  $X_V := \mathrm{Spec}(R_V)$: associated derived scheme of V.
- So far we encountered Poisson and vertex Poisson algebras in dg setting. We will explain those in the following parts.

1. Introduction
2. Vertex algebras in derived setting
3. Dg Poisson algebras and derived Hamiltonian reduction
   - Dg n-Poisson algebras
   - Derived Hamiltonian reduction of dg Poisson algebras
   - Comparison to classical BRST reduction.
4. Dg vertex Poisson algebras and derived arc spaces
5. Chiral derived gluing

- For $n \in \mathbb{Z}$, a dg n-Poisson algebra $(R, \cdot, \{\ ,\ \})$ consist of
  - commutative dg algebra $(R, \cdot)$
  - morphism of complexes $\{\ ,\ \} : R \otimes R \longrightarrow R[1-n]$ (n-Poisson bracket)

  satisfying
  - $\{\ ,\ \}$ is a Lie bracket on $R[n-1]$.
  - $\{f, g \cdot h\} = \{f, g\} \cdot h + (-1)^{|g||h|}\{f, h\} \cdot g$ for homogeneous $f, g, h \in R$.

  In the case $n = 1$, we call it a dg Poisson algebra.
- Examples. $\mathfrak{l}$: dg Lie algebra.
  - Kirillov-Kostant dg Poisson algebra $(\mathrm{Sym}(\mathfrak{l}), \{\ ,\ \}_{\mathrm{KK}})$
  - Chevalley-Eilenberg complex $\mathrm{CE}(\mathfrak{l}, \mathrm{Sym}(\mathfrak{l})) = \mathrm{Sym}(\mathfrak{l}^*[-1]) \otimes \mathrm{Sym}(\mathfrak{l})$ is a dg 2-Poisson algebra with $\cup$ product and Schouten bracket.

- Derived Hamiltonian reduction via coisotropic intersection
  - R: dg Poisson algebra, $\mathfrak{l}$: dg Lie algebra.
    A morphism $\mu : \mathfrak{l} \to R$ of dg Lie algebras is called momentum map.
    It induces $\mathrm{CE}(\mu) : \mathrm{CE}(\mathfrak{l}, \mathrm{Sym}(\mathfrak{l})) \to \mathrm{CE}(\mathfrak{l}, R)$.
    Taking $R = \mathbb{C}$, trivial dg Poisson algebra, we also have
    $\mathrm{CE}(0) : \mathrm{CE}(\mathfrak{l}, \mathrm{Sym}(\mathfrak{l})) \to \mathrm{CE}(\mathfrak{l}, \Bbbk)$
  - By Safronov ["Poisson reduction as a coisotropic intersection" 2017],
    $\mathrm{CE}(0)$ and $\mathrm{CE}(\mu)$ are coisotropic, and the derived tensor product as commutative dg algebras

$$R /\!\!/^{\mathbb{L}}_{\mu} \mathrm{Sym}(\mathfrak{l}) := \mathrm{CE}(\mathfrak{l}, R) \otimes^{\mathbb{L}}_{\mathrm{CE}(\mathfrak{l}, \mathrm{Sym}(\mathfrak{l}))} \mathrm{CE}(\mathfrak{l}, \Bbbk)$$

    is a (homotopy) dg Poisson algebra.

- Derived Hamiltonian reduction and classical BRST reduction:
  $\mathfrak{l}$: dg Lie algebra, R: dg Poisson algebra,
  $\mu : \mathfrak{g} \to R$: momentum map.
  - classical Clifford algebra $\overline{\mathrm{Cl}}(\mathfrak{l}) = (\mathrm{Sym}(\mathfrak{l}[1] \oplus \mathfrak{l}^*[-1]), d_{\overline{\mathrm{Cl}}(\mathfrak{l})})$: dg Poisson algebra
  - classical BRST complex $\mathrm{BRST}_{\mathrm{cl}}(\mathfrak{l}, R, \mu) = (\overline{\mathrm{Cl}}(R) \otimes R, d_{\overline{\mathrm{Cl}}(\mathfrak{l}) \otimes R} + \{\overline{Q}, -\})$: tensor product as graded Poisson algebras and BRST differential
  - Proposition. For $\mathfrak{l} = \mathfrak{g}$ finite dimensional Lie algebra,

$$R /\!\!/_{\mu}^{\mathbb{L}} \mathrm{Sym}(\mathfrak{g}) \simeq \mathrm{BRST}_{\mathrm{cl}}(\mathfrak{g}, R, \mu)$$

    as (homotopy) dg Poisson algebras.

  Thus composition $R_{23} \,\widetilde{\circ}\, R_{12} := \left(R_{12}^{\mathrm{op}} \otimes R_{23}\right) /\!\!/_{\mu}^{\mathbb{L}} \mathrm{Sym}(\mathfrak{g}_2)$ in MT can be regarded as classical BRST reduction.

4. Dg vertex Poisson algebras and derived arc spaces

1. Introduction
2. Vertex algebras in derived setting
3. Dg Poisson algebras and derived Hamiltonian reduction
4. Dg vertex Poisson algebras and derived arc spaces
    - Example of dg vertex Poisson algebra from derived arc space
    - Coisson BRST reduction for dg Poisson vertex algebra
5. Chiral derived gluing

- dg Poisson vertex algebra $(P^\bullet, d, |0\rangle, T, Y_+, Y_-)$
  := dg commutative vertex algebra $(P^\bullet, d, |0\rangle, T, Y_+)$
  + dg vertex Lie algebra $(P^\bullet, d, T, Y_-)$
- dgVA: category of dg vertex algebras
  dgVP: category of dg Poisson vertex algebras
  dgPA: category of dg Poisson algebras
- Constructions in part 2 give

$$\begin{aligned} R_{(-)} : \text{dgVA} &\xrightarrow{\mathrm{gr}^F} \text{dgVP} \xrightarrow{R^{\mathrm{co}}_{(-)}} \text{dgPA}, \\ V &\longmapsto \oplus_p F^p V / F^{p+1} V \longmapsto R_V := F^0 V / F^1 V \\ &\qquad\qquad\qquad \simeq R^{\mathrm{co}}_{\mathrm{gr}^F V} := (\mathrm{gr}^F V)/(\mathrm{Im}\, T). \end{aligned}$$

- Derived coordinate ring of arc space
  - For cdga A, there is cdga $J_\infty(A)$ with 0-derivation T s.t.

$$\mathrm{Map}_{\mathbf{dSch}}(-, \operatorname{Spec} J_\infty(A)) \simeq \mathrm{Map}_{\mathbf{dSch}}(- \times^{\mathbb{L}} \operatorname{Spec} \mathbb{C}[[z]], \operatorname{Spec} A).$$

  - Proposition (level 0 vertex Poisson structure).

    [Arakawa, "A remark on the C_2 cofiniteness condition on vertex algebras", 2012] for non-dg case

    For $(R, \{\ ,\ \}_R) \in$ dgPA, we have $J_\infty(R) \in$ dgVP with

$$u_{(n)}(T^l v) = \begin{cases} \frac{l!}{(l-n)!} T^{l-n}\{u, v\}_R & (l \geq n), \\ 0 & (l < n). \end{cases}$$

    ($u, v \in R \subset J_\infty(R)$, $l \in \mathbb{N}$)
    It satisfies $R^{\mathrm{co}}_{J_\infty(R)} = R$.
  - Sub-example: $\mathfrak{l}$: dg Lie algebra,
    $J_\infty(\mathrm{Sym}(\mathfrak{l})) \simeq \mathrm{Sym}(J_\infty(\mathfrak{l})) = \mathrm{Sym}(\mathfrak{l}[[t]]) \in$ dgVP.
    We have $R^{\mathrm{co}}_{J_\infty(\mathrm{Sym}(\mathfrak{l}))} = \mathrm{Sym}(\mathfrak{l})$.

- Coisson BRST reduction
  - $\mathfrak{l}$: dg Lie algebra, $P \in$ dgVP,
    $J_\infty(\mathfrak{l}) := \mathfrak{l}[[t]]$: dg vertex Lie algebra.
    $\mu_{\mathrm{co}} : J_\infty(\mathrm{Sym}(\mathfrak{l})) \to P$: morphism in dgVP, coisson momentum map.
  - $\mathrm{Cl}_{\mathrm{co}}(J_\infty(\mathfrak{l})) = \mathrm{Sym}(J_\infty(\mathfrak{l})[1] \oplus J_\infty(\mathfrak{l})^*[-1]) \in$ dgVP:
    Clifford vertex Poisson algebra.
  - $\mathrm{BRST}_{\mathrm{co}}(\mathfrak{l}, P, \mu_{\mathrm{co}}) = (\mathrm{Cl}_{\mathrm{co}}(J_\infty(\mathfrak{l})) \otimes P, d_{\mathrm{co}})$: coisson BRST complex.
  - Proposition. Given momentum map $\mu : \mathfrak{l} \to R$, we have coisson momentum map $\mu_{\mathrm{co}} = J_\infty(\mu) : J_\infty(\mathrm{Sym}(\mathfrak{l})) \to J_\infty(R)$, and

$$R^{\mathrm{co}}_{\mathrm{BRST}_{\mathrm{co}}(J_\infty(\mathfrak{l}), J_\infty(R), J_\infty(\mu))} \simeq \mathrm{BRST}(\mathfrak{l}, R, \mu).$$

Thus we can define the category MT$^{\mathrm{co}}$ of dg vertex Poisson algebras equipped with coisson momentum maps using coisson derived gluing. (Details are omitted).
$R^{\mathrm{co}}_{(-)}$ gives a functor MT$^{\mathrm{co}} \to$ MT.

5. Chiral derived gluing

1. Introduction
2. Vertex algebras in derived setting
3. Dg Poisson algebras and derived Hamiltonian reduction
4. Dg vertex Poisson algebras and derived arc spaces
5. Chiral derived gluing
    - BRST reduction and compatibility with coisson/classical BRST reduction.
    - Definition of chiral derived gluing (composition in $\mathsf{MT}^{\mathrm{ch}}$)
    - Main statement

- BRST reduction
  - Back to setting in part 1.
    G: simply connected semisimple group. $\mathfrak{g} = \mathrm{Lie}(G)$.
    $V_k(\mathfrak{g})$: universal affine vertex algebra at level $k \in \mathbb{C}$
  - $V \in$ dgVA with $\mu : V_k(\mathfrak{g}) \to V$ (chiral momentum map),
    $\mathrm{BRST}(\widehat{\mathfrak{g}}_k, V, \mu) = (V \otimes \bigwedge^{\frac{\infty}{2}}(\mathfrak{g}), d_{\mathrm{ch}}) \in$ dgVA: BRST complex.
  - Proposition.
    $\mathrm{gr}^F \mathrm{BRST}(\widehat{\mathfrak{g}}_k, V, \mu) \simeq \mathrm{BRST}_{\mathrm{co}}(J_\infty(\mathfrak{g}), \mathrm{gr}^F V, \mathrm{gr}^F \mu)$ in dgVP,
    $R_{\mathrm{BRST}(\widehat{\mathfrak{g}}_k, V, \mu)} \simeq \mathrm{BRST}_{\mathrm{cl}}(\mathfrak{g}, R_V, R_\mu)$ in dgPA.

- Definition of $\mathrm{MT}^{\mathrm{ch}}$.
  - Objects: simply connected semi-simple groups G.
  - Morphism $(V, \mu_V) : G_1 \to G_2$: $V \in$ dgVA with chiral momentum map $\mu_V : V_k(\mathfrak{g}_1) \otimes V_l(\mathfrak{g}_2) \to V$.
  - Composition of $(V, \mu_V) : G_1 \to G_2$, $(W, \mu_W) : G_2 \to G_3$ (chiral derived gluing):

$$W \mathbin{\widetilde{\circ}} V := \mathrm{BRST}(\widehat{\mathfrak{g}}_{2l+m}, V^{\mathrm{op}} \otimes W, \mu).$$

  V^{op} with $Y_{V^{\mathrm{op}}}(a, z) := Y_V(a, -z)$, $\mu := -\mu_V^2 + \mu_W^1$.

- Theorem.
  The functors gr^F : dgVA $\to$ dgVP, R^{co} : dgVA $\to$ dgPA and R : dgVP $\to$ dgPA give a commutative diagram

$$\begin{array}{ccc} \mathrm{MT}^{\mathrm{ch}} & \xrightarrow{\mathrm{gr}^F} & \mathrm{MT}^{\mathrm{co}} \\ {\scriptstyle R}\downarrow & & \downarrow{\scriptstyle R^{\mathrm{co}}} \\ \mathrm{MT} & = & \mathrm{MT} \end{array}$$

Thank you.