Geometric derived Hall algebra

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§0 Introduction

The derived Hall algebra introduced by Toën (2006) is a version of Ringel-Hall algebra. Roughly it is a "Hall algebra for complexes".

In the case of ordinary Ringel-Hall algebra, we know Lusztig's geometric formulation using the theory of equivariant derived categories on schemes.

I will explain a geometric formulation of derived Hall algebras using the theory of derived categories of constructible sheaves on derived stacks.

Contents

§1.1 Ringel-Hall algebra

A: an $\mathbb{F}_q$-linear abelian category of finite global dimension

$\mathbb{Q}_c(\mathsf{A})$: the linear space of $\mathbb{Q}$-valued func. on $\mathrm{Iso}(\mathsf{A})$ w/ finite supports

$1_{[M]}$: the characteristic function of $[M] \in \mathrm{Iso}(\mathsf{A})$

Fact (Ringel)

$\mathrm{Hall}(\mathsf{A}) := \big(\mathbb{Q}_c(\mathsf{A}), *, 1_{[0]}\big)$ is a unital associative $\mathbb{Q}$-algebra, where

$$1_{[M]} * 1_{[N]} := \sum_{[R] \in \mathrm{Iso}(\mathsf{A})} g_{M,N}^{R} 1_{[R]},$$

$$g_{M,N}^{R} := a_M^{-1} a_N^{-1} e_{M,N}^{R}, \quad a_M := |\mathrm{Aut}(M)|$$

$$e_{M,N}^{R} := |\{0 \to N \to R \to M \to 0 \mid \text{exact in } \mathsf{A}\}|$$

Recall the following different definition of $g_{M,N}^R$:

$$g_{M,N}^R = \left|\mathcal{G}_{M,N}^R\right|, \quad \mathcal{G}_{M,N}^R := \{N' \subset R \mid N' \simeq N,\ R/N \simeq M\}.$$

Thus $g_{M,N}^R$ counts the number of the inclusion pairs $N \subset R$.

Assume basic knowledge on the theory of model category

$\mathsf{C}(\mathbb{F}_q)$: the model category of complexes of $\mathbb{F}_q$-modules

: a fibration is defined to be an epimorphism

a weak equivalence is defined to be a quasi-isomorphism

D: a dg-category over $\mathbb{F}_q$

M(D): the model dg-category of dg-modules over D^{op}

: a dg-module means a $\mathsf{C}(\mathbb{F}_q)$-enriched functor $\mathsf{D}^{\mathsf{op}} \to \mathsf{C}(\mathbb{F}_q)$.

: the model structure is induced by that of $\mathsf{C}(\mathbb{F}_q)$

Assume basic knowledge on the simplicial homotopy theory

Set_Δ: the category of simplicial sets and simplicial maps

: having Kan model structure where a fibration is a Kan fibration and a weak equivalence is a homotopy equivalence of geom. realizations

$\mathcal{H}$: the homotopy category of the model category Set_Δ

: called the homotopy category of spaces

The category $\mathsf{M}(\mathsf{D})$ is $\mathsf{C}(\mathbb{F}_q)$-enriched, so one can attach a simplicial set

$$\mathrm{Map}_{\mathsf{M}(\mathsf{D})}(X,Y) := \mathrm{N}(\mathrm{Hom}_{\mathsf{Mod}(\mathsf{D})}(X,Y)) \in \mathsf{Set}_\Delta$$

where N denotes the nerve construction.

Definition

A dg-module $X \in \mathsf{M}(\mathsf{D})$ is perfect if for any filtered system $\{Y_i\}_{i \in I}$ in $\mathsf{M}(\mathsf{D})$ the natural morphism

$$\varinjlim_{i \in I} \mathrm{Map}_{\mathsf{M}(\mathsf{D})}(X, Y_i) \longrightarrow \mathrm{Map}_{\mathsf{M}(\mathsf{D})}(X, \varinjlim_{i \in I} Y_i)$$

is an isomorphism in $\mathcal{H}$.

$\mathsf{P}(\mathsf{D})$: the sub-dg-cat. of $\mathsf{M}(\mathsf{D})$ of cofibrant and perfect objects and of weak equivalences

$\mathsf{G}'(\mathsf{D}) := \mathsf{Fun}(\Delta^1, \mathsf{M}(\mathsf{D}))$ where $I = \Delta^1$ is the 1-simplex

: with the model structure induced levelwise by $\mathsf{M}(\mathsf{D})$

$\mathsf{G}(\mathsf{D})$: the sub-dg-cat. of $\mathsf{G}'(\mathsf{D})$ of cofibrant and perfect objects

: considered as the category of cofibrations $X \hookrightarrow Y$

For an object $u : X \to Y$ in G(D),

$$s(u) := X, \quad c(u) := Y, \quad t(u) := Y \coprod^{X} 0,$$

which yield the diagram of dg-categories:

$$\begin{array}{ccc} \mathsf{G}(\mathsf{D}) & \xrightarrow{\ c\ } & \mathsf{P}(\mathsf{D}) \\ {\scriptstyle s\times t}\downarrow & & \\ \mathsf{P}(\mathsf{D}) \times \mathsf{P}(\mathsf{D}) & & \end{array} \qquad \begin{array}{ccc} (X \hookrightarrow Y) & \longmapsto & Y \\ \downarrow & & \\ (X, \text{“}Y/X\text{”}) & & \end{array}$$

Define $X^{(0)}(\mathsf{D}), X^{(1)}(\mathsf{D}) \in \mathcal{H}$ by

$$X^{(0)}(\mathsf{D}) := [\mathrm{N}_{\mathsf{dg}}(\mathsf{P}(\mathsf{D}))], \quad X^{(1)}(\mathsf{D}) := [\mathrm{N}_{\mathsf{dg}}(\mathsf{G}(\mathsf{D}))],$$

where N_{dg} denotes the dg nerve construction and $[\cdot] : \mathsf{Set}_{\Delta} \to \mathcal{H}$.
Then we have the diagram of homotopy types

$$\begin{array}{ccc} X^{(1)}(\mathsf{D}) & \xrightarrow{\quad c \quad} & X^{0}(\mathsf{D}) \\ {\scriptstyle s\times t}\downarrow & & \\ X^{(0)}(\mathsf{D}) \times X^{(0)}(\mathsf{D}) & & \end{array}$$

Lemma

If the dg-category D is locally finite, then c is proper and the homotopy types $X^{(i)}(\mathsf{D}) \in \mathcal{H}$ are locally finite.

In the above Lemma we used

Definition

A dg-category D is called locally finite
if the chain complex $\mathrm{Hom}_{\mathsf{D}}(x, y)$ is homologically bounded
with finite-dimensional homology groups for any $x, y \in \mathsf{D}$.

Definition

A homotopy type $X \in \mathcal{H}$ is called locally finite
if for any $x \in X$ the group $\pi_i(X, x)$ is finite and there exists an $n \in \mathbb{N}$
such that $\pi_i(X, x)$ is trivial for $i > n$.

$\mathcal{H}^{\mathsf{lf}}$: the full subcategory of $\mathcal{H}$ spanned by locally finite objects

For a morphism $f : X \to Y$ in $\mathcal{H}^{\mathrm{lf}}$, define $f^* : \mathbb{Q}_c(Y) \to \mathbb{Q}_c(X)$ by

$$f^*(\alpha)(x) := \alpha(f(x)) \quad (\alpha \in \mathbb{Q}_c(Y),\ x \in \pi_0(X)).$$

Also define a linear map $f_! : \mathbb{Q}_c(X) \to \mathbb{Q}_c(Y)$ by

$$f_!(\alpha)(y) := \sum_{x \in \pi_0(X),\, f(x)=y} \alpha(x) \cdot \prod_{i>0} \Big(|\pi_i(X,x)|^{(-1)^i} \, |\pi_i(Y,y)|^{(-1)^{i+1}} \Big).$$

Fact (Toën 2006)

Let D be a locally finite dg-category over $\mathbb{F}_q$. Then

$$\mathrm{Hall}(\mathsf{D}) = \mathbb{Q}_c(X^{(0)}(\mathsf{D}))$$

has a structure of a unital associative $\mathbb{Q}$-algebra with the multiplication

$$\mu := c_! \circ (s \times t)^* : \mathrm{Hall}(\mathsf{D}) \otimes_{\mathbb{Q}} \mathrm{Hall}(\mathsf{D}) \longrightarrow \mathrm{Hall}(\mathsf{D}).$$

We call $\mathrm{Hall}(\mathsf{D})$ the derived Hall algebra of D.

§2.1. Outline of the construction

D: a locally finite dg-category over $\mathbb{F}_q$

Fact (Toën-Vaquié (2009))

We have the moduli stack $\mathcal{P}(\mathsf{D})$ of perfect dg-modules over D^{op}.
It is a derived stack, locally geometric and locally of finite type.

We can also construct the moduli stack of cofibrations $X \to Y$ of perfect-modules over D^{op}, denoted by $\mathcal{G}(\mathsf{D})$.
There exist morphisms

$$s, c, t : \mathcal{G}(\mathsf{D}) \longrightarrow \mathcal{P}(\mathsf{D})$$

of derived stacks which send $u : X \to Y$ to

$$s(u) = X, \quad c(u) = Y, \quad t(u) = Y \coprod^{X} 0.$$

where s, t are smooth and c is proper.

Thus we have the diagram

$$\begin{array}{ccc} \mathcal{G}(\mathsf{D}) & \xrightarrow{c} & \mathcal{P}(\mathsf{D}) \\ {\scriptstyle p}\downarrow & & \\ \mathcal{P}(\mathsf{D}) \times \mathcal{P}(\mathsf{D}) & & \end{array}$$

of derived stacks with smooth $p := s \times t$ and proper c.

Next let $\Lambda := \overline{\mathbb{Q}}_\ell$ where ℓ and q are assumed to be coprime.

We have the derived category $\mathsf{D}^b_{\mathsf{c}}(\mathcal{X}, \Lambda)$ of constructible lisse-étale Λ-sheaves over a locally geometric derived stack $\mathcal{X}$.
We also have derived functors (§4).

Applying the general theory to the present situation, we have

$$\begin{array}{ccc} \mathsf{D}^b_{\mathsf{c}}(\mathcal{G}(\mathsf{D}), \Lambda) & \xrightarrow{c_!} & \mathsf{D}^b_{\mathsf{c}}(\mathcal{P}(\mathsf{D}), \Lambda) \\ \uparrow{\scriptstyle p^*} & & \\ \mathsf{D}^b_{\mathsf{c}}(\mathcal{P}(\mathsf{D}) \times \mathcal{P}(\mathsf{D}), \Lambda) & & \end{array}$$

Now we set

$$\mu : \mathsf{D}^b_{\mathsf{c}}(\mathcal{P}(\mathsf{D}) \times \mathcal{P}(\mathsf{D}), \Lambda) \longrightarrow \mathsf{D}^b_{\mathsf{c}}(\mathcal{P}(\mathsf{D}), \Lambda), \quad M \longmapsto c_! p^*(M)[\dim p]$$

Theorem

μ is associative.

§3 Moduli stack of complexes

§3.1. Derived stacks

Assume basic knowledge on the theory of ∞-categories and ∞-topoi.

- $\Lambda^n_j \subset \Delta^n$ denotes the j-th horn of the n-simplex Δ^n $(0 \leq j \leq n)$.
- An ∞-category is a simplicial set K such that for any $n \in \mathbb{N}$ and any $0 < i < n$, any map $f_0 : \Lambda^n_i \to K$ of simplicial sets admits an extension $f : \Delta^n \to K$.

k: a fixed commutative ring

sCom: the category of simplicial commutative k-algebra

sCom_∞: the ∞-category obtained by localizing sCom by the set of weak equivalences in $\mathrm{sCom} \subset \mathrm{Set}_\Delta$

Definition

We call $\mathrm{dAff}_\infty := (\mathrm{sCom}_\infty)^{\mathrm{op}}$ the ∞-category of affine derived schemes.

A rough explanation of derived schemes

$\mathsf{sCom}_\infty(X)$: the ∞-category of sheaves on a topological space X with coefficients in sCom_∞

dRgSp_∞: the ∞-category of derived ringed spaces whose object is a pair $(X, \mathcal{O}_X)$ of a topological space X and $\mathcal{O}_X \in \mathsf{sCom}_\infty(X)$

dSch_∞: the ∞-category of derived schemes

: the full sub-∞-category of dRgSp_∞ spanned by $(X, \mathcal{O}_X)$ s.t.

- the truncation $(X, \pi_0(\mathcal{O}_X))$ is a scheme
- $\pi_n(\mathcal{O}_X)$ is a quasi-coherent sheaf of $\pi_0(\mathcal{O}_X)$-modules for any $n \in \mathbb{N}$.

Turn to the definition of derived stacks.

A morphism $A \to B$ in sCom_∞ is called étale [smooth] if

- the induced $\pi_0(A) \to \pi_0(B)$ is an étale [smooth] map of com. k-alg.,
- the induced $\pi_i(A) \otimes_{\pi_0(A)} \pi_0(B) \to \pi_i(B)$ is an isomorphism for any i.

Étale morphisms endow dAff_∞ with a Grothendieck topology et.

Definition

The ∞-category of derived stacks is defined to be

$$\mathsf{dSt}_\infty := \mathsf{Sh}_{\infty,\mathsf{et}}(\mathsf{dAff}_\infty)^\wedge \subset \mathsf{PSh}_\infty(\mathsf{dAff}_\infty) := \mathsf{Fun}_\infty((\mathsf{dAff}_\infty)^{\mathsf{op}}, \mathcal{S})$$

with $\mathcal{S}$ the ∞-category of spaces. Its object is called a derived stack.

§3.2. Geometric derived stacks

For $n \in \mathbb{Z}_{\geq -1}$, we define an n-geometric derived stack, inductively on n. At the same time we also define an n-atlas, a n-representable morphism and a n-smooth morphism of derived stacks.

- Let $n = -1$.
  1. A (-1)-geometric derived stack is defined to be an affine derived scheme.
  2. A morphism $f : \mathcal{X} \to \mathcal{Y}$ of derived stacks is called (-1)-representable if for any affine derived scheme U and any morphism $U \to \mathcal{Y}$ of derived stacks, the pullback $\mathcal{X} \times_{\mathcal{Y}} U$ is an affine derived scheme.
  3. A morphism $f : \mathcal{X} \to \mathcal{Y}$ of derived stacks is called (-1)-smooth if it is (-1)-representable, and if for any affine scheme U and any morphism $U \to \mathcal{Y}$ of derived stacks, the induced morphism $\mathcal{X} \times_{\mathcal{Y}} U \to U$ is a smooth morphism of affine derived schemes.
  4. A (-1)-atlas of a stack $\mathcal{X}$ is defined to be the one-member family $\{\mathcal{X}\}$.

- Let $n \in \mathbb{N}$.
  1. Let $\mathcal{X}$ be a derived stack. An n-atlas of $\mathcal{X}$ is a small family $\{U_i \to \mathcal{X}\}_{i \in I}$ of morphisms of derived stacks satisfying the following three conditions.
     - Each U_i is an affine derived scheme.
     - Each morphism $U_i \to \mathcal{X}$ is $(n-1)$-smooth.
     - The morphism $\coprod_{i \in I} U_i \to \mathcal{X}$ is an epimorphism.
  2. A derived stack $\mathcal{X}$ is called n-geometric if the following two conditions are satisfied.
     - The diagonal morphism $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is $(n-1)$-representable.
     - There exists an n-atlas of $\mathcal{X}$.
  3. A morphism $f : \mathcal{X} \to \mathcal{Y}$ of derived stacks is called n-representable if for any affine derived scheme U and for any morphism $U \to \mathcal{Y}$ of derived stacks, the derived stack $\mathcal{X} \times_{\mathcal{Y}} U$ is n-geometric.
  4. A morphism $f : \mathcal{X} \to \mathcal{Y}$ of derived stacks is called n-smooth if for any derived affine scheme U and any morphism $U \to \mathcal{Y}$ of stacks, there exists an n-atlas $\{U_i\}_{i \in I}$ of $\mathcal{X} \times_{\mathcal{Y}} U$ such that for each $i \in I$ the composition $U_i \to \mathcal{X} \times_{\mathcal{Y}} U \to U$ is a smooth morphism of affine derived schemes.

To an algebraic stack $\mathcal{X}$ in the ordinary sense, one can attach a derived stack $j(\mathcal{X})$ functorially.

Fact (Toën-Vezzosi (2008))

For an algebraic stack $\mathcal{X}$, the derived stack $j(\mathcal{X})$ is 1-geometric.

§4 Constructible sheaves on derived stacks

§4.1. Lisse-étale ∞-site

We will introduce the lisse-étale ∞-site for a geometric derived stack, which is an analogue of the lisse-étale site for a algebraic stack introduced by Laumon and Moret-Bailly (2000).

$\mathsf{dAff}_\infty/\mathcal{X}$: the full sub-$\infty$-category of the over-$\infty$-category $(\mathsf{dSt}_\infty)_{/\mathcal{X}}$ spanned by affine derived schemes

Definition

Let $n \in \mathbb{Z}_{\geq -1}$ and $\mathcal{X}$ be an n-geometric derived stack.
The lisse-étale ∞-site

$$\mathsf{Lis\text{-}Et}^n_\infty(\mathcal{X}) = (\mathsf{Lis}^n_\infty(\mathcal{X}), \mathsf{lis\text{-}et})$$

on $\mathcal{X}$ is the ∞-site given by the following description.

- $\mathsf{Lis}^n_\infty(\mathcal{X})$ is the full sub-∞-category of $\mathsf{dAff}_\infty/\mathcal{X}$ spanned by (U, u) where the morphism $u : U \to \mathcal{X}$ is n-smooth.
- The set $\mathrm{Cov}_{\mathsf{lis\text{-}et}}(U, u)$ of covering sieves on (U, u) consists of $\{(U_i, u_i) \to (U, u)\}_{i\in I}$ in $\mathsf{Lis}^n_\infty(\mathcal{X})$ s.t. $\{U_i \to U\}_{i\in I}$ is étale covering.

§4.2. Constructible lisse-étale sheaves

Recall the notion of a constructible sheaf on an ordinary scheme:

A sheaf $\mathcal{F}$ on a scheme X is called constructible if for any affine Zariski open $U \subset X$ there is a finite decomposition $U = \cup_i U_i$ into constructible locally closed subschemes U_i such that $\mathcal{F}|_{U_i}$ is a locally constant sheaf with value in a finite set.

We will introduce a derived analogue of this notion.

Let $\mathcal{X}$ be a geometric derived stack.

Definition

An object of the ∞-category $\mathsf{Sh}_{\infty,\text{lis-et}}(\mathsf{Lis}_{\infty}(\mathcal{X}))$ is called a lisse-étale sheaf.

For an affine derived scheme U, we denote by $\pi_0(U)$ the associated affine scheme.

Definition

A lisse-étale sheaf $\mathcal{F}$ on $\mathcal{X}$ is called constructible if it is cartesian and for any $U \in \mathsf{Lis\text{-}Et}_{\infty}(\mathcal{X})$ the restriction $\pi_0(\mathcal{F})|_{\pi_0(U)}$ is a constructible sheaf on $\pi_0(U)$.

Let Λ be a commutative ring.

Definition

A lisse-étale sheaf of simplicial Λ-modules is an object of the ∞-category

$$\mathsf{Mod}_\infty(\mathcal{X}_{\text{lis-et}}, \Lambda) := \mathsf{Sh}_{\infty,\text{lis-et}}(\mathsf{Lis}_\infty(\mathcal{X}), \Lambda\text{-}\mathsf{sMod}_\infty)$$

For $* \in \{+, -, b\}$ we denote by

$$\mathsf{Mod}^*_\infty(\mathcal{X}_{\text{lis-et}}, \Lambda) \subset \mathsf{Mod}_\infty(\mathcal{X}_{\text{lis-et}}, \Lambda)$$

the full sub-∞-category spanned by sheaves with homologies left bounded (resp. right bounded, resp. bounded).

A constructible lisse-étale sheaf of simplicial Λ-modules is similarly defined. The ∞-category of constructible lisse-étale sheaves of simplicial Λ-modules is denoted by $\mathsf{Mod}^{\mathsf{c}}_\infty(\mathcal{X}_{\text{lis-et}}, \Lambda)$. We also define

$$\mathsf{Mod}^{\mathsf{c},*}_\infty(\mathcal{X}_{\text{lis-et}}, \Lambda) := \mathsf{Mod}^{\mathsf{c}}_\infty(\mathcal{X}_{\text{lis-et}}, \Lambda) \cap \mathsf{Mod}^*_\infty(\mathcal{X}_{\text{lis-et}}, \Lambda) .$$

§4.3. Derived categories and derived functors

Proposition

The ∞-category $\mathsf{Mod}_{\infty}^{\mathsf{c},*}(\mathcal{X}_{\mathsf{lis\text{-}et}}, \Lambda)$ is stable in the sense of Lurie.

Thus the homotopy category $\mathrm{h}\mathsf{Mod}_{\infty}^{\mathsf{c},*}(\mathcal{X}_{\mathsf{lis\text{-}et}}, \Lambda)$ has a structure of a triangulated category.

Definition

For a commutative ring Λ, we set

$$\mathsf{D}_{\mathsf{c}}^{*}(\mathcal{X}, \Lambda) := \mathrm{h}\mathsf{Mod}_{\infty}^{\mathsf{c},*}(\mathcal{X}_{\mathsf{lis\text{-}et}}, \Lambda) \quad (* \in \{\emptyset, +, -, b\})$$

and call it the (resp. left bounded, resp.right bounded, resp.bounded) derived category of constructible sheaves of Λ-modules on $\mathcal{X}$.

Using this derived category one can construct an analogue of Grothendieck's six derived functors. For a morphism $f : \mathcal{X} \to \mathcal{Y}$ of geometric derived stacks (which are locally of finite type) we have

$$\mathrm{R}f_* : \mathsf{D}^+_{\mathsf{c}}(\mathcal{X}) \longrightarrow \mathsf{D}^+_{\mathsf{c}}(\mathcal{Y}), \quad \mathrm{R}f_! : \mathsf{D}^-_{\mathsf{c}}(\mathcal{X}) \longrightarrow \mathsf{D}^-_{\mathsf{c}}(\mathcal{Y}),$$
$$\mathrm{L}f^* : \mathsf{D}_{\mathsf{c}}(\mathcal{Y}) \longrightarrow \mathsf{D}_{\mathsf{c}}(\mathcal{X}), \quad \mathrm{R}f^! : \mathsf{D}_{\mathsf{c}}(\mathcal{Y}) \longrightarrow \mathsf{D}_{\mathsf{c}}(\mathcal{X}).$$

Also we have RHom and $\otimes^{\mathrm{L}}$.

By construction these derived categories and functors are compatible with those for algebraic stacks developed by Laszlo and Olsson (2008).

Remark

On the construction of pullback $\mathrm{L}f^*$ for general f.
For $f : \mathcal{X} \to \mathcal{Y}$, we have an adjunction

$$f^{-1} : \text{Lis-Et}_\infty(\mathcal{Y}) \rightleftarrows \text{Lis-Et}_\infty(\mathcal{X}) : f_*$$

where f_* sends a sheaf M to the sheaf $U \mapsto M(U \times_{\mathcal{Y}} \mathcal{X})$ and f^{-1} sends a sheaf N to the sheaf

$$V \longmapsto \varinjlim_{(V \to U) \in (\mathsf{dSt}_\infty)_{/f}} N(U).$$

It turns out that f^{-1} is not left exact, so that the pair (f^{-1}, f_*) is not a geometric morphism of ∞-topoi in the sense of Lurie.
Quite the same phenomenon occurs in the non-derived case, which causes a little bit complication in the construction of $\mathrm{L}f^*$.