AGT conjecture and
Zamolodchikov-type recursive formula

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Synthesis of integrabilities in the context of
duality between string theory and gauge theories
@ HSE and Steklov
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Abstract

I shall explain a proof of 4-dim pure gauge SU(2) AGT relation.

(Mainly based on the works by Poghossian, Fateev-Litvinov and Hasadz-Jaskólski-Suchanek.)

In this strategy an important role is played by a recursive formula, which has oringin in the classical analysis of conformal block by Al. Zamolodchikov in 1984.

**I will also discuss a strange term appearing in the recursive formula.
It is nothing but the norm of logarithmic primary field,
which also appeared in the studies of (super) Liouville field theory in 2000s.**

Contents

§1. Quick review of AGT relation

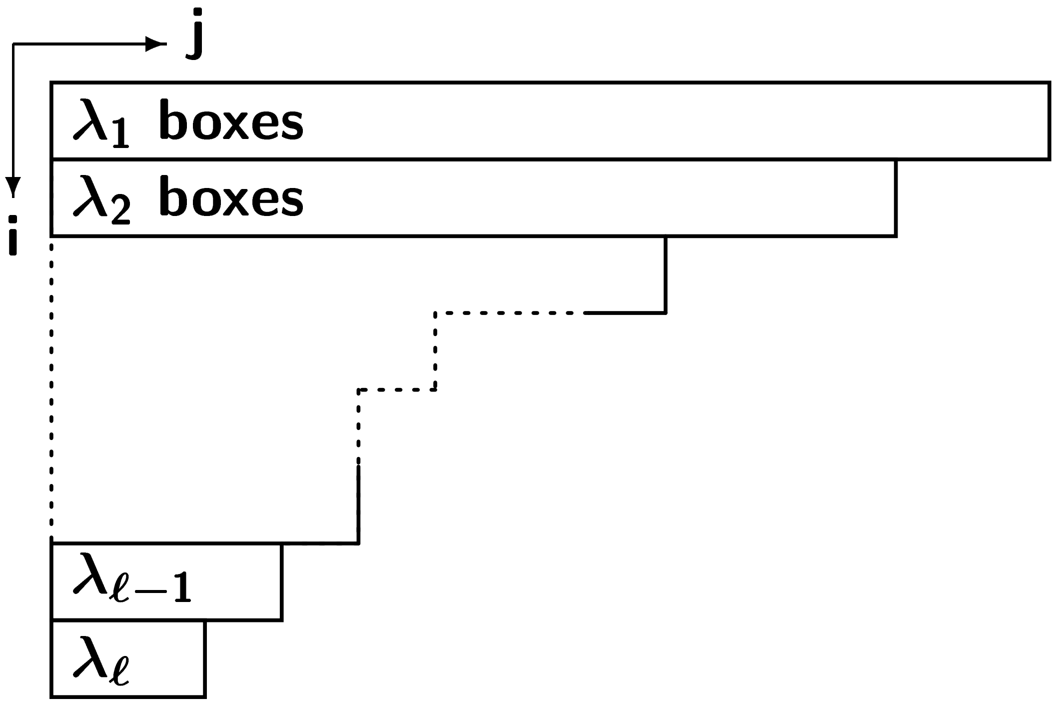
§1.0. Notations of partitions and Young diagram

- Partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$: empty sequence $\emptyset = () = (0)$ or non-increasing sequence of natural number $(\lambda_1 \geq \cdots \lambda_k \geq 1)$

  $|\lambda| := \lambda_1 + \cdots + \lambda_k$, $\quad \ell(\lambda) := k$: length

  $\lambda \vdash n \overset{\text{def}}{\Longleftrightarrow} \lambda$ is a partition s.t. $|\lambda| = n$.

- Young diagram of the partition λ:



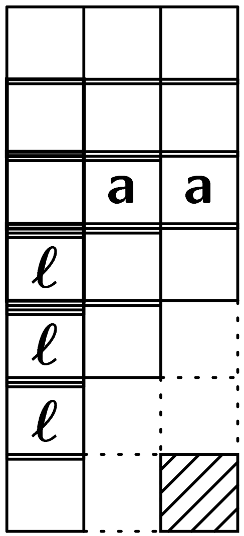
- **arm and leg**

λ : partiton, $\quad \square = (i,j)$: box at the location (i,j)

$$a_\lambda(\square) := \lambda_i - j : \text{arm}, \quad \ell_\lambda(\square) := {\lambda^\vee}_j - i : \text{leg}$$

where $\lambda^\vee$ is the transposition of λ, $\quad \lambda_i := \begin{cases} \lambda_i & (i \leq \ell(\lambda)) \\ 0 & (i > \ell(\lambda)) \end{cases}$

<u>E.g.</u> $\lambda = (3,2,1,1) \quad \lambda^\vee = (4,2,1)$



$(3,2,1,1)$
$(3,2,1,1)$
$(3,2,1,1)$
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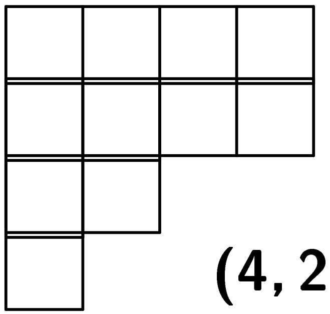

$(4,2,1)$
$(4,2,1)$

$\square = (1,1) \quad a_\lambda(\square) = \lambda_1 - 1 = 2, \quad \ell_\lambda(\square) = {\lambda^\vee}_1 - 1 = 3$

§1.1. Combinatorial defn. of pure SU(r) Nekrasov function

- $r \in \mathbb{Z}_{\geq 1}$

  $x, \epsilon_1, \epsilon_2, \vec{a} = (a_1, \dots, a_r)$: indeterminates

$$Z^{\text{4-dim}}_{SU(r)}(x; \epsilon_1, \epsilon_2, \vec{a}) = \sum_{\vec{Y}} \frac{x^{|\vec{Y}|}}{\prod_{1\leq \alpha,\beta \leq r} n^{\vec{Y}}_{\alpha,\beta}(\epsilon_1, \epsilon_2, \vec{a})}$$

$\vec{Y} = (Y_1, \dots, Y_r)$: r-tuple of partitions, $\quad |\vec{Y}| := |Y_1| + \cdots + |Y_r|$,

$$n^{\vec{Y}}_{\alpha,\beta}(\epsilon_1, \epsilon_2, \vec{a}) := \prod_{\square \in Y_\alpha} [-\ell_{Y_\beta}(\square)\epsilon_1 + (a_{Y_\alpha}(\square) + 1)\epsilon_2 + a_\beta - a_\alpha] \times \prod_{\blacksquare \in Y_\beta} [(\ell_{Y_\alpha}(\blacksquare) + 1)\epsilon_1 - a_{Y_\beta}(\blacksquare)\epsilon_2 + a_\beta - a_\alpha].$$

§1.2. Gaiotto state

§1.2.1 Notations on Virasoro algebra

- **Vir :Virasoro alg.**

  generators : L_n $(n \in \mathbb{Z})$, $\quad C$ (central)

  relaiton : $[L_n, L_m] = (n-m)L_{n+m} + \frac{1}{12}Cn(n^2-1)\delta_{n+m,0}$

- **Triangular decomposition :** $\mathrm{Vir} = \mathrm{Vir}_+ \oplus \mathrm{Vir}_0 \oplus \mathrm{Vir}_-$

  $$\mathrm{Vir}_\pm := \oplus_{\pm n \in \mathbb{Z}_{>0}} \mathbb{C}L_n, \quad \mathrm{Vir}_0 := \mathbb{C}L_0 \oplus \mathbb{C}C$$

- **(Our) PBW basis of U(Vir) :**

  $$\{L_{-\lambda}L_0^n L_\mu C^m \mid n, m \in \mathbb{Z}_{\geq 0},\ \lambda, \mu : \text{partition}\},$$

where for a partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ **we set**

$$L_\lambda := L_{\lambda_\ell} \cdots L_{\lambda_1}, \quad L_{-\lambda} := L_{-\lambda_1} \cdots L_{-\lambda_\ell}$$

§1.2.2 Verma module

• $c, h \in \mathbb{C}$: central charge and highest weight

$$M(c,h) := \mathrm{Ind}_{\mathrm{Vir}_+ \oplus \mathrm{Vir}_0}^{\mathrm{Vir}} \mathbb{C}_{c,h}$$

$\mathbb{C}_{c,h} := \mathbb{C}\,|c,h\rangle$: one dim. $(\mathrm{Vir}_+ \oplus \mathrm{Vir}_0)$-representation

$$L_n\,|c,h\rangle = 0 \ (n > 0), \quad L_0\,|c,h\rangle = h\,|c,h\rangle, \quad C\,|c,h\rangle = c\,|c,h\rangle$$

L_0-weight decomposition :

$$M(c,h) = \oplus_{n \in \mathbb{Z}_{\geq 0}} M(c,h)_n,$$
$$M(c,h)_n := \{v \in M(c,h) \mid L_0 v = (h+n)v\}$$

• Dual Verma module $\quad M^*(c,h) := \mathrm{Ind}_{\mathrm{Vir}_- \oplus \mathrm{Vir}_0}^{\mathrm{Vir}} \mathbb{C}^*_{c,h}$

$\mathbb{C}^*_{c,h} := \mathbb{C}\,\langle c,h|$: one dim. $(\mathrm{Vir}_- \oplus \mathrm{Vir}_0)$-right repr.

• $\cdot : M^*(c,h) \times M(c,h) \to \mathbb{C}$: Shapovalov form

Determined by $\langle c,h| \cdot |c,h\rangle = 1$, $\langle c,h|\, u \cdot L_n v\,|c,h\rangle = \langle c,h|\, uL_n \cdot v\,|c,h\rangle$.

§1.2.3. Kac determinant

- Set the matrix of Shapovalov form as

$$\mathbf{K_n} := (\langle c, h|\, L_\lambda L_{-\mu}\, |c, h\rangle)_{\lambda,\mu \vdash n},$$

then

$$\det K_n \propto \prod_{\substack{r,s\in\mathbb{Z}_{\geq 1} \\ 1\leq rs\leq n}} (h - h_{r,s})^{p(n-rs)} \qquad (p(m) := \#\{\lambda \mid \lambda \vdash m\})$$

$$h_{r,s} := \frac{1}{48}[(13 - c)(r^2 + s^2) - 24rs - 2(1 - c) + \sqrt{(1 - c)(25 - c)}(r^2 - s^2)]$$

§1.2.4. Gaiotto state

- $\Lambda \in \mathbb{C}$ fix. Gaiotto state $|G\rangle \in M(c,h)$ is an element such that

$$L_1 |G\rangle = \Lambda^2 |G\rangle, \quad L_n |G\rangle = 0 \ (n \geq 2),$$

$$|G\rangle = |c,h\rangle + \cdots$$

(The homogeneous component of $|G\rangle$ in $M(c,h)_0$ is $|c,h\rangle$).

- Dual Gaiotto state $\langle G| \in M^*(c,h)$ is similarly defined :

$$\langle G| L_{-1} = \Lambda^2 \langle G|, \quad \langle G| L_{-n} = 0 \ (n \geq 2),$$

$$\langle G| = \langle c,h| + \cdots$$

§1.3. pure gauge SU(2) AGT relation

[Gaiotto, arXiv:0908.0307]

- For generic c, h

$$\langle G \mid G \rangle \stackrel{?}{=} Z^{\text{4-dim}}_{SU(2)}(x; \epsilon_1, \epsilon_2, \vec{a}),$$

where parameters are related as

Virasoro	Nekrasov
c	$13 + 6(\epsilon_1/\epsilon_2 + \epsilon_2/\epsilon_1)$
h	$(\epsilon_1/\epsilon_2 + \epsilon_2/\epsilon_1 + 2)/4 - (a_2 - a_1)^2/\epsilon_1\epsilon_2$
Λ	$x^{1/4}/(\epsilon_1\epsilon_2)$

<u>Restatement</u> [Marshakov-Mironov-Morozov, 2009]

Since $\langle G \mid G \rangle = \sum_{n=0}^{\infty} \Lambda^{4n} (K_n^{-1})_{(1^n),(1^n)}$, we should have

$$(K_n^{-1})_{(1^n),(1^n)} \stackrel{?}{=} (\epsilon_1\epsilon_2)^{4n} Z_n(\epsilon_1, \epsilon_2; \vec{a}),$$

where Z_n is a homogeneous component of $Z^{\text{4-dim}}_{SU(2)}$:

$$Z^{\text{4-dim}}_{SU(2)}(x; \epsilon_1, \epsilon_2, \vec{a}) = \sum_{n=0}^{\infty} x^n Z_n(\epsilon_1, \epsilon_2, \vec{a}).$$

§2. Proof via recursive formula

• The strategy I shall treat is showing that Z_n and $(K_n)^{-1}_{(1^n),(1^n)}$ satisfy the same recursive formula.

Poghossian JHEP 0912 (2009), arXiv:0909.3412

Fateev-Litvinov JHEP 1002 (2010), arXiv:0912.0504

Hasadz-Jaskólski-Suchanek JHEP 1006 (2010), arXiv:1004.1841

§2.1 Recursive formula for Nekrasov partition function

Fact [Fateev-Litvinov, Hasadz-Jaskólski-Suchanek]

Set $a := a_1 - a_2$. Then $Z_n(\epsilon_1, \epsilon_2, a)$ satisfies

$$Z_n(\epsilon_1, \epsilon_2, a) = \delta_{n,0} + \sum_{\substack{r,s \in \mathbb{Z}_{\geq 1} \\ 1 \leq rs \leq n}} R_{r,s}(\epsilon_1, \epsilon_2) \frac{Z_{n-rs}(\epsilon_1, \epsilon_2, (r\epsilon_1 - s\epsilon_2)/2)}{4a^2 - (r\epsilon_1 + s\epsilon_2)^2} \qquad (1)$$

with

$$R_{r,s}(\epsilon_1, \epsilon_2) := 2^{-1} \prod_{\substack{1-r \leq j \leq r \\ 1-s \leq k \leq s \\ (j,k) \neq (0,0),(r,s)}} (j\epsilon_1 + k\epsilon_2)^{-1}$$

Remark

(1) The proof uses an integral expression of Z_n.

(2) Fateev-Litvinov showed the adjoint matter case.
Hasadz-Jaskólski-Suchanek showed the cases with $N_f \leq 2$.

§2.2 Recursive formula in the Virasoro side

<u>Fact</u> [Hasadz-Jaskólski-Suchanek]

Set

$$c = c(t) := 13 - 6(t + t^{-1}),$$

$$h_{r,s}(t) := h_{r,s}|_{c=c(t)} = \frac{(rt-s)^2 - (t-1)^2}{4t}$$

Then $f_n(t,h) := (K_n(c(t),h)^{-1})_{(1^n),(1^n)}$ satisfies

$$f_n(t,h) = \delta_{0,n} + \sum_{\substack{(r,s)\in\mathbb{Z}^2_{>0},\\ 1\leq rs\leq n}} \left[\lim_{h\to h_{r,s}(t)} \frac{N_{r,s}(t,h)}{h - h_{r,s}(t)}\right]^{-1} \frac{f_{n-rs}(t, h_{r,s}(t) + rs)}{h - h_{r,s}(t)}. \qquad (2)$$

- $N_{r,s}(t,h)$ will be defined in the next subsection.
- By rewriting this formula (2) in the Nekrasov parameter,

one recovers (1) except the term $R_{r,s}(\epsilon_1, \epsilon_2)$.

§2.3. Norm of Logarithmic Primary Field

§2.3.1 Singular vector

- $v \in M(c,h)_n$ is a singular vector $\overset{\text{def}}{\Longleftrightarrow} L_k v = 0$ for any $k \in \mathbb{Z}_{>0}$.

- $M(c(t), h_{r,s}(t))_{rs}$ has a unique singular vector $|\chi_{r,s}\rangle$

up to normalization, and it is expressed as :

$$|\chi_{r,s}\rangle = P_{r,s}(t)\,|c(t), h_{r,s}(t)\rangle,$$

$$P_{r,s}(t) = L_{-1}^{rs} + \sum_{\lambda \vdash rs,\ \lambda \neq (1^{rs})} c_\lambda(t) L_{-\lambda}$$

$$\in U(\mathrm{Vir}_-) \otimes \mathbb{C}[t, t^{-1}]$$

For a proof see : D. B. Fuchs, "Singular vectors over the Virasoro algebra and extended Verma modules", in "Unconventional Lie algebras", Adv. Soviet Math., 17, AMS, 1993.

Examples

$$\begin{aligned}
P_{1,1}(t) &= L_{-1}, \\
P_{2,1}(t) &= L_{-1}^2 - tL_{-2}, \\
P_{3,1}(t) &= L_{-1}^3 - 4tL_{-2}L_{-1} + 2t(2t-1)L_{-3}, \\
P_{4,1}(t) &= L_{-1}^4 - 10tL_{-3}L_{-1} + 9t^2L_{-2}^2 + 2t(12t-5)L_{-3}L_{-1} \\
&\quad - 6t(6t^2-4t+1)L_{-4}, \\
P_{2,2}(t) &= L_{-1}^4 - 2(t+t^{-1})L_{-3}L_{-1} + (t^2-2+t^{-2})L_{-2}^2 \\
&\quad - 2(t-3+t^{-1})L_{-3}L_{-1} - 3(t-2+t^{-1})L_{-4}
\end{aligned}$$

We also have $P_{r,s}(t) = P_{s,r}(t^{-1})$.

§2.3.2. Norm of Logarithmic Primary Field

- Define the anti-autmorphism † on U(Vir) by $L_n^\dagger = L_{-n}$, $C^\dagger = C$.

- By the definition of the singular vector, its norm vanishes :

$$\langle \chi_{r,s} \mid \chi_{r,s} \rangle = \langle c(t), h_{r,s}(t) | \, [P_{r,s}(t)]^\dagger P_{r,s}(t) \, | c(t), h_{r,s}(t) \rangle = 0.$$

- Let us define the norm of logarithmic primary field by :

$$N_{r,s}(t,h) := \langle c(t), h | \, [P_{r,s}(t)]^\dagger P_{r,s}(t) \, | c(t), h \rangle \, .$$

Examples

$$N_{1,1}(t,h) = 2h \; N_{1,1}(t,h) = 2(h - h_{1,1}(t))$$

$$N_{2,1}(t,h) = -(-t(4h + 13/2 - 3t - 3/t) + 6h)t - 6th + 4h(1 + 2h)$$
$$N_{2,1}(t,h) = 4(t^2 - 1)(h - h_{2,1}(t)) + 8(h - h_{2,1}(t))^2$$
$$N_{1,2}(t,h) = 4(t^{-2} - 1)(h - h_{1,2}(t)) + 8(h - h_{1,2}(t))^2$$

$$N_{3,1}(t,h) = 24(t^2 - 1)(4t^2 + 1)(h - h_{3,1}(t)) + 8(16t^2 - 9)(h - h_{3,1}(t))^2 + 48(h - h_{3,1}(t))^3$$
$$N_{1,3}(t,h) = 24(t^{-2} - 1)(4t^{-2} + 1)(h - h_{1,3}(t)) + 8(16t^{-2} - 9)(h - h_{1,3}(t))^2 + 48(h - h_{1,3}(t))^3$$

$$N_{4,1}(t,h) = 288(t^2 - 1)(4t^2 - 1)(9t^2 - 1)(h - h_{4,1}(t)) + (1056 - 7696t^2 + 9504t^4)(h - h_{4,1}(t))^2 + 128(25t^2 - 9)(h - h_{4,1}(t))^3 + 384(h - h_{4,1}(t))^4$$
$$N_{2,2}(t,h) = -8(t^2 - 1)(t^2 - 4)(t^{-2} - 1)(t^{-2} - 4)(h - h_{2,2}(t)) + 16(2t^{-4} - 33t^{-2} + 91 - 33t^2 + 2t^4)(h - h_{2,2}(t))^2 + 128(t^2 + 3t + 1))(t^{-2} - 3t^{-1} + 1)(h - h_{2,2}(t))^3 + 384(h - h_{2,2}(t))^4$$

Observation (Al. Zamolodchikov)

$$\mathsf{A}_{r,s}(t) := \lim_{h\to h_{r,s}(t)} \frac{N_{r,s}(t,h)}{h - h_{r,s}(t)} \text{ has the next form :}$$

$$A_{r,s}(t) \overset{?}{=} 2 \prod_{\substack{1-r\leq j\leq r,\ 1-s\leq k\leq s,\\ (j,k)\neq(0,0),(r,s)}} (jt^{1/2} + kt^{-1/2})$$

- This is nothing but the factor $R_{r,s}$ in the recursive formula (1) of Nekrasov partition function.

Thus the AGT relation $\langle G \mid G\rangle \overset{?}{=} Z^{\text{4-dim}}_{SU(2)}$ is reduced to the above equation.

c.f. Al. Zamolodchikov : “Higher equations of motion in Liouville field theory”, Int. J. Mod. Phys. A 19 (2004); hep-th/0312279.

§3 How to calculate $A_{r,s}(t)$?

Step 1. Degree estimate of $A_{r,s}(t)$

Step 2. Determine the zero set S of $A_{r,s}(t) \in \mathbb{C}[t, t^{-1}]$.

(The observation predicts
$\{t = -k/j \mid 1 - r \leq j \leq r,\ 1 - s \leq k \leq s,\ j \neq 0,\ k \neq 0,\ (j, k) \neq (r, s)\}$.)

2-1. Bound S from above : from the degree estimate in Step 1

2-2. Bound S from below

2-2-1. Bosonise $P_{r,s}(t) \left|c(t), h\right\rangle$.

2-2-2. From 2-2-1, S contains

$$S' := \{t = k/j \mid 1 \leq j \leq r,\ 1 \leq k \leq s,\ (j, k) \neq (r, s)\}$$
$$\cup \{t = k/j \mid 1 \leq j \leq r - 1,\ 1 \leq k \leq s - 1\}.$$

2-2-3. Show that $-S'$ is also contained in S.

2-3. From 2-1 and 2-2, one has $S = S' \cup -S'$.

Step 3. Determine the leading term of t in $A_{r,s}(t)$

§3.1 Step 1. Degree estimate of $A_{r,s}(t)$

Fact [Astashkevich-Fuchs, 1997] Expanding $P_{r,s}(t)$ by t, one has

$$P_{r,s}(t) = [(r-1)!]^{2s} L_{-r}^{s} t^{(r-1)s} + \cdots + [(s-1)!]^{2r} L_{-s}^{r} t^{-(s-1)r}.$$

This fact and the degree counting of $\langle c(t), h | L_{\mu} L_{-\lambda} | c(t), h \rangle$ with respect to t, h gives

Lemma 1 Defining the max / min degree of the Laurent polynomial $A_{r,s}(t) = \sum_k a_k t^k$ by

$$\max\deg A_{r,s}(t) := \max\{k \mid a_k \neq 0\},$$
$$\min\deg A_{r,s}(t) := \min\{k \mid a_k \neq 0\}$$

we have

$$\max\deg A_{r,s}(t) \leq 2(r-1)s,$$
$$\min\deg A_{r,s}(t) \geq -2(s-1)r.$$

§3.2. Step 2-2-1. Bound S from below (1) Bosonization

§3.2.1 Heisenberg algebra

- **Heisenberg alg.** $\mathcal{H}$

generator : a_n $(n \in \mathbb{Z})$ **relation** $[a_n, a_m] = n\delta_{n+m,0}a_0$.

Triangular decomposition : $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_0 \oplus \mathcal{H}_-$ $\mathcal{H}_\pm := \oplus_{\pm n \in \mathbb{Z}_{>0}} \mathbb{C}a_n$, $\mathcal{H}_0 := \mathbb{C}a_0$

- $\mathcal{F}_\alpha$ **: Fock module** $\mathcal{F}_\alpha := \mathrm{Ind}^{\mathcal{H}}_{\mathcal{H}_0 \oplus \mathcal{H}_+} \mathbb{C}_\alpha$

$\mathbb{C}_\alpha = \mathbb{C}|\alpha\rangle_{\mathcal{F}}$ **: one dim.** $(\mathcal{H}_0 \oplus \mathcal{H}_+)$**-rep.,**

$$a_0 |\alpha\rangle_{\mathcal{F}} = \alpha |\alpha\rangle_{\mathcal{F}}, \quad a_n |\alpha\rangle_{\mathcal{F}} = 0 \ (n \in \mathbb{Z}_{>0})$$

§3.2.2 Bosonization

- $\varphi : \mathsf{U}(\mathsf{Vir}) \to \widehat{\mathsf{U}}(\mathcal{H})$ is defined by :

$$\mathsf{L}_n \mapsto \mathcal{L}_n := \frac{1}{2}\sum_{m\in\mathbb{Z}} {}^{\circ}_{\circ} a_m a_{n-m} {}^{\circ}_{\circ} - (n+1)\rho a_n, \quad \mathsf{C} \mapsto 1 - 12\rho^2,$$

where ${}^{\circ}_{\circ}\ {}^{\circ}_{\circ}$ is the normal ordering of $\mathcal{H}$.

- The vector space homorphism

$$\psi : M(c,h) \to \mathcal{F}_\alpha, \quad L_{-\lambda}\,|c,h\rangle \mapsto \mathcal{L}_{-\lambda}\,|\alpha\rangle_{\mathcal{F}}$$

is compatible with φ, where $c \mapsto 1 - 12\rho^2$, $h \mapsto \alpha(\alpha - 2\rho)/2$.

- Set $\rho(t) := (t^{1/2} - t^{-1/2})/\sqrt{2}$,
  $\alpha_{r,s}(t) := \left[(r+1)t^{1/2} - (s+1)t^{-1/2}\right]/\sqrt{2}$.

Then $c(t) = 1 - 12\rho(t)^2$, $\quad h_{r,s}(t) = \alpha_{r,s}(t)(\alpha_{r,s}(t) - 2\rho(t))/2$.

Lemma 2 In the bosonized expression

$$\varphi(P_{r,s}(t))\,|\alpha\rangle \in \mathbb{C}[t^{\pm 1/2}] \otimes \mathbb{C}[a_{-1}, \cdots, a_{-rs}],$$

denote the coefficient of a_{-rs} by $e_{r,s}(t) \in \mathbb{C}[t^{\pm 1/2}]$.

Then the zeros of $e_{r,s}(t)$ w.r.t. t are in S (including multiplicities).

- Set $d_{r,s}(t,\alpha) := \alpha - \alpha_{r,s}(t)$, $\quad d^{\dagger}_{r,s}(t,\alpha) := \alpha - \alpha_{-r,-s}(t)$.

Then $h - h_{r,s}(t) = \frac{1}{2} d_{r,s}(t,\alpha) d^{\dagger}_{r,s}(t,\alpha)$.

Lemma 3 The bosonized $P_{r,s}(t)\,|c(t), h\rangle$ has the next form :

$$\varphi(P_{r,s}(t))\,|\alpha\rangle = d^{\dagger}_{r,s}(t,\alpha)\Big[g_0(t) + \sum_{k=1}^{rs-1} (d_{r,s}(t,\alpha))^k g_k(t)\Big]\,|\alpha\rangle$$

with $g_0(t) \in \mathbb{C}[t^{\pm 1/2}] \otimes \mathbb{C}[a_{-1}, \cdots, a_{-rs}]$,

$g_k(t) \in \mathbb{C}[t^{\pm 1/2}] \otimes \mathbb{C}[a_{-1}, \cdots, a_{-rs+1}]$ $(k \neq 0)$.

• From Lemma 3, the coefficient of a_{-rs} in $\varphi(P_{r,s}(t))\,|\alpha\rangle$ is the same as that in

$$\varphi(P_{r,s}(t))\,|\alpha\rangle\,|_{d_{r,s}(t,\alpha)=0} = \varphi(P_{r,s}(t))\,|\alpha_{r,s}(t)\rangle = \psi(|\chi_{r,s}\rangle),$$

that is, in the bosonized singular vector.

• Now it is time to recall the Jack symmetric function...

§3.2.4 Ring of symmetric functions

- $\Lambda = \mathbb{Z}[x_1, x_2, \dots,]^{\mathfrak{S}_\infty}$: ring of symmeric functions (à la Macdonald), $\Lambda_K := \Lambda \otimes K$.

- m_λ : monomial symmetric function
- $p_r := \sum_i x_i^r$, $\quad p_\lambda := p_{\lambda_1} \cdots p_{\lambda_\ell}$: power sum symmetric function.

- β : indeterminate. β-inner product on $\Lambda_{\mathbb{Q}(\beta)}$:

$$\langle\, p_\lambda, p_\mu \,\rangle_\beta := \delta_{\lambda,\mu} z_\lambda \beta^{\ell(\lambda)}.$$

where

$$z_\lambda := \prod_{i \in \mathbb{Z}_{\geq 1}} i^{m_i(\lambda)} m_i(\lambda)!, \quad \text{with} \quad m_i(\lambda) := \#\{1 \leq j \leq \ell(\lambda) \mid \lambda_j = i\},$$

§3.2.5 Jack symmetric function

- (monic) Jack symmetric function $P_\lambda^{(\beta)} \in \Lambda_{\mathbb{Q}(\beta)}$ is characterized by

$$\text{(i)} \quad P_\lambda^{(\beta)} = \sum_{\mu \le \lambda} c_{\lambda,\mu}(\beta) m_\mu, \quad c_{\lambda,\mu}(\beta) \in \mathbb{Q}(\beta), \; c_{\lambda,\lambda}(\beta) = 1$$

$$\text{(ii)} \quad \langle P_\lambda^{(\beta)}, P_\mu^{(\beta)} \rangle_\beta = 0 \quad \text{if } \lambda \ne \mu.$$

Here the ordering in (i) is the dominance semi-ordering :

$$\lambda \ge \mu \overset{\text{def}}{\Longleftrightarrow} |\lambda| = |\mu|, \; \textstyle\sum_{k=1}^{i} \lambda_k \ge \sum_{k=1}^{i} \mu_k \; (i = 1, 2, \ldots).$$

- (integral) Jack symmetric function $J_\lambda^{(\beta)}$:

$$J_\lambda^{(\beta)} := P_\lambda^{(\beta)} \cdot \prod_{\square \in \lambda} (\beta a_\lambda(\square) + \ell_\lambda(\square) + 1)$$

§3.2.6 Fock module and symmetric functions

- Consider the isomorphism of vector spaces

$$\iota : \mathcal{F}_\alpha \otimes \mathbb{C}[t^{\pm 1/2}] \to \Lambda_{\mathbb{C}[t^{\pm 1/2}]}, \quad a_{-\lambda} |\alpha\rangle \mapsto (t/2)^{\ell(\lambda)/2} p_\lambda$$

Key Fact

(1) [Mimachi-Yamada, 1995] The singular vector expressed in the symmetric function $\iota \circ \psi(|\chi_{r,s}\rangle)$ is proportional to Jack symemtric function :

$$\iota \circ \psi(|\chi_{r,s}\rangle) \propto J^{(1/t)}_{(s^r)}.$$

(2) [Sakamoto-Shiraishi-Arnaudon-Frappat-Ragoucy, 2005] The proportional constant is

$$B(r,s,t) := \prod_{j=1}^{r} \prod_{k=1}^{s} (jt - k).$$

§3.2.7. Back to the proof

<u>Fact</u> [c.f. Hanlon-Stanley-Stembridge, 1992]

Let λ be a partition of n. Expanding Jack by power sum

$$J_\lambda^{(\beta)} = \sum_{\mu \vdash n} \theta_\lambda^\mu(\beta) p_\mu,$$

we have

$$\theta_\lambda^{(n)}(\beta) = n \prod_{(i,j)\in\lambda,\ (i,j)\neq(1,1)} [\beta(i-1)-(j-1)].$$

• From Lemmas 2,3, Key Fact and formulas for $\theta_{(s^r)}^{(n)}(1/t)$, $B(r,s,t)$:

<u>Prop</u> $A_{r,s}(t)$ has zeros at

$$\begin{aligned} S' :=& \{t = k/j \mid 1 \le j \le r,\ 1 \le k \le s,\ (j,k) \neq (r,s)\} \\ &\cup \{t = k/j \mid 1 \le j \le r-1,\ 1 \le k \le s-1\}. \end{aligned}$$

(including multiplicites)

§3.2.8 Duality of $t \leftrightarrow -t$

• The zero set of $A_{r,s}(t)$ is invariant under the transformation $t \mapsto -t$.

This is the consequence of

<u>Fact</u> [Feigin-Fuchs, 1990]

Defining the anti auto-morphsm σ on $U(Vir_-)$ by $\sigma : L_{-n} \mapsto (-1)^{n-1} L_{-n}$, then

$$\sigma(P_{r,s})(t) = P_{r,s}(-t).$$

• Now we have $S' \cup (-S') \subset S$ and $\#[S' \cup (-S')] = 2(2rs - r - s)$.

But Step 1 saids $\#S \leq 4rs - 2r - 2s$

This is the end of Step 2.

§3.2.9 The end of the proof (Step 3.)

• The leading term of $A_{r,s}(t)$ can be calculated from the asymptotic behavior $P_{r,s}(t) = L_{-r}^{s} t^{(r-1)s} + \cdots$.

§4. Comments and (naive) questions

• Pure SU(2) AGT relation can be shown by recursive formuals.

The recursive formula has representation theoretic meanings, but it also has a strange term $A_{r,s}(t)$ which reflects the connection between singular vectors and Jack symemtric functions.

• Question 1. Does recursive formula has any geometric meaning ?

• Question 2. Can we prove analogous AGT conjectures by the same method ?

e.g. q-analogue (K-theoretic SU(2) Nekrasov $\leftrightarrow$ deformed Virasoro),
higer rank analogue (SU(n) Nekrasov $\leftrightarrow$ W_n algebra),
affine Lie algebra version...

Thank you

$\S\infty$: Announcement of the tomorrow's talk

- **5-dim (or q-deformed) pure SU(2) AGT conjecture (due to Awata-Yamada)**
  - **What is the K-theoretic Nekrasov function ?**
  - **What is the deformed Virasoro algebra ?**
  - **What is the deformed Gaiotto state ?**
  - **Does the recursive formula work ?**
- **Nakajima-Yoshioka blow-up formula**
  - **What is it?**
  - **What does it imply for CFT ?**