AGT conjectures and Zamolodchikov-type recursive formula

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Abstract

What is AGT conjecture ?

L. F. Alday, D. Gaiotto, Y. Tachikawa,
"Liouville Correlation Functions from Four-dimensional Gauge Theories",
Lett. Math. Phys. 91 (2010), arXiv:0906.3219.

Physical meaning :

4 dimensional $\mathcal{N} = 2$ U(2) Super Yang-Mills theory
||?
2 dimensional Liouville conformal field theory

In today's talk, I only mention the (mathematically) simple part :

rank 2 pure gauge Nekrasov partition function
(algebraic geometry / combinatorics)
||?
norm of Whittaker vector of Virasoro algebra
(representation theory)

Contents

§1. Nekrasov partition function

Nakajima-Yoshioka, Invent. Math. 162 (2005),
Nekrasov, Adv. Theor. Math. Phys. 7 (2003)

§1.1. Geometric definition

- $M(r,n)$: moduli of framed torsion free sheaves over $\mathbb{P}^2$ (r : rank, n : c_2)
- ADHM description :

$$M(r,n) = \left\{ (B_1, B_2, j, k) \;\middle|\; \begin{array}{l} B_1, B_2 \in \mathrm{End}(\mathbb{C}^n), \\ j \in \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^n), \ \text{s.t. } (1)\&(2) \\ k \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^r) \end{array} \right\} \Big/ GL_n(\mathbb{C})$$

(1)$[B_1, B_2] + jk = 0$

(2)there is no subspace $S \subset \mathbb{C}^n$ s.t. $B_i(S) \subset S$ and $\mathrm{Im} j \subset S$

- Action of $G := \mathbb{T}^2 \times \mathbb{T}^{r-1}$ ($\mathbb{T} := \mathbb{C}^\times$):

$$(B_1, B_2, j, k) \mapsto (t_1 B_1, t_2 B_2, j s^{-1}, t_1 t_2 s k)$$
$$t_i \in \mathbb{T}, \ s = (s_\alpha)_{\alpha=1}^r \in \mathbb{T}^r, \ \prod_\alpha s_\alpha = 1$$

- Partition function of "4-dim U(r) pure gauge theory"

$$
\begin{aligned}
Z^{\mathcal{N}_f=0,\mathrm{inst}}_{\mathrm{rank}=r}(x;\epsilon_1,\epsilon_2,\vec{a}) &= Z(x;\epsilon_1,\epsilon_2,\vec{a}) \\
&:= \sum_{n=0}^{\infty} x^n \int_{M(r,n)} 1 \qquad (1 \in H^*_G(M(r,n))) \\
&= \sum_{n=0}^{\infty} x^n \lim_{\hbar\to 0} \hbar^{2nr} \sum_{i=0}^{2nr} (-1)^i \mathrm{ch}[H^i(M(r,n),\mathcal{O})]\Big|_{\substack{t_1=e^{-\hbar\epsilon_1}\\ t_2=e^{-\hbar\epsilon_2}\\ s_\alpha=e^{-\hbar a_\alpha}}}
\end{aligned}
$$

- Localization theorem of G-equivariant cohomology gives

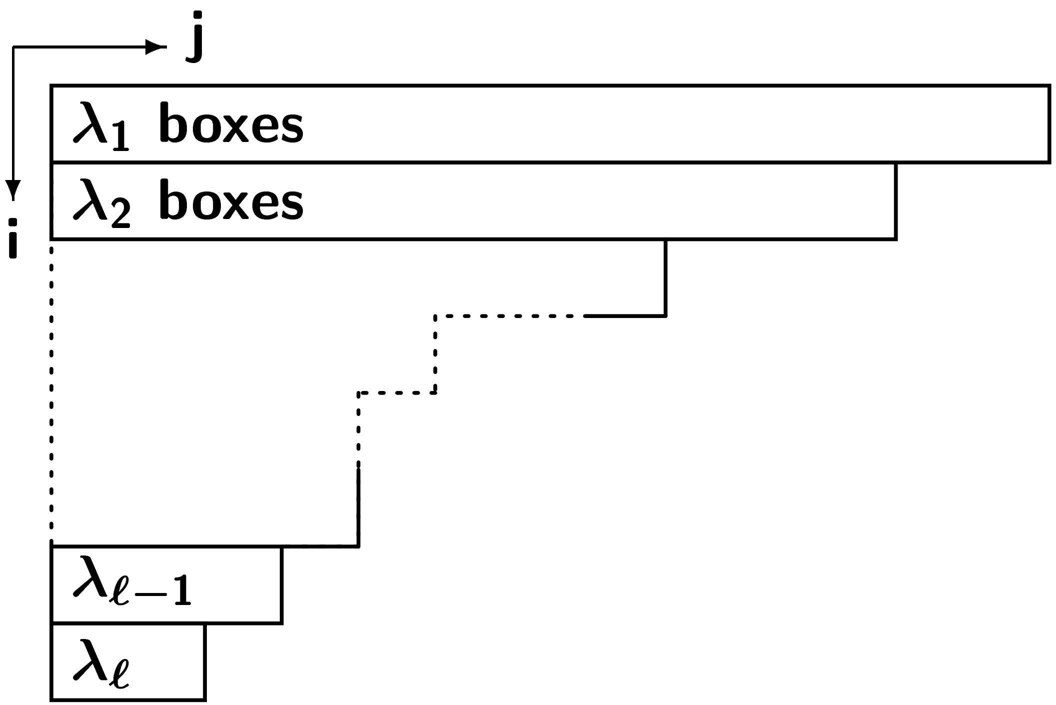
$$
Z(x;\epsilon_1,\epsilon_2,\vec{a}) = \sum_{\vec{Y}} \frac{x^{|\vec{Y}|}}{e(T_{\vec{Y}})},
$$

where $\vec{Y} = (Y_1,\ldots,Y_r)$: r-tuple of partitions, $|\vec{Y}| := |Y_1| + \cdots + |Y_r|$.

§1.2. Partition, Young diagram

- **Partition** $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$: empty sequence $\emptyset = () = (0)$ or
  non-increasing sequence of natural number ($\lambda_1 \geq \cdots \lambda_k \geq 1$)
  $|\lambda| := \lambda_1 + \cdots + \lambda_k$, $\ell(\lambda) := k$: length
  $\lambda \vdash n \overset{\text{def}}{\Longleftrightarrow} \lambda$ is a partition s.t. $|\lambda| = n$.

- **Young diagram** of the partition λ:



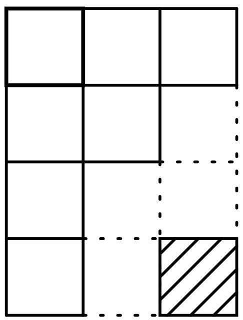
• (relative) arm and leg

λ : partition, $\square = (i, j)$: box located at (i.j)

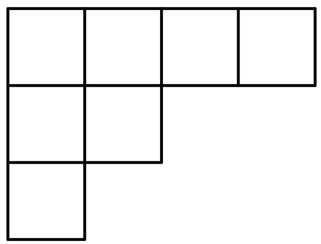
$$a_\lambda(\square) := \lambda_i - j : \text{arm}, \quad \ell_\lambda(\square) := \lambda_j^\vee - i : \text{leg},$$

where $\lambda^\vee$ is the transpose of λ, $\quad \lambda_i := \begin{cases} \lambda_i & (i \leq \ell(\lambda)) \\ 0 & (i > \ell(\lambda)) \end{cases}$

<u>E.g.</u> $\lambda = (3, 2, 1, 1) \quad \lambda^\vee = (4, 2, 1)$



(3, 2, 1, 1)



(4, 2, 1)

$\square = (1, 1) \quad a_\lambda(\square) = \lambda_1 - 1 = 2, \quad \ell_\lambda(\square) = \lambda_1^\vee - 1 = 3$

$\blacksquare = (4, 3) \quad a_\lambda(\blacksquare) = \lambda_4 - 3 = -2, \quad \ell_\lambda(\blacksquare) = \lambda_3^\vee - 4 = -3$

§1.3 Combinatorial definition

- $r \in \mathbb{Z}_{\geq 1}$: rank, $x, \epsilon_1, \epsilon_2, \vec{a} = (a_1, \ldots, a_r)$: indeterminant

$$Z(x; \epsilon_1, \epsilon_2, \vec{a}) = \sum_{\vec{Y}} \frac{x^{|\vec{Y}|}}{\prod_{1 \leq \alpha, \beta \leq r} n^{\vec{Y}}_{\alpha,\beta}(\epsilon_1, \epsilon_2, \vec{a})}$$

$\vec{Y} = (Y_1, \ldots, Y_r)$: r-tuple of partitions, $|\vec{Y}| := |Y_1| + \cdots + |Y_r|$,

$$n^{\vec{Y}}_{\alpha,\beta}(\epsilon_1, \epsilon_2, \vec{a}) := \prod_{\square \in Y_\alpha} [-\ell_{Y_\beta}(\square)\epsilon_1 + (a_{Y_\alpha}(\square) + 1)\epsilon_2 + a_\beta - a_\alpha]$$

$$\times \prod_{\blacksquare \in Y_\beta} [(\ell_{Y_\alpha}(\blacksquare) + 1)\epsilon_1 - a_{Y_\beta}(\blacksquare)\epsilon_2 + a_\beta - a_\alpha].$$

- **Abbreviation :**

$$Z_{\vec{Y}}(\epsilon_1, \epsilon_2, \vec{a}) := [\prod_{1 \le \alpha, \beta \le r} n^{\vec{Y}}_{\alpha,\beta}(\epsilon_1, \epsilon_2, \vec{a})]^{-1},$$

$$Z_n(\epsilon_1, \epsilon_2, \vec{a}) := \sum_{|\vec{Y}|=n} Z_{\vec{Y}}(\epsilon_1, \epsilon_2, \vec{a}).$$

E.g. $r = 1$: $\quad Z_Y = \dfrac{1}{n^Y_{11}(\epsilon_1, \epsilon_2)}$

$n^Y_{11} = \prod_{\square \in Y}[-\ell_Y(\square)\epsilon_1 + (a_Y(\square) + 1)\epsilon_2][(\ell_Y(\square) + 1)\epsilon_1 - a_Y(\square)\epsilon_2]$

$Z_0 = Z_{(\emptyset)} = 1, \quad Z_1 = Z_{(1)} = \dfrac{1}{\epsilon_1\epsilon_2}$

$$Z_2 = Z_{(2)} + Z_{(1,1)} = \frac{1}{2\epsilon_2(\epsilon_1 - \epsilon_2)\epsilon_2\epsilon_1} + \frac{1}{\epsilon_2\epsilon_1(\epsilon_2 - \epsilon_1)2\epsilon_1} = \frac{1}{2\epsilon_1^2\epsilon_2^2}$$

$$Z_3 = Z_{(3)} + Z_{(2,1)} + Z_{(1,1,1)} = \cdots = \frac{1}{6\epsilon_1^3\epsilon_2^3}$$

• If $r = 1$, then

$$Z_{r=1}(x; \epsilon_1, \epsilon_2) = \sum_{n=0} x^n Z_n(\epsilon_1, \epsilon_2) = \exp(\frac{x}{\epsilon_1 \epsilon_2})$$

E.g. 2. $r = 2$

$Z_1 = Z_{(1),\emptyset} + Z_{\emptyset,(1)}$

$= \dfrac{1}{\epsilon_1\epsilon_2(a_1 - a_2)(\epsilon_1 + \epsilon_2 + a_2 - a_1)} + \dfrac{1}{(\epsilon_1 + \epsilon_2 + a_1 - a_2)(a_2 - a_1)\epsilon_2\epsilon_1}$

$Z_2 = Z_{(2),\emptyset} + Z_{(1,1),\emptyset} + Z_{(1),(1)} + Z_{\emptyset,(1,1)} + Z_{\emptyset,(2)}$

There seems no ‘simple’ formula for the case $r = 2$ as opposed to the case $r = 1$...

§1.2. Nakajima-Yoshioka Blow-up formula

- Z satisfies the next bilinear differential equation.
- $\gamma_{\epsilon_1,\epsilon_2}(\xi, x) := \frac{d}{du}\Big|_{u=0} \frac{x^u}{\Gamma(u)} \int_0^\infty \frac{dt}{t} t^u \frac{e^{-t\xi}}{(e^{\epsilon_1 t} - 1)(e^{\epsilon_2 t} - 1)}$

$$\widetilde{Z}(x; \epsilon_1, \epsilon_2, \vec{a}) := \exp[-\sum_{1 \le \alpha \ne \beta \le r} \gamma_{\epsilon_1,\epsilon_2}(a_\alpha - a_\beta; x)] Z(x; \epsilon_1, \epsilon_2, \vec{a})$$

$$(D_x^{(\epsilon_1,\epsilon_2)})^m (f, g) := \left(\frac{d}{dy}\right)^m f(x + \epsilon_1 y) g(x + \epsilon_2 y)\Big|_{y=0}$$

-

$$\sum_{\vec{k}} [D_{\log x}^{(\epsilon_1,\epsilon_2)} - \frac{1}{12}(\epsilon_1 + \epsilon_2)(r - 1)]^d$$

$$(\widetilde{Z}(x; \epsilon_1, \epsilon_2 - \epsilon_1, \vec{a} + \epsilon_1 \vec{k}), \widetilde{Z}(x; \epsilon_1 - \epsilon_2, \epsilon_2, \vec{a} + \epsilon_2 \vec{k}))$$

$$= \begin{cases} 0 & 1 \le d \le 2r - 1 \\ \widetilde{Z}(x; \epsilon_1, \epsilon_2, \vec{a}) & d = 0 \end{cases}$$

The index $\overrightarrow{k}$ runs in the range $\{\overrightarrow{k} = (k_1, \dots, k_r) \in \mathbb{Z}^r \mid \sum_{\alpha=1}^{r} k_\alpha = 0\}$.

- This system of equations is called blow-up formula.
- The proof uses a description of moduli over the blown-up $\widehat{\mathbb{P}}^2$.
- The blow-up formula determines Z uniquely.

(The equations with $d = 1, 2$ give a recursice formula for Z_n.)

- Nakajima and Yoshioka used this blow-up formula

to solve the Nekrasov conjecture.

§2. Virasoro algebra

§2.1 Definition

- **Vir$_c$** : Lie algebra with central extension

$c \in \mathbb{C}$: central charge,

generators : L_n $(n \in \mathbb{Z})$

relations : $[L_n, L_m] = (n-m)L_{n+m} + \frac{1}{12}cn(n^2-1)\delta_{n+m,0}$

- Triangular decomposition : $Vir_c = Vir_{c,+} \oplus Vir_{c,0} \oplus Vir_{c,-}$

$$Vir_{c,\pm} := \oplus_{\pm n \in \mathbb{Z}_{>0}} \mathbb{C}L_n, \quad Vir_{c,0} := \mathbb{C}L_0 \oplus \mathbb{C}$$

- PBW basis of $\mathcal{U}(Vir_c)$: $\{L_{-\lambda}L_0^n L_\mu \mid n \in \mathbb{Z}_{\geq 0},\ \lambda, \mu : \text{partition}\}$

Here we denoted for a parition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$

$$L_\lambda := L_{\lambda_\ell} \cdots L_{\lambda_1}, \ L_{-\lambda} := L_{-\lambda_1} \cdots L_{-\lambda_\ell}$$

§2.2 Verma module M_h

- $h \in \mathbb{C}$: highest weight

  $\mathbb{C}_h := \mathbb{C}|h\rangle$: one-dimensional $(Vir_{c,+} \oplus Vir_{c,0})$ representation

  $L_n |h\rangle = 0 \ (n > 0), \quad L_0 |h\rangle = h |h\rangle$
- $M_h := \mathrm{Ind}^{Vir_c}_{Vir_{c,+} \oplus Vir_{c,0}} \mathbb{C}_h$
- weight decomposition :

  $M_h = \oplus_{n \in \mathbb{Z}_{\geq 0}} M_{h,n}, \quad M_{h,n} := \{v \in M_h \mid L_0 v = (h+n)v\}$
- basis of $M_{h,n}$: $\{L_{-\lambda} |h\rangle \, ; \lambda \vdash n\}$

Dual Verma module $\quad M_h^* := \mathrm{Ind}^{Vir_c}_{Vir_{c,-} \oplus Vir_{c,0}} \mathbb{C}_h^*$

$\mathbb{C}_h^* := \mathbb{C}\langle h|$: 1-dim $(Vir_{c,-} \oplus Vir_{c,0})$ rep.

$\langle h| L_n = 0 \ (n < 0), \quad \langle h| L_0 = h \langle h|$

weight decomp. : $M_h^* = \oplus_{n \in \mathbb{Z}_{\geq 0}} M_{h,n}^*, \quad M_{h,n}^* := \{v \in M_h^* \mid v L_0 = (h-n)v\}$

basis of $M_{h,n}^*$: $\{\langle h| L_\lambda ; \lambda \vdash n\}$

§2.3 Shapovalov form

- $\cdot : M_h^* \times M_h \to \mathbb{C}$: bilinear form

$$\langle h| \cdot |h\rangle := 1,\ \langle h| L_\lambda \cdot L_{-\mu} |h\rangle := \sum_{\nu_1,\nu_2,n} \delta_{\nu_1,\emptyset}\delta_{\nu_2,\emptyset} c_{\nu_1,\nu_2,n} h^n$$

$$\left(L_\lambda L_{-\mu} \overset{\mathrm{PBW}}{=} \sum_{\substack{\nu_1,\nu_2:\mathrm{partition} \\ n\in\mathbb{Z}_{\geq 0}}} c_{\nu_1,\nu_2,n} L_{-\nu_1} L_0^n L_{\nu_2}\right)$$

- Then $uL_n \cdot v = u \cdot L_n v \quad (u \in M_h^*,\ v \in M_h)$.

Below we simply write $\langle h| L_\lambda L_{-\mu} |h\rangle := \langle h| L_\lambda \cdot L_{-\mu} |h\rangle$, $uv := u \cdot v$.

- $\langle h| L_\lambda L_{-\mu} |h\rangle = 0$ unless $|\lambda| = |\mu|$
- $\langle h| L_\lambda L_{-\mu} |h\rangle = \langle h| L_{-\mu} L_\lambda |h\rangle$.

§2.4. Kac determinant

- Define a matrix $K_n := (\langle h| L_\lambda L_{-\mu} |h\rangle)_{\lambda,\mu \vdash n}$. Then

$$\det K_n = \prod_{\substack{r,s\in\mathbb{Z}_{\geq 1} \\ 1\leq rs\leq n}} (h - h_{r,s})^{p(n-rs)},$$

where $p(m) := \#\{\lambda \mid \lambda \vdash m\}$,

$$h_{r,s} := \frac{1}{48}[(13-c)(r^2+s^2) - 24rs - 2(1-c) \\ + \sqrt{(1-c)(25-c)}(r^2-s^2)]$$

Proof needs free field realization (Feigin-Fuchs, 1980's.)

- Single pole phenomena [K. Brown, J. Algebra (2003)]

Each element of K_n^{-1} has at most simple poles with respect to h.

Remark Similar phenomena holds for finite dimensional Lie algebra. (Ostapenko, J. Algebra 147 (1992))

§3. Gaiotto conjecture

• The original AGT conjecture claims that

the conformal block (four-point correlation function of CFT) coincides with the Nekrasov parition function (with $\mathcal{N}_f = 4$ gauge matters).

• Here we only mention a degenerate version.

Ref. : Gaiotto arXiv:0908.0307

§3.1 Gaiotto state

• $\Lambda \in \mathbb{C}$ fix.

Gaiotto state $|G\rangle$ is an element of M_h satisfying :

$$L_1 |G\rangle = \Lambda^2 |G\rangle, \; L_n |G\rangle = 0 \; (n \geq 2), \quad |G\rangle = |h\rangle + \cdots$$

• Dual Gaiotto state $\langle G| \in M_h^*$ is similarly defined:

$$\langle G| L_{-1} = \Lambda^2 \langle G|, \; \langle G| L_{-n} = 0 \; (n \geq 2), \quad \langle G| = \langle h| + \cdots$$

- $|\mathsf{G}\rangle$ is a (degenerate) Whittaker vector of Virasoro algebra.

- Definition of Whittaker vector (Kostant, Invent. Math. 48 (1978))

$\mathfrak{g}$: fin. dim. Lie alg., $\mathfrak{n}$: maximal nilpotent Lie subalg. of $\mathfrak{g}$

$\eta : \mathfrak{n} \to \mathbb{C}$: homomorphism of Lie alg. (central character)

V : $\mathcal{U}(\mathfrak{g})$-module

w $\in$ V Whittaker vector w.r.t. η

$$\overset{\text{def}}{\Longleftrightarrow} \text{ for any } x \in \mathfrak{n} \text{ one has } xw = \eta(x)w$$

- In the case of Virasoro alg., replace $\mathfrak{g}$ with Vir_c, and $\mathfrak{n}$ with $\mathrm{Vir}_{c,+}$.

η is determined by the images of L_1 and L_2.

Gaiotto state is a Whittaker vector w.r.t. η with $\eta(L_1) = \Lambda^2$, $\eta(L_2) = 0$.

§3.2 Gaiotto cojecture for pure Gauge partition function

Conjecture

$$\langle G \mid G \rangle \overset{?}{=} Z_{r=2}(x; \epsilon_1, \epsilon_2, \vec{a}).$$

Here the parameters are related as :

Virasoro	Nekrasov
c	$13 + 6(\epsilon_1/\epsilon_2 + \epsilon_2/\epsilon_1)$
h	$(\epsilon_1/\epsilon_2 + \epsilon_2/\epsilon_1 + 2)/4 - (a_2 - a_1)^2/\epsilon_1\epsilon_2$
Λ	$x^{1/4}/(\epsilon_1\epsilon_2)$

Restatement (Marshakov-Mironov-Morozov, Phys. Lett. B 682 (2009))

One has $\langle G \mid G \rangle = \sum_{n=0}^{\infty} \Lambda^{4n} (K_n^{-1})_{(1^n,1^n)}$, so that

$$(K_n^{-1})_{(1^n,1^n)} \overset{?}{=} (\epsilon_1\epsilon_2)^{4n} Z_n(\epsilon_1, \epsilon_2; \vec{a})$$

§4. Whittaker vector via Jack symmetric function

• Singular vecors of M_h can be expressed by Jack symmetric functions. (Mimachi-Yamada, Comm. Math. Phys. 174 (1995))

Problem : Can $|G\rangle$ be also expressed by Jack symmetric functions ?

(Observed by Awata-Yamada. They actually conjectued that the deformed Gaiotto state is expressed via Macdonald symmetric functions.)

§4.1 Heisenberg algbra

- Heisenberg algebra $\mathcal{H}$

generators : a_n $(n \in \mathbb{Z})$ relations $[a_n, a_m] = n\delta_{n+m,0}a_0$.

Triang. decomp. : $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_0 \oplus \mathcal{H}_-$ $\mathcal{H}_\pm := \oplus_{\pm n \in \mathbb{Z}_+} \mathbb{C}a_n$, $\mathcal{H}_0 := \mathbb{C}a_0$

- $\mathcal{F}_\alpha$: Fock representation $\mathcal{F}_\alpha := \mathrm{Ind}^{\mathcal{H}}_{\mathcal{H}_0 \oplus \mathcal{H}_+} \mathbb{C}_\alpha$

Here $\mathbb{C}_\alpha = \mathbb{C}|\alpha\rangle_{\mathcal{F}}$ is the 1-dim. $(\mathcal{H}_0 \oplus \mathcal{H}_+)$ rep. given by

$$a_0 \,|\alpha\rangle_{\mathcal{F}} = \alpha \,|\alpha\rangle_{\mathcal{F}},\ a_n \,|\alpha\rangle_{\mathcal{F}} = 0 \ (n \in \mathbb{Z}_{>0})$$

basis of $\mathcal{F}_\alpha$: $\{a_{-\lambda} \,|\alpha\rangle_{\mathcal{F}} \mid \lambda : \text{partition}\}$

weight decomp. : $\mathcal{F}_\alpha = \oplus_{n \geq 0} \mathcal{F}_{\alpha,n}$, $\mathcal{F}_{\alpha,n} := \{v \in \mathcal{F}_\alpha \mid a_0 v = (n+\alpha)v\}$

§4.2 Feigin-Fuchs bosonization

- The next map is called Feigin-Fuchs bosonization of Virasoro algebra.

$$\begin{aligned} \iota_{\alpha_0} : \ \mathcal{U}(\mathrm{Vir}_{c(\alpha_0)}) &\to \widehat{\mathcal{U}}(\mathcal{H}) \\ L_n &\mapsto \mathcal{L}_n := \frac{1}{2}\sum_{m\in\mathbb{Z}} {}^{\circ}_{\circ} a_m a_{n-m} {}^{\circ}_{\circ} - (n+1)\alpha_0 a_n, \end{aligned}$$

where $c(\alpha_0) := 1 - 12\alpha_0^2$, ${}^{\circ}_{\circ}\ {}^{\circ}_{\circ}$ is the normal ordering.

- Moreorever the linear isomorphism

$$\begin{aligned} f_{\alpha_0,\alpha} : \quad M_{h(\alpha,\alpha_0)} &\to \mathcal{F}_\alpha \\ L_{-\lambda}\,|h(\alpha,\alpha_0)\rangle &\mapsto \mathcal{L}_{-\lambda}\,|\alpha\rangle_{\mathcal{F}} \end{aligned}$$

is compatible with ι_{α_0}. Here $h(\alpha,\alpha_0) := \frac{1}{2}[(\alpha-\alpha_0)^2 - \alpha_0^2]$.

§4.3 Symmetric functions

- $\Lambda^{(N)}$: ring of N-variable $\mathbb{Z}$-coefficient symmetric polynomials
- $\Lambda^{(N)}_n$: space of n-degree symm. polynom., $\Lambda^{(N)} = \bigoplus_{n\geq 0} \Lambda^{(N)}_n$
- $m^{(N)}_\lambda(x)$: monomial symmetric polynom.

  If $n \geq N$, then $\{m_\lambda \mid \lambda \vdash n\}$ is a $\mathbb{Z}$-basis of $\Lambda^{(N)}_n$.
- $p^{(N)}_k := \sum_{i=1}^N x_i^k \in \Lambda^{(N)}_k$: power sum symm. polynom.
- $\Lambda^{(N)}_{n,\mathbb{Q}} := \Lambda^{(N)}_n \otimes_{\mathbb{Z}} \mathbb{Q}$, $\Lambda^{(N)}_{n,\mathbb{C}} := \Lambda^{(N)}_n \otimes_{\mathbb{Z}} \mathbb{C}$.
- If $n \leq N$, then $\{p^{(N)}_\lambda \mid \lambda \vdash n\}$ is a $\mathbb{Q}$-basis of $\Lambda^{(N)}_{n,\mathbb{Q}}$.

§4.4 Jack symmetric polynomial

- Jack symmetric polynomial $P_\lambda^{(N)}(x;\beta) \in \Lambda_{\mathbb{C}}^{(N)}$ is determined by

$$\text{(i)}\quad P_\lambda^{(N)}(x;\beta) = \sum_{\mu\le\lambda} c_{\lambda,\mu}(\beta) m_\mu^{(N)}(x), \quad c_{\lambda,\mu}(\beta)\in\mathbb{C},\ c_{\lambda,\lambda}(\beta)=1$$

$$\text{(ii)}\quad H_\beta^{(N)} P_\lambda^{(N)}(x;\beta) = \epsilon_\lambda^{(N)}(\beta) P_\lambda^{(N)}(x;\beta),$$

$$H_\beta^{(N)} := \sum_{i=1}^{N}\Big(x_i\frac{\partial}{\partial x_i}\Big)^2 + \beta \sum_{1\le i<j\le N} \frac{x_i+x_j}{x_i-x_j}\Big(x_i\frac{\partial}{\partial x_i} - x_j\frac{\partial}{\partial x_j}\Big),$$

$$\epsilon_\lambda^{(N)}(\beta) := \sum_i (\lambda_i^2 + \beta(N+1-2i)\lambda_i).$$

Here the ordering in (i) is the dominance semi-ordering.

$$\lambda \ge \mu \overset{\text{def}}{\Longleftrightarrow} |\lambda| = |\mu|,\ \textstyle\sum_{k=1}^{i}\lambda_k \ge \sum_{k=1}^{i}\mu_k\ (i=1,2,\ldots).$$

In (ii), the operator $H_\beta^{(N)}$ is equiv. to the Calogero-Sutherland hamiltonian.

- If $N\ge n$, then $\{P_\lambda^{(N)}(x;\beta)\}_{\lambda\vdash n}$ is a basis of $\Lambda_{n,\mathbb{C}}^{(N)}$.

§4.5 Fock module and symmetric polynomial

• If $n \leq N$ and $\beta' \neq 0$, then the next map determines a linear isomorphism:

$$\begin{array}{rcl} g^{(N)}_{\beta',n} : \bigoplus_{k=1}^{n} \mathcal{F}_{\alpha,k} & \to & \bigoplus_{k=1}^{n} \Lambda^{(N)}_{k,\mathbb{C}} \\ u\,|\alpha\rangle_{\mathcal{F}} & \mapsto & {}_{\mathcal{F}}\langle\alpha|\exp(\beta' \sum_{n>0} \frac{1}{n} a_n p^{(N)}_n) u\,|\alpha\rangle_{\mathcal{F}} \end{array}$$

• Moreover the next $\mathbb{C}$-alg. hom.

$$\begin{array}{rcl} j_{\beta,n} : \bigoplus_{k=1}^{n} \widehat{\mathcal{U}}(\mathcal{H})_k & \to & \mathrm{End}(\Lambda_{\mathbb{C}}) \\ a_{-n} & \mapsto & \beta' p^{(N)}_n \\ a_n & \mapsto & \frac{n}{\beta'} \frac{\partial}{\partial p^{(N)}_n} \quad (n > 0) \end{array}$$

(Here $\widehat{\mathcal{U}}(\mathcal{H})_k$ is the k-deg. homog. part of $\widehat{\mathcal{U}}(\mathcal{H})$ with $\deg a_n := -n$.)

is compatible with $g^{(N)}_{\beta',n}$.

§4.6. Split form of Calogero-Sutherland hamiltonian

Awata-Matsuo-Odake-Shiraishi, Nucl. Phys. B 449 (1995)

- Introduce $|\lambda;\beta,\beta'\rangle_{\mathcal{F}} \in \mathcal{F}_\alpha$ by $g^{(N)}_{\beta',n}(|\lambda;\beta,\beta'\rangle_{\mathcal{F}}) = P^{(N)}_\lambda(x;\beta)$.
- Define an element $\widehat{E}_{\beta,\beta'}$ of $\widehat{\mathcal{U}}(\mathcal{H})$ by

$$\widehat{E}_{\beta,\beta'} := \sum_{n,m>0}\left(\beta' a_{-m-n}a_m a_n + \frac{\beta}{\beta'}a_{-m}a_{-n}a_{m+n}\right) + \sum_{n>0} n(1-\beta)a_{-n}a_n.$$

Then one has

$$\widehat{E}_{\beta,\beta'}\left|\lambda;\beta,\beta'\right\rangle_{\mathcal{F}} = \epsilon_\lambda(\beta)\left|\lambda;\beta,\beta'\right\rangle_{\mathcal{F}},\ \ \epsilon_\lambda(\beta) := \sum_i(\lambda_i^2 + \beta(1-2i)\lambda_i).$$

Set $\widehat{E}_\beta := \widehat{E}_{\beta,\sqrt{\beta/2}}$, and $\alpha_0 = (\beta-1)/\sqrt{2\beta}$, then

$$\widehat{E}_\beta = \sqrt{2\beta}\sum_{n>0} a_{-n}\mathcal{L}_n + \sum_{n>0} a_{-n}a_n\left(\beta - 1 - \sqrt{2\beta}a_0\right)$$

§4.7. bosonization of the Gaiotto state

- Decompose the Gaiotto state as $|G\rangle = \sum_{n\in\mathbb{Z}_{\geq 0}} \Lambda^{2n}|G,n\rangle$ ($|G,n\rangle \in M_{h,n}$).
- $|\mathcal{G},n\rangle_{\mathcal{F}} = f_{\alpha_0,\alpha}(|G,n\rangle) \in \mathcal{F}_{\alpha,n}$, $|\lambda;\beta\rangle_{\mathcal{F}} := |\lambda;\beta,\sqrt{\beta/2}\,\rangle_{\mathcal{F}}$ とする.

Proposition [Y. arXiv:1003.1049]

The Gaiotto state exists uniquely.

Expand this state as $|\mathcal{G},n\rangle_{\mathcal{F}} = \sum_{\lambda\vdash n} c_\lambda(\alpha,\beta)\,|\lambda;\beta\rangle_{\mathcal{F}}$, then

$$c_\lambda(\alpha,\beta) = \prod_{\square\in\lambda} \frac{1}{a_\lambda(\square)+1+\beta\ell_\lambda(\square)} \times \prod_{(i,j)\in\lambda\setminus\{(1,1)\}} \frac{\beta}{(j+1)+\sqrt{2\beta}\alpha-(i+1)\beta}$$

- The proof use the split form $\widehat{E}_\beta$ and the Pieri formula for Jack symmm. poly. (only need the version of adding one box) to derive a recursion formula for c_λ, and check that the function above satisfies the formula.

§5. Zamolodchikov-type recursion formula

§5.1. Strategy for the proof of Gaiotto conjecture

- In today's talk, I will only mention a strategy using some representation theoretic observation.

To show the coincidence of Z_n and $(K_n)^{-1}_{(1^n,1^n)}$, it is enough to prove that both equations satisfy the same recursive formula.

Ref. :

Poghossian, JHEP 0912 (2009), arXiv:0909.3412

Fateev-Litvinov, JHEP 1002 (2010), arXiv:0912.0504

Hasadz-Jaskólski-Suchanek, arXiv:1004.1841

§5.2 Zamolodchikov-type recursion formula

Fact. [Fateev-Litvinov, Hasadz-Jaskólski-Suchanek]

Set $a := a_1 - a_2$. Then $Z_n(\epsilon_1, \epsilon_2, a)$ satisfies the next recursion formula:

$$Z_n(\epsilon_1, \epsilon_2, a) = \delta_{n,0} + \sum_{\substack{r,s \in \mathbb{Z}_{\geq 1} \\ 1 \leq rs \leq n}} \frac{R_{r,s}(\epsilon_1, \epsilon_2) Z_{n-rs}(\epsilon_1, \epsilon_2, (r\epsilon_1 - s\epsilon_2)/2)}{4a^2 - (r\epsilon_1 + s\epsilon_2)^2},$$

where

$$R_{r,s}(\epsilon_1, \epsilon_2) := 2 \prod_{\substack{1-r \leq j \leq r \\ 1-s \leq k \leq s \\ (r,s) \neq (0,0)}} (j\epsilon_1 + k\epsilon_2)^{-1}.$$

Remark

(1) The proof uses an integral expression of Z_n.

(2) Fateev-Litvinov showed the version with adjoint matter, and Hasadz-Jaskólski-Suchanek showed the versions with $\mathcal{N}_f \leq 2$.

- Thus Gaiotto conjecture is reduced to the proof of the recursive formula

$$f_n(h,c) = \delta_{n,0} + \sum_{\substack{r,s\in\mathbb{Z}_{\geq 1} \\ 1\leq rs\leq n}} \frac{\widetilde{R}_{r,s}(c) f_{n-rs}(h_{r,-s},c)}{h-h_{r,s}}$$

for $(K_n)^{-1}_{(1^n,1^n)}$.

<u>Remark</u>

This formula is related to the "Zamolodchikov recursive formula" for the 4-point conformal block :

$$\mathcal{F}(x;c,h;h_1,h_2,h_3,h_4) = A(x,h_i)H(x;c,h;h_i),$$
$$H(x;c,h;h_i) = 1 + \sum_{r,s} \frac{R_{r,s}(x;r,s,c;h_i)}{h-h_{r,s}} H(x;c,h_{r,-s};h_i)$$

c.f. Al. Zamolodchikov, CMP 96 (1984); Theor. Math. Phys. 73 (1987)

§6. K theoretic version

Both Nekrasov partition function and Virasoro algebra have q-analogue. Thus it is natural to propose a q-analogue of AGT conjecture...

Awata-Yamada, JHEP 1001 (2010), arXiv:0910.4431

§6.1 K theoreic Nekrasov partition function

Nakajima-Yoshioka, Transform. Groups 10 (2005)

$$Z^K(x;\epsilon_1,\epsilon_2,\vec{a}) := \sum_{n=0}^{\infty}(xe^{-r(\epsilon_1+\epsilon_2)/2})^n \sum_i (-1)^i \mathrm{ch}[H^i(M(r,n),\mathcal{O})]$$

$$= \sum_{\vec{Y}} \frac{x^{|\vec{Y}|}}{\prod_{1\le\alpha,\beta\le r} N^{\vec{Y}}_{\alpha,\beta}},$$

where $N^{\vec{Y}}_{\alpha,\beta} := \prod_{\square\in Y_\alpha}(1-\exp[\ell_{Y_\beta}(\square)\epsilon_1-(a_{Y_\alpha}(\square)+1)\epsilon_2-a_\beta+a_\alpha)]$

$$\prod_{\blacksquare\in Y_\beta}(1-\exp[-(\ell_{Y_\beta}(\blacksquare+1)\epsilon_1+a_{Y_\alpha}(\blacksquare)\epsilon_2-a_\beta+a_\alpha])$$

§§6.2 Deformed Virasoro algebra $\mathrm{Vir}_{q,t}$

- $q, t \in \mathbb{C}$, $p := qt^{-1}$

generator : $T_n (n \in \mathbb{Z})$,

$$\text{relation : } [T_n, T_m] = -\sum_{\ell=1}^{\infty} f_\ell (T_{n-\ell} T_{m+\ell} - T_{m-\ell} T_{n+\ell}) - \frac{(1-q)(1-t^{-1})}{1-p}(p^n - p^{-n})\delta_{n+m,0},$$

$$\text{where} \quad \sum_{k=0}^{\infty} f_k z^k = \exp[\sum_{n=1}^{\infty} \frac{(1-q^n)(1-t^{-n})}{1+p^n} \frac{z^n}{n}].$$

- In the limit $\hbar \to 0$ with $t = q^\beta$ and $q = e^\hbar$, the $\hbar$-expansion of $T(z) := \sum_{n \in \mathbb{Z}} T_n z^{-n}$ as

$$T(z) = 2 + \beta \hbar^2 (z^2 L(z) + \frac{(1-\beta)^2}{4\beta}) + O(\hbar^4) \quad (L(z) = \sum L_n z^{-n-2})$$

gives the relations of Vir_c among $\{L_n\}$ with $c = 1 - 6(1-\beta)^2/\beta$.

§6.3 deformed Gaiotto state

- $h \in \mathbb{C}$, M_h : Verma module of $Vir_{q,t}$

  : generated by $|h\rangle$ s.t. $T_n |h\rangle = 0$ $(n > 0)$, $T_0 |h\rangle = h |h\rangle$.

  M_h is graded as $M_h = \oplus_{n \in \mathbb{Z}_{\geq 0}} M_{h,n}$ by $\deg T_n := -n$.

- Dual Verma modules M_h^* and its generator $\langle h|$ is similarly defined.

- Deformed Gaiotto state $|G_{q,t}\rangle \in M_h$ is an element satisfying

  $$T_1 |G_{q,t}\rangle = \Lambda^2 |G_{q,t}\rangle, \ T_n |G_{q,t}\rangle = 0 \ (n \geq 2), \ |G_{q,t}\rangle = |h\rangle + \cdots$$

- Dual state $\langle G_{q,t}|$ is similarly defined.

§6.4. K theoretic AGT conjecture

Conjecture (Awata-Yamada)

$$\langle G_{q,t} \mid G_{q,t} \rangle \stackrel{?}{=} Z^{K}_{r=2}(x; \epsilon_1, \epsilon_2, \vec{a}),$$

where the parameters are related as

$Vir_{q,t}$	K theoretic Nekrasov
q	$e^{-\epsilon_1}$
t	e^{ϵ_2}
h	$e^{(a_1-a_2)/2} + e^{(a_2-a_1)/2}$
Λ	$x^{1/4}$

Remark In the higher rank case ($r \geq 3$), the deformed W algebra $\mathcal{W}_{q,t}(sl_r)$ should come into the game, but no explicit conjecture has been obtained yet...

§6.5 Recursion formula

Proposition [Y. arXiv:1005.0216]

Expanding the K-theoreic Nekrasov parition function as

$$Z^K(\Lambda; q, t, Q) = \sum_n (\Lambda q/t)^n Z^K_n(q, t, Q), \quad (Q := e^{a_1 - a_2})$$

then one has the next recursive formula.

$$Z^K_n(q, t, Q) = \delta_{n,0} + \sum_{\substack{r,s \in \mathbb{Z} \\ 1 \leq rs \leq n}} \frac{R^K_{r,s}(q, t) Z^K_{n-rs}(q, t, q^r t^s)}{Q - q^r t^{-s}}.$$

Here

$$R^K_{r,s}(q, t) := -(\mathrm{sgn}(r))\, q^r t^{-s} \prod_{\substack{1-|r| \leq j \leq |r| \\ 1-|s| \leq k \leq |s| \\ (j,k) \neq (0,0)}} (1 - q^j t^{-k})^{-1}.$$

- The proof uses the next integral expression.

Setting $q_1 := q, q_2 := t^{-1}, q_3 := p^{-1} = q^{-1}t$, one has

$$Z_n^K(q,t,Q) = \frac{1}{n!}\left(\frac{1-q_3^{-1}}{(1-q_1)(1-q_2)}\right)^n \times \oint\cdots\oint \prod_{i=1}^{n}\frac{dx_i}{2\pi\sqrt{-1}}\prod_{k=1}^{n} P(x_k;Q^{1/2},q_3)\prod_{i<j}\omega(x_j/x_i;q_1,q_2,q_3)$$

with

$$P(x;a,q) := \frac{x}{(x-a)(x-a^{-1})(x-qa)(x-qa^{-1})},$$

$$\omega(y;q_1,q_2,q_3) := \frac{(y-1)^2(y-q_3)(y-q_3^{-1})}{(y-q_1)(y-q_1^{-1})(y-q_2)(y-q_2^{-1})}.$$

§6.6 Deformed Gaiotto state and Macdonald symmetric function

- **Awata-Yamada conjecturally expressed $|G_{q,t}\rangle$ via Macdonald function.**
- **First we need the bosonization of the defomed Virasoro algebra.**

$$
\begin{aligned}
T(z) &= \Lambda_1(z) + \Lambda_2(z),\\
\Lambda_1(z) &= p^{1/2}\exp[-\sum_{n=1}^{\infty}\frac{1-t^n}{1+p^n}\frac{b_{-n}}{n}t^{-n}p^{-n/2}z^n]\\
&\qquad\times\exp[-\sum_{n=1}^{\infty}(1-t^n)\frac{b_n}{n}p^{n/2}z^{-n}]q^{\beta b_0},\\
\Lambda_2(z) &= p^{-1/2}\exp[\sum_{n=1}^{\infty}\frac{1-t^n}{1+p^n}\frac{b_{-n}}{n}t^{-n}p^{n/2}z^n]\\
&\qquad\times\exp[\sum_{n=1}^{\infty}(1-t^n)\frac{b_n}{n}p^{-n/2}z^{-n}]q^{-\beta b_0}.
\end{aligned}
$$

Here b_n is the boson satisfying $[b_n, b_m] = n\dfrac{1-q^{|n|}}{1-t^{|n|}}\delta_{n+m,0}b_0$.

- $\mathcal{F}_0$: Fock module generated by $|0\rangle$ with $b_n |0\rangle = 0$ $(n > 0)$.

  Then T_n acts as $T_0 |0\rangle = h |0\rangle$, $T_n |0\rangle = 0$ $(n > 0)$.

- $b_{-\lambda} |0\rangle \mapsto p_\lambda$ gives an isomorphism of $\mathcal{F}_0$ and $\Lambda_{\mathbb{C}}$. Under this identification

<u>Conjecture</u> (Awata-Yamada)

$$|G_{q,t}\rangle \stackrel{?}{=} \sum_{\lambda} \Lambda^{2|\lambda|} P_\lambda(x; q, t) \prod_{(i,j)\in\lambda} \frac{Q^{1/2}}{1 - Qq^j t^{-i}} \frac{q^{\lambda_i - j}}{1 - q^{\lambda_i - j + 1} t^{\lambda_j^\vee - i}},$$

where $P_\lambda(x; q, t)$ is Macdonald symmetric function.

<u>Remark</u>

The strategy used in the Virasoro case doesn't work. There exists a split form of the Macdonald difference operator E_1, but the difficulty lies in the treatment of "the action of T_0".