Macdonald symmetric functions, Feigin-Odesskii algebra and Ding-Iohara algebra

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§0 Abstract and Contents

Macdonald symmetric functions are joint eigen-functions of a commutative family of difference operators.

Using bosonization of these operators, we obtain a commutative algebra, which is an analogue of the algebra introduced by B. Feigin and V. Odesskii.

We also discuss the relation between Macdonald functions, Ding-Iohara quantum algebra and deformed W-algebras of type A.

[FHHSY] Feigin, Hashizume, Hoshino, Shiraishi, Y.,
"A commutative algebra on degenerate $\mathbb{CP}^1$ and Macdonald polynomials",
J. Math. Phys., 50, 095215 (2009); arXiv:0904.2291.

[FHSSY] B. Feigin, Hoshino, Shibahara, Shiraishi, Y.,
"Kernel function and quantum algebras",
RIMS Kokyuroku No. 1689 153-163 (2010); arXiv:1002.2485.

§1. Macdonald symmetric function

Reference: I.G.Macdonald, "Symmetric function and Hall polynomials".

§1.1. Notation on symmetric polynomials

- n : positive integer, q, t : indeterminants
- $\mathbb{F} := \mathbb{Q}(q,t)$
- $x = (x_1, x_2, \ldots, x_n)$: set of variables
- $\Lambda_n := \mathbb{Z}[x_1, \ldots, x_n]^{\mathfrak{S}_n}$: space of n-variable symmetric polynomial

  $\Lambda_{n,\mathbb{F}} := \Lambda_n \otimes \mathbb{F}$
- $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) = (\lambda_1, \ldots, \lambda_\ell, 0, 0, \ldots)$: partition

  (increasing sequence of positive integers, $|\lambda| := \sum_i \lambda_i$)
- $\geq$: dominance semi-ordering of partitions

  $$\lambda \geq \mu \overset{\text{def}}{\iff} |\lambda| = |\mu|, \sum_{k=1}^{i} \lambda_k \geq \sum_{k=1}^{i} \mu_k \ (i = 1, 2, \ldots).$$

§1.2. Macdonald difference operators

- T_{q,x_i} : q-shift operator

$$T_{q,x_i} f(x_1, x_2, \ldots, x_n) = f(x_1, x_2, \ldots, qx_i, \ldots, x_n).$$

- $D_r^{(n)}$: Macdonald difference operator on $\Lambda_{n,\mathbb{F}}$ $(1 \le r \le n)$

$$D_r^{(n)} := \sum_{\substack{J \subset \{1,2,\ldots,n\} \\ \#J = r}} \left[t^{r(r-1)/2} \prod_{\substack{j \in J \\ k \notin J}} \frac{tx_j - x_k}{x_j - x_k} \prod_{j \in J} T_{q,x_j} \right].$$

- $m_\lambda^{(n)}(x) := \sum_{\alpha:\text{permutation of } \lambda} x^\alpha$: n-var. monomial symm. polynomial.

$$e_r^{(n)}(x) := \sum_{1 \le i_1 < i_2 < \cdots < i_r \le n} x_{i_1} x_{i_2} \cdots x_{i_r},$$

$$e_\lambda^{(n)}(x) := e_{\lambda_1}(x) e_{\lambda_2}(x) \cdots e_{\lambda_\ell}(x)$$

: n-var. elementary symmetric polynomial.

§1.3. Macdonald symmetric polynomial

- $\lambda = (\lambda_1, \dots, \lambda_\ell)$: partition s.t. $\ell \leq n$.

n-variable Macdonald symmetric polynomial $P^{(n)}_\lambda(x; q, t) \in \Lambda_{n,\mathbb{F}}$ is determined by the following 2 conditions:

$$P^{(n)}_\lambda = m^{(n)}_\lambda + \sum_{\lambda > \mu} c^{(n)}_{\lambda,\mu} m^{(n)}_\mu \qquad (c^{(n)}_{\lambda,\mu} \in \mathbb{F}),$$

$$D^{(n)}_1 P^{(n)}_\lambda(x; q, t) = P^{(n)}_\lambda(x; q, t) \cdot e^{(n)}_1(t^n s^\lambda),$$

where

$$s^\lambda := (q^{\lambda_1} t^{-1}, q^{\lambda_2} t^{-2}, \dots, q^{\lambda_\ell} t^{-\ell}, t^{-\ell-1}, t^{-\ell-2}, \dots)$$

: spectral parameter

$$e^{(n)}_1(t^n s^\lambda) = e^{(n)}_1(q^{\lambda_1} t^{n-1}, \dots, q^{\lambda_\ell} t^{n-\ell}, t^{n-\ell-1}, \dots, 1)$$
$$= \sum_{i=1}^n q^{\lambda_i} t^{n-i}.$$

§1.4. Commuting difference operators and joint eigen-functions

• $P_\lambda^{(n)}(x;q,t)$ is a joint eigen-function of the family of Macdonald difference operators $\{D_r^{(n)} \mid 1 \le r \le n\}$:

$$D_r^{(n)} P_\lambda^{(n)}(x;q,t) = P_\lambda^{(n)}(x;q,t) \cdot e_r^{(n)}(t^n s^\lambda).$$

• $D_r^{(n)}$'s are commutative.

$$[D_r^{(n)}, D_s^{(n)}] = 0 \quad (1 \le r, s \le n).$$

§1.5. Ring of symmetric functions

- Restriction map $\rho_{m,n} : \Lambda_m \to \Lambda_n$ $(m \geq n)$

$$\rho_{m,n} f(x_1, \dots, x_m) = f(x_1, \dots, x_n, 0, \dots, 0).$$

$\{(\Lambda_m)_m, (\rho_{m,n})_{m,n}\}$: projective system

- $\Lambda = \mathbb{Z}[x_1, x_2, \dots]^{\mathfrak{S}_\infty} := \varprojlim_n \Lambda_n$: ring of symmetric functions,

  $\Lambda_{\mathbb{F}} := \Lambda \otimes \mathbb{F}$.

- $m_\lambda(x) := \sum_{\alpha:\text{permutation of } \lambda} x^\alpha$: monomial symmetric function,

  $e_r(x) := \sum_{i_1 < i_2 < \cdots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$, $\quad e_\lambda(x) := e_{\lambda_1} \cdots e_{\lambda_\ell}$:

  : elementary symmetric function,

  $p_r(x) := \sum_i x_i^r$, $\quad p_\lambda(x) := p_{\lambda_1} \cdots p_{\lambda_\ell}$: power sum function.

- $\{m_\lambda\}$ and $\{e_\lambda\}$ are $\mathbb{Z}$-bases of Λ, $\{p_\lambda\}$ is a $\mathbb{Q}$-basis of $\Lambda \otimes \mathbb{Q}$.

§1.6. Macdonald symmetric function

- For each partition λ, $P_\lambda(x; q, t) \in \Lambda_{\mathbb{F}}$ is determined by

$$P_\lambda = m_\lambda + \sum_{\mu<\lambda} c_{\lambda,\mu} m_\mu \qquad (c_{\lambda,\mu} \in \mathbb{F}),$$

$$\langle P_\lambda, P_\mu \rangle_{q,t} = 0 \text{ for } \mu \neq \lambda,$$

where $\langle \cdot, \cdot \rangle_{q,t}$ is an inner product on $\Lambda_{\mathbb{F}}$ given by

$$\langle p_\lambda, p_\mu \rangle_{q,t} := \delta_{\lambda,\mu} \prod_{j\geq 1} j^{z_j} z_j! \prod_{k\geq 1} \frac{1-q^{\lambda_k}}{1-t^{\lambda_k}},$$

$$z_j := \#\{k \mid \lambda_k = j\}.$$

- $\{P_\lambda(x; q, t)\}$ is a $\mathbb{F}$-basis of $\Lambda_{\mathbb{F}}$.

§1.7. Difference operators on the ring of symmetric functions

Ref. : [FHHSY, §3]

- $D_r^{(n)}$ is not compatible with the restriction map $\rho_{m,n}$.

Introduce the difference operator $E_r^{(n)}$ on $\Lambda_{n,F} = \mathbb{F}[x_1, \cdots, x_n]^{\mathfrak{S}_n}$ by

$$E_r^{(n)} := \sum_{j=0}^{r} \frac{t^{-nr-\binom{r-j+1}{2}}}{(t^{-1};t^{-1})_{r-j}} D_j^{(n)}.$$

Then

$$\rho_{n,n-1} \circ E_r^{(n)} = E_r^{(n-1)} \circ \rho_{n,n-1}.$$

Therefore $E_r := \varprojlim_n E_r^{(n)} : \Lambda_{\mathbb{F}} \to \Lambda_{\mathbb{F}}$ is well defined.

- $E_r P_\lambda(x;q,t) = P_\lambda(x;q,t) \cdot e_r(s^\lambda)$.

Remark. Macdonald defined E_1 only.

§2. Bosonization of Macdonald difference oparator

Ref : Shiraishi, 「量子可積分系入門」

§2.1. Free field relization (bosonization)

- Heisenberg algebra $\mathfrak{h}_{q,t}$:

  generators : a_n $(n \in \mathbb{Z})$,

  relations : $[a_n, a_m] = n\frac{1-q^{|n|}}{1-t^{|n|}}\delta_{n+m,0}\, a_0$.

- Triangular decomposition : $\mathfrak{h}_{q,t} = \mathfrak{h}^+_{q,t} \oplus \mathfrak{h}^0_{q,t} \oplus \mathfrak{h}^-_{q,t}$

  $(\mathfrak{h}^{\pm}_{q,t} = \oplus_{\pm n>0}\mathbb{F}a_n,\ \mathfrak{h}^0_{q,t} = \mathbb{F}a_0)$.

- Fock representation $\mathcal{F}_{q,t} := \mathrm{Ind}^{\mathfrak{h}_{q,t}}_{\mathfrak{h}^+_{q,t}\oplus\mathfrak{h}^0_{q,t}}(\mathbb{F}\cdot 1)$,

  where $\mathbb{F}\cdot 1$ is the one-dimensional $\mathfrak{h}^+_{q,t} \oplus \mathfrak{h}^0_{q,t}$ representation with

  $a_n \cdot 1 = 0\ (n > 0)$, $a_0 \cdot 1 = 1$.

• Identify $\Lambda_{\mathbb{F}}$ with $\mathcal{F}_{q,t}$ by the following isomorphism :

$$\mathcal{F}_{q,t} \xrightarrow{\sim} \Lambda_{\mathbb{F}} \qquad a_{-\lambda_1} a_{-\lambda_2} \cdots a_{-\lambda_\ell} \cdot 1 \mapsto p_\lambda(x).$$

• For $E \in \mathrm{End}(\Lambda_{\mathbb{F}})$, a realization $\widehat{E}$ in $\widetilde{U}(\mathfrak{h}_{q,t}) \subset \mathrm{End}(\mathcal{F}_{q,t})$ is called a bosonization (or free field realization, FFR) of E.

Example 1. Bosonization of $E_1 \in \mathrm{End}\, \Lambda_{\mathbb{F}}$.

Introduce the vertex operator $\eta(z)$ by

$$\eta(z) := \exp\Big(\sum_{n>0} \frac{1-t^{-n}}{n} a_{-n} z^n\Big) \exp\Big(-\sum_{n>0} \frac{1-t^{n}}{n} a_{n} z^{-n}\Big).$$

Then the zero-mode η_0 of the expansion $\eta(z) = \sum_{n\in\mathbb{Z}} \eta_n z^{-n}$ gives the bosonization of E_1 :

$$\widehat{E}_1 = \frac{\eta_0 - 1}{t-1}.$$

§2.2. Bosonization of E_r

Ref. Shiraishi, CMP 263 (2006) 439–460.

- Macdonald difference operator E_r has the next FFR :

$$\widehat{E}_r = \frac{[r]_{t^{-1}}!}{r!}\Big[\prod_{1\le i<j\le r} \varpi(z_j/z_i) :\eta(z_1)\eta(z_2)\cdots\eta(z_r): \Big]_1,$$

where

$$[r]_x := \frac{1-x^r}{1-x}, \quad [r]_x! := [r]_x \cdot [r-1]_x \cdots [1]_x,$$

$$\varpi(y) := \frac{(1-y)(1-y^{-1})}{(1-t^{-1}y)(1-t^{-1}y^{-1})},$$

$: \ :$: normal ordering in $\mathfrak{h}_{q,t}$,

$[f(z_1,\dots,z_n)]_1$: constant term of the Laurent series f.

§2.3. Problem

(A) Find the FFR of the composition of operators (e.g. $E_r \circ E_s$) .

(B) As a generalization of the difference equation

$$E_r P_\lambda(x; q, t) = P_\lambda(x; q, t) \cdot e_r(s^\lambda),$$

find the FFR of the operator $\mathcal{O}_\mu$ satisfying

$$\mathcal{O}_\mu P_\lambda(x; q, t) = P_\lambda(x; q, t) \cdot P_\mu(s^\lambda; q, t) \quad (\forall\lambda : \text{partition}).$$

for any partition μ.

c.f.1. $e_r(y) = P_{(1^r)}(y; q, t)$.

c.f.2. An operator acting on the finite-variable Macdonald polynomial

Ref. : Noumi, 2010 年度数学会 無限可積分系セッション特別講演

「可換差分作用素と核函数」p. 18

(to be continued)

c.f.2. (continued)

$$H_r^{(n)} := \sum_{\substack{\nu\in\mathbb{N}^n \\ |\nu|=r}} \Big[\prod_{1\le i<j\le n} \frac{q^{\nu_i}x_i - q^{\nu_j}x_j}{x_i - x_j} \Big] \Big[\prod_{i,j=1}^{n} \frac{(tx_i/x_j;q)_{\nu_i}}{(qx_i/x_j;q)_{\nu_i}} \Big] T_{q,x}^{\nu}$$

is a member of $\mathbb{F}[D_1^{(n)}, \dots, D_n^{(n)}]$.

And for any partition λ of length $\le n$

$$H_r^{(n)} P_\lambda^{(n)}(x;q,t) = P_\lambda^{(n)}(x;q,t) \cdot g_r^{(n)}(s^\lambda;q,t).$$

Here $g_r^{(n)}(y;q,t)$ $(1 \le r \le n)$ is a symmetric polynomial given by

$$\sum_{r\ge 0} g_r^{(n)}(y;q,t)u^r = \exp\Big[\sum_{m\ge 1} \frac{1}{m}\frac{1-t^m}{1-q^m}\sum_{i=1}^{n} y_i^m u^m\Big]$$

It is known that $\varprojlim_n g_r^{(n)}(y;q,t) = P_{(r)}(y;q,t)$.

§3. Feigin-Odesskii algebra $\mathcal{A}$ on $\mathbb{CP}^1$

Ref. : [FHHSY, §2]

§3.1. Definition of $\mathcal{A}$

- q_1, q_2 : indeterminants, $F := \mathbb{Q}(q_1, q_2)$, $q_3 := q_1^{-1} q_2^{-1}$.
- Define the F-vector space $\mathcal{A}_n = \mathcal{A}_n(q_1, q_2, q_3)$ by the next conditions :

(i) $\mathcal{A}_0 := F$. For $n \geq 1$, each element $f = f(z_1, \dots, z_n)$ of $\mathcal{A}_n$ is a n-variable rational symmetric functions over F.

(ii) The poles of f are at most of degree two and on the big diagonal.

(By (i) and (ii), each $f \in \mathcal{A}_n$ is of the form :

$$f = \frac{\text{symmetric polynomial of } z_1, \dots, z_n}{\prod_{i<j}(z_i - z_j)^2})$$

(to be continued)

(iii) For any $0 \le k \le n$, $\partial^{(0,k)} f = \partial^{(\infty,k)} f$,

where $\partial^{(\alpha,k)} f := \lim_{\xi \to \alpha} f(z_1, \ldots, z_{n-k}, \xi z_{n-k+1}, \ldots, \xi z_n)$.

(degenerate $\mathbb{CP}^1$ condition.)

(By (i) - (iii), each $\mathcal{A}_n$ is of finite dimension.)

(iv) For $n \ge 3$, each $f \in \mathcal{A}_n$ satisfies the next wheel condition:

$$f(z_1, q_1 z_1, q_1 q_2 z_1, z_4, \cdots) = f(z_1, q_2 z_1, q_1 q_2 z_1, z_4, \cdots) = 0.$$

Example 2.

- $\mathcal{A}_0 = \mathsf{F}$.
- $\mathcal{A}_1 = \mathsf{F}$ (By (iii), only the constant functions are allowed.)
- $\mathcal{A}_2 = \mathsf{F}\dfrac{z_1^2 + z_2^2}{(z_1 - z_2)^2} \oplus \mathsf{F}\dfrac{z_1 z_2}{(z_1 - z_2)^2}$.
- $\mathcal{A}_3 = \left\langle \epsilon_{(3)}(z), \epsilon_{(2,1)}(z), \epsilon_{(1,1,1)}(z) \right\rangle$.
- $\dim_{\mathsf{F}} \mathcal{A}_4 = 5$, $\dim_{\mathsf{F}} \mathcal{A}_5 = 7$, $\dim_{\mathsf{F}} \mathcal{A}_6 = 11$, $\cdots$.

  These numbers are the partition numbers... $\dim_{\mathsf{F}} \mathcal{A}_n \stackrel{?}{=} p(n)$.

§3.2. Shuffle product

- For $f \in \mathcal{A}_n$ and $g \in \mathcal{A}_m$, define the **shuffle product** $*$ by

$$(f * g)(z_1, \cdots, z_{n+m}) :=$$

$$\mathrm{Sym}\left(f(z_1, \cdots, z_n)\, g(z_{n+1}, \cdots, z_{n+m}) \prod_{\substack{1 \le \alpha \le n \\ n+1 \le \beta \le n+m}} \omega(z_\alpha, z_\beta) \right).$$

Here Sym is the symmetrization operator, and ω is given by

$$\omega(z_\alpha, z_\beta; q_1, q_2, q_3) := \frac{(z_\alpha - q_1 z_\beta)(z_\alpha - q_2 z_\beta)(z_\alpha - q_3 z_\beta)}{(z_\alpha - z_\beta)^3}.$$

§3.3. Structure theorem

- Define the graded vector space $\mathcal{A} := \bigoplus_{n \geq 0} \mathcal{A}_n$.

<u>**Theorem 1.**</u>

(1) The operation $*$ is closed in $\mathcal{A}$.

(2) $(\mathcal{A}, *)$ is a unital associative algebra over F.

(3) $(\mathcal{A}, *)$ is commutative.

(4) $\dim_F \mathcal{A}_n$ is equal to the partition number of n.

(5) There are three bases $\{\epsilon_\lambda(z; q_i) \mid \lambda : \text{partition}\}$ $(i = 1, 2, 3)$, where

$$\epsilon_n(z; q_i) := \prod_{1 \leq j < k \leq n} \frac{(z_k - q_i z_j)(z_k - q_i^{-1} z_j)}{(z_k - z_j)^2},$$

$$\epsilon_\lambda(z; q_i) := \epsilon_{\lambda_1}(z; q_i) * \epsilon_{\lambda_2}(z; q_i) * \cdots * \epsilon_{\lambda_\ell}(z; q_i).$$

To show the theorem, we need the next Gordon filtration.

§3.4. Gordon filtration

- For a partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ of n, define a linear map

  (called specialization map) $\varphi_\lambda^{(q_i)} : \mathcal{A}_n \to F(y_1, \cdots, y_\ell)$ by

$$
\begin{aligned}
f(x_1, \cdots, x_n) \ \mapsto \ & f(y_1, q_i y_1, \cdots, q_i^{\lambda_1 - 1} y_1, \\
& y_2, q_i y_2, \cdots, q_i^{\lambda_2 - 1} y_2, \\
& \cdots \\
& y_\ell, q_i y_\ell, \cdots, q_i^{\lambda_\ell - 1} y_\ell)
\end{aligned}
$$

Next we introduce a subspace of $\mathcal{A}_n$ by

$$
\mathcal{A}_{n,\lambda}^{(q_i)} := \bigcap_{\substack{\mu:\text{partition of } n \\ \mu \not\leq \lambda}} \ker \varphi_\mu^{(q_i)}.
$$

Here $<$ is the dominance semi-ordering.

Example 3. $q := q_1$

- $\mathcal{A}_2 = \mathcal{A}^{(q)}_{2,(2)} = \left\langle \epsilon_{(2)}(z;q), \epsilon_{(1,1)}(z;q) \right\rangle \supsetneq \mathcal{A}^{(q)}_{2,(1,1)} = \ker \varphi^{(q)}_{(2)} = \left\langle \epsilon_{(2)} \right\rangle.$

- $\mathcal{A}_3 = \mathcal{A}^{(q)}_{3,(3)} = \left\langle \epsilon_{(3)}(z;q), \epsilon_{(2,1)}(z;q), \epsilon_{(1,1,1)}(z;q) \right\rangle$

  $\supsetneq \mathcal{A}^{(q)}_{3,(2,1)} = \ker \varphi^{(q)}_{(3)} = \left\langle \epsilon_{(3)}(z;q), \epsilon_{(2,1)}(z;q) \right\rangle$

  $\supsetneq \mathcal{A}^{(q)}_{3,(1,1,1)} = \ker \varphi^{(q)}_{(2,1)} = \left\langle \epsilon_{(3)}(z;q) \right\rangle.$

- $\mathcal{A}_4 = \mathcal{A}^{(q)}_{4,(4)} = \left\langle \epsilon_{(4)}, \epsilon_{(3,1)}, \epsilon_{(2,2)}, \epsilon_{(2,1,1)}, \epsilon_{(1,1,1,1)} \right\rangle$

  $\supsetneq \mathcal{A}^{(q)}_{4,(3,1)} = \ker \varphi^{(q)}_{(4)} = \left\langle \epsilon_{(4)}, \epsilon_{(3,1)}, \epsilon_{(2,2)}, \epsilon_{(2,1,1)} \right\rangle$

  $\supsetneq \mathcal{A}^{(q)}_{4,(2,2)} = \ker \varphi^{(q)}_{(3,1)} = \left\langle \epsilon_{(4)}, \epsilon_{(3,1)}, \epsilon_{(2,2)} \right\rangle$

  $\supsetneq \mathcal{A}^{(q)}_{4,(2,1,1)} = \ker \varphi^{(q)}_{(2,2)} = \left\langle \epsilon_{(4)}, \epsilon_{(3,1)} \right\rangle$

  $\supsetneq \mathcal{A}^{(q)}_{4,(1,1,1,1)} = \ker \varphi^{(q)}_{(2,1,1)} = \left\langle \epsilon_{(4)} \right\rangle.$

Example 3. (continued)

- $\circ$ $\mathcal{A}_5 = \mathcal{A}^{(q)}_{5,(5)} \supsetneqq \mathcal{A}^{(q)}_{5,(4,1)} \supsetneqq \mathcal{A}^{(q)}_{5,(3,2)} \supsetneqq \mathcal{A}^{(q)}_{5,(3,1^2)} \supsetneqq \mathcal{A}^{(q)}_{5,(2^2,1)}$
$\supsetneqq \mathcal{A}^{(q)}_{5,(2,1^3)} \supsetneqq \mathcal{A}^{(q)}_{5,(1^5)}$.

- $\circ$ $\mathcal{A}_6 = \mathcal{A}^{(q)}_{6,(6)} \supsetneqq \mathcal{A}^{(q)}_{6,(5,1)} \supsetneqq \mathcal{A}^{(q)}_{6,(4,2)} \begin{smallmatrix} \supsetneqq \mathcal{A}^{(q)}_{6,(4,1^2)} \supsetneqq \\ \supsetneqq \mathcal{A}^{(q)}_{6,(3^2)} \supsetneqq \end{smallmatrix} \mathcal{A}^{(q)}_{6,(3,2,1)} \begin{smallmatrix} \supsetneqq \mathcal{A}^{(q)}_{6,(3,1^3)} \supsetneqq \\ \supsetneqq \mathcal{A}^{(q)}_{6,(2^3)} \supsetneqq \end{smallmatrix} \mathcal{A}^{(q)}_{6,(2^2,1^2)}$
$\supsetneqq \mathcal{A}^{(q)}_{6,(2,1^4)} \supsetneqq \mathcal{A}^{(q)}_{6,(1^6)}$.

- The proof of Theorem 1 is a purely algebraic one, and based on this Gordon filtration and some combinatorics.

§4. Intersection space and bosonization

Ref. : [FHHSY, §3]

§4.1. Intersection space

<u>Theorem 2.</u> For any partition λ of n, we have

$$\dim_{\mathbb{C}} \left(\mathcal{A}_{n,\lambda}^{(q^{-1})} \cap \mathcal{A}_{n,\lambda'}^{(t)} \right) = 1.$$

Here λ' is the transposed partition of λ. We also changed notation :

$$q_1 = q^{-1} \in \mathbb{C}, \quad q_2 = t \in \mathbb{C}, \quad F = \mathbb{Q}(q_1, q_2) \mapsto \mathbb{C}.$$

$$|q| < 1, \quad |t| > 1, \quad q^i t^j \neq 1 \; \forall \, (i, j) \in \mathbb{Z}^2 \setminus \{(0, 0)\}.$$

• The statement of Theorem only contains algebraic properties of $\mathcal{A}$. But our proof requires some arguments on Macdonald symmetric functions, with the aid of analytic FFR construction. Therefore we changed the coefficient field F with $\mathbb{C}$.

§4.2. FFR and the algebra $\mathcal{A}$

- Recall the vertex operator

$$\eta(z) = \circ\!\!\circ \exp\Big(-\sum_{n\neq 0} \frac{1-t^n}{n} a_n z^{-n}\Big) \circ\!\!\circ.$$

For $f \in \mathcal{A}_n(q^{-1}, t)$ we define $\mathcal{O}(f) \in \widetilde{U}(\mathfrak{h}_{q,t})$ as follows :

$$\mathcal{O}(f) := \oint_{C_n} \Big(\prod_{j=1}^{n} \frac{dz_j}{2\pi i\, z_j}\Big) \frac{f(z_1, \cdots, z_n)}{\prod_{k<\ell} \omega(z_k, z_\ell; q^{-1}, t, qt^{-1})} \eta(z_1) \cdots \eta(z_n),$$

where $C_n := \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid |z_i| = 1\}$.

Proposition 1. (1) $\mathcal{O}(f * g) = \mathcal{O}(f)\mathcal{O}(g)$. (Answer to Problem (A))

(2) $\mathcal{O}$ is injective.

(3) $\mathcal{O}(\epsilon_n(z; q^{-1})) = \widehat{E}_n$.

(4) $\mathrm{Im}\,\mathcal{O}(\cong \mathcal{A})$ is isomorphic to $\mathbb{C}[\widehat{E}_1, \widehat{E}_2, \cdots]$.

(5) For each $f \in \mathcal{A}$, $\mathcal{O}(f)$ has $P_\lambda(x; q, t)$ as eigen-functions.

§4.3. Operator $\widehat{G}_n$

- $\widehat{G}_n := \dfrac{(-1)^n q^{\binom{n}{2}}}{(q;q)_n} \dfrac{[n]_q!}{n!} \mathcal{O}(\epsilon_n(z;t))$.

- $\widehat{E}_r$ and $\widehat{G}_s$ satisfy the next Wronski relation :

$$\sum_{k=0}^{n} (-1)^k (1 - q^k t^{n-k}) \widehat{E}_{n-k} \widehat{G}_k = 0.$$

Proposition 2. $\widehat{G}_r$ is the FFR of the next difference operator G_r :

$$G_r P_\lambda(x;q,t) = P_\lambda(x;q,t) \cdot g_r(s^\lambda;q,t), \quad (\forall \lambda : \text{partition})$$

where $g_r(y;q,t) := \varprojlim_n g_r^{(n)}(y;q,t)$ (p.15).

Remark. The operator G_r is also realized as $G_r = \varprojlim_n G_r^{(n)}$ with

$$G_r^{(n)} := \frac{t^{-rn} q^{\binom{r}{2}}}{(-1)^r (q;q)_r} \sum_{k=0}^{r} (-1)^k q^{-\binom{k}{2}} q^{-k(r-k)} (q^{r-k+1};q)_k H_k^{(n)}.$$

Here $H_k^{(n)}$ is the difference operator given in p.15.

§4.4 Refinement of Theorem 2

- **Theorem 2 is proved in the next refined form.**

Theorem 2'. There exists a **unique** element $f_\mu \in \mathcal{A}_n(q^{-1}, t)$ satisfying

$$\mathcal{O}(f_\mu)P_\lambda(x; q, t) = P_\lambda(x; q, t) \cdot P_\mu(s^\lambda; q, t) \quad (\forall \lambda : \text{partition}).$$

And It spans the intersection space $\mathcal{A}_{n,\mu}^{(q^{-1})} \cap \mathcal{A}_{n,\mu'}^{(t)}$.

(Answer to Problem (B))

- In the proof we used Proposition 1, Proposition 2 and a triangularity of P_λ obtained by Haiman.

§5. Relation to Ding-Iohara algebra and deformed $\mathcal{W}$ algebra

Ref. : [FHHSY, §3-F, Appendix], [FHSSY]

- $\eta(z)$ appears in the level 1 representation of Ding-Iohara algebra.

§5.1. Ding-Iohara algebra

- Introduce an associative algebra $\mathcal{U} = \mathcal{U}(q,t)$ as follows :

generators : $x^{\pm}(z) = \sum_{n\in\mathbb{Z}} x^{\pm}_n z^{-n}$, $\psi^{\pm}(z) = \sum_{n\in\mathbb{Z}_{\geq 0}} \psi^{\pm}_n z^{-n}$, $\gamma^{\pm 1/2}$ (central)

relation : $\psi^{\pm}(z)\psi^{\pm}(w) = \psi^{\pm}(w)\psi^{\pm}(z)$, $\quad \psi^{+}(z)\psi^{-}(w) = \dfrac{g(\gamma^{+1}w/z)}{g(\gamma^{-1}w/z)}\psi^{-}(w)\psi^{+}(z)$,

$\psi^{+}(z)x^{\pm}(w) = g(\gamma^{\mp 1/2}w/z)^{\mp 1}x^{\pm}(w)\psi^{+}(z)$, $\quad \psi^{-}(z)x^{\pm}(w) = g(\gamma^{\mp 1/2}z/w)^{\pm 1}x^{\pm}(w)\psi^{-}(z)$,

$$[x^{+}(z), x^{-}(w)] = \frac{(1-q)(1-1/t)}{1-q/t}\Big(\delta(\gamma^{-1}z/w)\psi^{+}(\gamma^{1/2}w) - \delta(\gamma z/w)\psi^{-}(\gamma^{-1/2}w)\Big),$$

$G^{\mp}(z/w)x^{\pm}(z)x^{\pm}(w) = G^{\pm}(z/w)x^{\pm}(w)x^{\pm}(z)$.

where $g(z) := \dfrac{G^{+}(z)}{G^{-}(z)}$, $G^{\pm}(z) := (1-q^{\pm 1}z)(1-t^{\mp 1}z)(1-q^{\mp 1}t^{\pm 1}z)$.

Fact. (Ding-Iohara) $\mathcal{U}$ has a (formal) Hopf algebra structure.

The coproduct is given as follows.

$$\Delta(\gamma^{\pm 1/2}) = \gamma^{\pm 1/2} \otimes \gamma^{\pm 1/2},$$

$$\Delta(x^+(z)) = x^+(z) \otimes 1 + \psi^-(\gamma_{(1)}^{1/2} z) \otimes x^+(\gamma_{(1)} z),$$

$$\Delta(x^-(z)) = x^-(\gamma_{(2)} z) \otimes \psi^+(\gamma_{(2)}^{1/2} z) + 1 \otimes x^-(z),$$

$$\Delta(\psi^{\pm}(z)) = \psi^{\pm}(\gamma_{(2)}^{\pm 1/2} z) \otimes \psi^{\pm}(\gamma_{(1)}^{\mp 1/2} z),$$

where $\gamma_{(1)}^{\pm 1/2} = \gamma^{\pm 1/2} \otimes 1$, $\gamma_{(2)}^{\pm 1/2} = 1 \otimes \gamma^{\pm 1/2}$.

Remark. Ding and Iohara introduced their Hopf algebra $U_q(g, \mathfrak{sl}_n)$ as a generalization of the Drinfeld (new) realization of $U_q(\widehat{\mathfrak{sl}}_n)$.

Here $g = \{g_{i,j} \mid 1 \leq i, j \leq n-1\}$ is a family of analytic functions satisfying $g_{i,j}(z) = g_{i,j}(z^{-1})^{-1}$.

Our algebra $\mathcal{U}$ is the case $n = 2$ with the "structure function" g given in the previous page.

§5.2. Level 1 representation

- A representation of $\mathcal{U}$ is called level k if $\gamma^{\pm 1/2}$ is realized as $(t/q)^{\pm k/4}$.

- Introduce the following vertex operators.

$$\eta(z) := {}^{\circ}_{\circ} \exp\Big(-\sum_{n\neq 0} \frac{1-t^n}{n} a_n z^{-n} \Big) {}^{\circ}_{\circ},$$

$$\xi(z) := \exp\Big(\sum_{n>0} \frac{t^{-n}-1}{n} (t/q)^{n/2} a_{-n} z^n \Big) \exp\Big(\sum_{n>0} \frac{1-t^n}{n} (t/q)^{n/2} a_n z^{-n} \Big),$$

$$\varphi^+(z) := \exp\Big(-\sum_{n>0} \frac{1-t^n}{n} (1-(t/q)^n)(t/q)^{-n/4} a_n z^{-n} \Big),$$

$$\varphi^-(z) := \exp\Big(\sum_{n>0} \frac{1-t^{-n}}{n} (1-(t/q)^n)(t/q)^{-n/4} a_{-n} z^n \Big).$$

Proposition On the Fock rep. $\mathcal{F}_{q,t}$ of the Heisinberg alg. $\mathfrak{h}_{q,t}$ (p. 11), one has a level 1 representation ρ_c of $\mathcal{U}$ by setting

$$\rho_c(\gamma^{\pm 1/2}) = (t/q)^{\pm 1/4}, \quad \rho_c(\psi^{\pm}(z)) = \varphi^{\pm}(z),$$
$$\rho_c(x^{+}(z)) = c\,\eta(z), \quad \rho_c(x^{-}(z)) = c^{-1}\xi(z),$$

with $c \in \mathbb{Q}(q^{1/2}, t^{1/2}) \setminus \{0\}$. (Later we denote this rep. space as $\mathcal{F}_c$)

Remark. The Ding-Iohara algebra with 2(or 3) parameter $\mathcal{U}$ appears in several other works at the same period.

B. Feigin, A. Tsymbaliuk, "Heisenberg action in the equivariant K-theory of Hilbert schemes via Shuffle Algebra", arXiv:0904.1679.

O. Schiffmann, E. Vasserot, "The elliptic Hall algebra and the equivariant K-theory of the Hilbert scheme of $\mathbb{A}^2$", arXiv:0905.2555.

B. Feigin, E. Feigin, M. Jimbo, T. Miwa, E. Mukhin, "Quantum continuous $\mathfrak{gl}_\infty$", arXiv:1002.3100, 1002.3113.

§5.3. Intertwining operator and Macdonald difference operator

- We have a level 0 rep. π_x of $\mathcal{U}$ on $V_x := \mathbb{Q}(q^{1/2}, t^{1/2})[x^{\pm 1}]$ by setting

$$\pi_x(\gamma^{\pm 1/2}) = 1, \quad \pi_x(x^{\pm}(z)) = c^{\pm 1}(1 - t^{\mp 1})\delta(q^{\mp 1/2}x/z)T_{q^{\mp 1},x},$$

$$\pi_x(\psi^{\pm}(z)) = \frac{(1 - q^{1/2}t^{-1}(x/z)^{\pm 1})(1 - q^{-1/2}t(x/z)^{\pm 1})}{(1 - q^{1/2}(x/z)^{\pm 1})(1 - q^{-1/2}(x/z)^{\pm 1})}$$

with $c \in \mathbb{Q}(q^{1/2}, t^{1/2})^{\times}$.

- Now we study the intertwining operator $\Phi^{\mathcal{F}_\gamma}_{V_{x,\alpha} \otimes \mathcal{F}_\beta} : V_{x,\alpha} \otimes \mathcal{F}_\beta \to \mathcal{F}_\gamma$

with the condition $\Phi^{\mathcal{F}_\gamma}_{V_{x,\alpha} \otimes \mathcal{F}_\beta} \Delta(a) = a\, \Phi^{\mathcal{F}_\gamma}_{V_{x,\alpha} \otimes \mathcal{F}_\beta}$ for any $a \in \mathcal{U}$.

Introduce the components $\Phi^{\gamma}_{\alpha,\beta,n}$ of $\Phi^{\mathcal{F}_\gamma}_{V_{x,\alpha} \otimes \mathcal{F}_\beta}$ by

$$\Phi^{\mathcal{F}_\gamma}_{V_{x,\alpha} \otimes \mathcal{F}_\beta}(x^n \otimes v) = \Phi^{\gamma}_{\alpha,\beta,n} v \ (v \in \mathcal{F}_\beta).$$

Set the generating function as $\Phi^{\gamma}_{\alpha,\beta}(y) = \sum_{n \in \mathbb{Z}} \Phi^{\gamma}_{\alpha,\beta,n} y^{-n}$.

<u>Proposition</u> $\Phi^{\mathcal{F}_\gamma}_{V_{x,\gamma}\otimes\mathcal{F}_{t^{-1}\gamma}}$ exists uniquly up to normalization, and its generating function $\Phi(y) := \Phi^{\gamma}_{\gamma,t^{-1}\gamma}(y)$ is given by $\Phi(y) = \widetilde{\Phi}(q^{1/2}y)$ with

$$\widetilde{\Phi}(y) := \exp\left(\sum_{n>0}\frac{1}{n}\frac{1-t^n}{1-q^n}t^{-n}a_{-n}y^n\right)\exp\left(-\sum_{n>0}\frac{1}{n}\frac{1-t^n}{1-q^n}q^n a_n y^{-n}\right).$$

• One can also calculate some relations between $\eta(z)$, $\phi(y)$ and $\widetilde{\Phi}(y)$'s, where we set

$$\phi(y) := \exp\left(\sum_{n\geq 1}\frac{1-t^n}{1-q^n}\frac{a_{-n}}{n}y^n\right) = \prod_{i\geq 1}\frac{(tx_iy;q)_\infty}{(x_iy;q)_\infty} = \sum_{n\geq 0} g_n(x;q,t)y^n.$$

For example, one has

$$:\widetilde{\Phi}(y_1)\cdots\widetilde{\Phi}(y_n):\cdot 1 = \phi(t^{-1}y_1)\cdots\pi(t^{-1}y_n)\cdot 1.$$

• **From those relations we recover the simplest construction of the bosonized difference operator :**

$$\eta_0\phi(y_1)\cdots\phi(y_n)\cdot 1 =$$
$$t^{-n}\phi(y_1)\cdots\phi(y_n)\cdot 1 + (1-t^{-1})t^{-n+1}D^1_{n,y}\phi(y_1)\cdots\phi(y_n)\cdot 1.$$

§5.4. Deformed W algebra $\mathcal{W}_{q,t}(\mathfrak{sl}_n)$

- $\rho_{y_1} \otimes \cdots \otimes \rho_{y_n}$: n-times tensor of Fock reps.

  $\Delta^{(2)} := \Delta, \quad \Delta^{(n)} := (\mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes \Delta) \circ \Delta^{(n-1)}.$

  $\rho_y^{(n)}$: level n representation defined by

$$\rho_y^{(n)} := \rho_{y_1} \otimes \cdots \otimes \rho_{y_n} \circ \Delta^{(n)}.$$

- Define a new current $t(z) := \alpha(z) x^+(z) \beta(z)$,

where $\alpha(z) := \exp\left(-\sum_{n=1}^{\infty} \frac{b_{-n} z^n}{\gamma^n - \gamma^{-n}} \right), \quad \beta(z) := \exp\left(\sum_{n=1}^{\infty} \frac{b_n z^{-n}}{\gamma^n - \gamma^{-n}} \right).$

Here b_n is the boson appearing in the expansion of $\psi^\pm$ as follows:

$$\psi^+(z) = \psi_0^+ \exp\left(+\sum_{n>0} b_n \gamma^{n/2} z^{-n} \right), \quad \psi^-(z) = \psi_0^- \exp\left(-\sum_{n>0} b_{-n} \gamma^{n/2} z^n \right).$$

These satisfies the next relations.

$$[b_m, b_n] = \frac{1}{m}(1 - q^{-m})(1 - t^m)(1 - q^m/t^m)(\gamma^m - \gamma^{-m})\gamma^{-|m|}\delta_{m+n,0}.$$

• Finally introduce the next function :

$$f_k(z) := \exp\Big(\sum_{n=1}^{\infty} \frac{(1-q^n)(1-t^{-n})(1-p^{(k-1)n})}{1-p^{kn}} z^n\Big).$$

Proposition 4. Define $\{\Lambda_i(z) \mid i = 1 \dots n\}$ by $\rho_y^{(n)}(t(z)) = \sum_{i=1}^n y_i \Lambda_i(z)$.

Then these satisfy

$$f_n(w/z)\Lambda_i(z)\Lambda_j(w) = {}^{\circ}_{\circ}\Lambda_i(z)\Lambda_j(w)^{\circ}_{\circ} \times \begin{cases} 1 & i = j, \\ \gamma_+(z, w; q, t) & i < j, \\ \gamma_-(z, w; q, t) & i > j. \end{cases}$$

with $\gamma_\pm(z, w; q, t) := \dfrac{(z - q^{\mp 1}w)(z - qt^{\mp 1}w)}{(z - w)(z - t^{\mp 1}w)}$.

These are the relations satisfied by the generators of $\mathcal{W}_{q,t}(\mathfrak{sl}_n)$.

Remark. $\mathcal{W}_{q,t}(\mathfrak{sl}_n)$ is defined by the vertex operators, and the whole relations are not discovered yet.

§6. Elliptic analogue

Ref. : [FHHSY §4, Appendix]

§6.1. Elliptic Feigin-Odesskii algebra $\mathcal{A}(p)$

- $q_1 = q^{-1}$, $q_2 = t$, $q_3 = q_1^{-1} q_2^{-1}$, $p \in \mathbb{C}$,

$|q| < 1, |t^{-1}| < 1, |p| < 1, |pq^{-1}t| < 1, q^i t^j p^k \neq 1 \ \forall (i,j,k) \in \mathbb{Z}^3 \setminus \{(0,0,0)\}$.

- Define a $\mathbb{C}$-vector space $\mathcal{A}_n(p) = \mathcal{A}_n(q_1, q_2, q_3, p)$ as follows :

(i) $\mathcal{A}_0(p) := \mathbb{C}$. For $n \geq 1$, $f(x_1, \ldots, x_n) \in \mathcal{A}_n(p)$ is symmetric and has two periods :

$$f(x_1, \ldots, e^{2\pi\sqrt{-1}} x_i, \ldots, x_n) = f(x_1, \ldots, p x_i, \ldots, x_n) = f(x_1, \ldots, x_n)$$

(ii) The poles of $f(x_1, \ldots, x_n)$ are at most of degree two, and on the big diagonal or its p-shift.

(iii) For $n \geq 3$, each $f \in \mathcal{A}_n(p)$ satisfies the wheel condition.

$f(x_1, q_1 x_1, q_1 q_2 x_1, x_4, \ldots) = 0$, $f(x_1, q_2 x_1, q_1 q_2 x_1, x_4, \ldots) = 0$.

• The shuffle product on $\mathcal{A}(p)$ is defined similarly as that on $\mathcal{A}$, with ω changed to

$$\omega(x, y; q_1, q_2, q_3, p) := \frac{\Theta_p(q_1 y/x)\Theta_p(q_2 y/x)\Theta_p(q_3 y/x)}{\Theta_p(y/x)^3},$$

$$\Theta_p(x) := (p; p)_\infty (x; p)_\infty (p/x; p)_\infty$$

Proposition 5. $(\mathcal{A}(p), *)$ is a commutative algebra,

and has bases $\{\epsilon_\lambda(x; q_i, p)\}$, where

$$\epsilon_\lambda := \epsilon_{\lambda_1} * \cdots * \epsilon_{\lambda_\ell} \qquad (\lambda = (\lambda_1, \dots, \lambda_\ell)),$$

$$\epsilon_n(x_1, \dots, x_n; q_i, p) := \prod_{1\le j<k\le n} \frac{\Theta_p(q_i x_j/x_k)\Theta_p(x_j/q_i x_k)}{\Theta_p(x_j/x_k)^2}.$$

§6.2. Ruijsenaars operator and elliptic deformation

• Introduce an elliptic analogue $\eta(z;p)$ of the vertex operator $\eta(z)$ by

$$\eta(z;p) = \exp\Big(\sum_{n>0} \frac{1-t^{-n}}{n} \frac{1-p^n q^{-n} t^n}{1-p^n} a_{-n} z^n\Big) \exp\Big(-\sum_{n>0} \frac{1-t^n}{n} a_n z^{-n}\Big).$$

• Recall that $[\eta(z)]_1$ is a FFR of E_1.

$[\eta(z;pq^{-1}t)]_1$ relates to the rank one Ruijsenaars operator

$$D_n^1(p) := \sum_{i=1}^{n} \prod_{j\neq i} \frac{\Theta_p(tx_i/x_j)}{\Theta_p(x_i/x_j)} T_{q,x_i}.$$

• In order to state that relation, we need to introduce the quasi-hopf twist technique...

§6.2.1 Quasi-Hopf twist

Ref. : [FFHSY, Appendix], Jimbo-Konno-Odake-Shiraishi CMP (1999)

• Recall the boson b_n introduced at p.35. Set $u^{\pm}(z;p) \in \mathcal{U} = \mathcal{U}(q,t)$ by

$$u^{+}(z;p) := \exp\Big(\sum_{n>0} \frac{-p^n\gamma^{-n}}{1-p^n\gamma^{-2n}} b_{-n} z^n\Big), \quad u^{-}(z;p) := \exp\Big(\sum_{n>0} \frac{p^n}{1-p^n} b_n z^{-n}\Big),$$

and set further

$$x^{+}(z;p) := u^{+}(z;p)x^{+}(z), \quad x^{-}(z;p) := x^{-}(z)u^{-}(z;p),$$
$$\psi^{\pm}(z;p) := u^{+}(\gamma^{\pm 1/2}z;p)\psi^{\pm}(z)u^{-}(\gamma^{\mp 1/2}z;p).$$

• These dressed Drinfeld currents $x^{\pm}(z;p), \psi^{\pm}(z;p) \in \mathcal{U}(q,t)$ enjoy elliptic permutation relations. For example,

$$\Theta_{p\gamma^{-2}}(q^{-1}z/w)\Theta_{p\gamma^{-2}}(tz/w)\Theta_{p\gamma^{-2}}(qt^{-1}z/w)x^{+}(z)x^{+}(w)$$
$$= -(z/w)^3\Theta_{p\gamma^{-2}}(q^{-1}w/z)\Theta_{p\gamma^{-2}}(tw/z)\Theta_{p\gamma^{-2}}(qt^{-1}w/z)x^{+}(w)x^{+}(z).$$

• Define the twistor $F(p)$ by

$$F(p) := \exp\left(\sum_{n>0} \frac{np^n\gamma_{(2)}^{-n}}{(1-q^{-n})(1-t^n)(1-q^nt^{-n})(1-p^n\gamma_{(2)}^{-2n})} b_n \otimes b_{-n}\right).$$

Proposition (1) $F(p)$ is invertible, and satisfies

$$(\varepsilon \otimes \mathrm{id})F(p) = (\mathrm{id} \otimes \varepsilon)F(p) = 1.$$

Hence $(\mathcal{U}(q,t), \Delta_p, \varepsilon, \Phi)$ is a quasi-bialgebra, where we set

$$\Delta_p(a) := F(p) \cdot \Delta(a) \cdot F(p)^{-1},$$
$$\Phi := (F^{(23)}(p)(\mathrm{id} \otimes \Delta)F(p)) \cdot (F^{(12)}(p)(\Delta \otimes \mathrm{id})F(p))^{-1}.$$

We denote the resulting quasi-bialgebra by $\mathcal{U}(q,t,p)$.

(2) $F(p)$ satisfies the shifted cocycle condition

$$F^{(23)}(p)(\mathrm{id} \otimes \Delta)F(p) = F^{(12)}(p\gamma_{(3)}^{-2})(\Delta \otimes \mathrm{id})F(p).$$

- The element Φ here is called the Drinfeld associator.

(Do not confuse with the intertwining operator $\Phi^{\mathcal{F}}_{V_x\otimes\mathcal{F}}$ or whose generating function $\Phi(y)$.)

- Dressed coproduct Δ_p can be calculted as

$$\Delta_p(\gamma^{\pm 1/2}) = \gamma^{\pm 1/2} \otimes \gamma^{\pm 1/2},$$
$$\Delta_p(x^+(z;p)) = x^+(z;p\gamma_{(2)}^{-2}) \otimes 1 + \psi^-(\gamma_{(1)}^{1/2}z;p\gamma_{(2)}^{-2}) \otimes x^+(\gamma_{(1)}z;p),$$
$$\Delta_p(x^-(z;p)) = x^-(\gamma_{(2)}z;p\gamma_{(2)}^{-2}) \otimes \psi^+(\gamma_{(2)}^{1/2}z;p) + 1 \otimes x^-(z;p),$$
$$\Delta_p(\psi^\pm(z;p)) = \psi^\pm(\gamma_{(2)}^{\pm 1/2}z;p\gamma_{(2)}^{-2}) \otimes \psi^\pm(\gamma_{(1)}^{\mp 1/2}z;p).$$

§6.2.2. Representations of dressed currents

__Lemma__ The level zero rep. $\pi_{x,c}$ of $\mathcal{U}$ on $V_{x,c} = \mathbb{Q}(q^{1/2}, t^{1/2})[x^{\pm 1}]$ written in terms of the dressed Drinfeld currents reads

$$\pi_{x,c}(\psi^{\pm}(z;p)) = \frac{\Theta_p(q^{1/2}t^{-1}(x/z)^{\pm 1})\Theta_p(q^{-1/2}t(x/z)^{\pm 1})}{\Theta_p(q^{1/2}(x/z)^{\pm 1})\Theta_p(q^{-1/2}(x/z)^{\pm 1})},$$

$$\pi_{x,c}(x^{\pm}(z;p)) = c^{\pm 1}(1-t^{\mp 1})\frac{(pt^{\mp 1};p)_\infty (pq^{\mp 1}t^{\pm 1};p)_\infty}{(p;p)_\infty (pq^{\mp 1};p)_\infty}\delta(q^{\mp 1/2}x/z)T_{q^{\mp 1},x}.$$

- Set elliptic deformed vertex operators as follows :

$$\eta(z;p) := \exp\left(\sum_{n>0}\frac{1-t^{-n}}{n}\frac{1-p^n}{1-p^nq^nt^{-n}}a_{-n}z^n\right)\exp\left(-\sum_{n>0}\frac{1-t^n}{n}a_nz^{-n}\right),$$

$$\xi(z;p) := \exp\left(-\sum_{n>0}\frac{1-t^{-n}}{n}(t/q)^{n/2}a_{-n}z^n\right)$$

$$\times \exp\left(\sum_{n>0}\frac{1-t^n}{n}\frac{1-p^nq^nt^{-n}}{1-p^n}(t/q)^{n/2}a_nz^{-n}\right),$$

(to be continued)

(continued)

$$\varphi^+(z;p) := \exp\left(\sum_{n>0} \frac{1-t^{-n}}{n}(1-t^nq^{-n})\frac{p^n}{1-p^nq^nt^{-n}}(t/q)^{-3n/4}a_{-n}z^n\right)$$
$$\times \exp\left(-\sum_{n>0} \frac{1-t^{n}}{n}(1-t^nq^{-n})\frac{1}{1-p^n}(t/q)^{-n/4}a_{n}z^{-n}\right),$$
$$\varphi^-(z;p) := \exp\left(\sum_{n>0} \frac{1-t^{-n}}{n}(1-t^nq^{-n})\frac{1}{1-p^nq^nt^{-n}}(t/q)^{-n/4}a_{-n}z^n\right)$$
$$\times \exp\left(-\sum_{n>0} \frac{1-t^{n}}{n}(1-t^nq^{-n})\frac{p^n}{1-p^n}(t/q)^{-3n/4}a_{n}z^{-n}\right).$$

Lemma The representation ρ_c on the Fock space $\mathcal{F}_{q,t}$ gives the images of dressed Drinfeld currents as

$$\rho_c(x^+(z;p)) = c\eta(z;p), \quad \rho_c(x^-(z;p)) = c^{-1}\xi(z;p),$$
$$\rho_c(\psi^{\pm}(z;p)) = \varphi^{\pm}(z;p).$$

§6.2.3. Elliptic intertwining operator

• Consider the intertwining operator $\Phi^{\mathcal{F}_\gamma}_{V_{x,\alpha}\otimes\mathcal{F}_\beta}(p) : V_{x,\alpha} \otimes \mathcal{F}_\beta \to \mathcal{F}_\gamma$ with respect to $\mathcal{U}(q,t,p)$. Namely, it satisfies the condition

$$\Phi^{\mathcal{F}_\gamma}_{V_{x,\alpha}\otimes\mathcal{F}_\beta}(p)\Delta_p(a) = a\,\Phi^{\mathcal{F}_\gamma}_{V_{x,\alpha}\otimes\mathcal{F}_\beta}(p) \text{ for any } a \in \mathcal{U}(q,t).$$

Introduce the components $\Phi^{\gamma}_{\alpha,\beta,n}(p)$ of $\Phi^{\mathcal{F}_\gamma}_{V_{x,\alpha}\otimes\mathcal{F}_\beta}(p)$ by

$$\Phi^{\mathcal{F}_\gamma}_{V_{x,\alpha}\otimes\mathcal{F}_\beta}(p)(x^n \otimes v) = \Phi^{\gamma}_{\alpha,\beta,n}(p)v \ (v \in \mathcal{F}_\beta),$$

and set the generating function as $\Phi^{\gamma}_{\alpha,\beta}(y;p) := \sum_{n\in\mathbb{Z}} \Phi^{\gamma}_{\alpha,\beta,n}(p)y^{-n}$.

<u>Proposition</u> $\Phi^{\mathcal{F}_\gamma}_{V_{x,\gamma}\otimes\mathcal{F}_{t^{-1}\gamma}}(p)$ uniquely exists up to normalization, and the generating function $\Phi(y;p) := \Phi^{\gamma,t^{-1}\gamma}_{\gamma}(y;p)$ is realized as

$$\Phi(y;p) = \exp\left(\sum_{n>0} \frac{1}{n}\frac{1-t^n}{1-q^n}\frac{1-p^n}{1-p^nq^nt^{-n}} q^{n/2}t^{-n}a_{-n}y^n\right)$$
$$\times \exp\left(-\sum_{n>0} \frac{1}{n}\frac{1-t^n}{1-q^n} q^{n/2}a_n y^{-n}\right).$$

§6.2.4. Elliptic interwtining operator and Ruijsenaars operator

Proposition Set $\widetilde{\Phi}(y;p) := \Phi(q^{-1/2}y;p)$, and

$$\phi(y_1,\ldots,y_n;p) := \prod_{1\le k\ne l\le n} \frac{(pty_l/y_k,q,p)_\infty}{(py_l/y_k,q,p)_\infty} : \widetilde{\Phi}(y_1;pq^{-1}t)\cdots\widetilde{\Phi}(y_n;pq^{-1}t) : .$$

Then we have

$$\begin{aligned}
&\Big[\eta(z;pq^{-1}t)\Big]_1 \phi(y_1,\ldots,y_n;p)\\
&= \phi(y_1,\ldots,y_n;p)\left[t^{-n}\prod_{i=1}^{n}\frac{\Theta_p(qt^{-1}z/y_i)}{\Theta_p(qz/y_i)}\frac{\Theta_p(tz/y_i)}{\Theta_p(z/y_i)}\eta(z;pq^{-1}t)\right]_1\\
&+(1-t^{-1})t^{-n+1}\frac{(p/t;p)_\infty(pt/q;p)_\infty}{(p;p)_\infty(p/q;p)_\infty}D^1_{n,y}(p)\phi(y_1,\ldots,y_n;p),
\end{aligned}$$

where $[\cdot]_1$ means the constant term in z.

- But we do not know how to treat higher rank Ruijsenaars operator in the framework of elliptic Feigin-Odesskii algebra...

§6.3. Okounkov-Pandharipande operator

- Consider the limit $\hbar \to 0$ with $q = e^{\hbar}$, $t = e^{\beta\hbar}$. (In this limit Macdonald polynomials reduce to Jack polynomial).

We use the next boson λ_n instead of a_n :

$$[\lambda_m, \lambda_n] = -\frac{1}{m}\frac{(1-q^m)(1-t^{-m})(1-p^m q^{-m} t^m)}{1-p^m}\delta_{m+n,0}.$$

$\eta(z;p)$ is simply written in terms of this λ_n :

$$\eta(z;p) = {}^{\circ}_{\circ} \exp\left(\sum_{n\neq 0} \lambda_n z^{-n}\right) {}^{\circ}_{\circ}.$$

- $\bar{a}_n$: boson in the Feigin-Fuchs FFR of Virasoro algebra.

$$[\bar{a}_m, \bar{a}_n] = m\delta_{m+n,0}.$$

$\bar{a}_n$ and λ_n are related by

$$\lambda_n = \frac{1}{|n|}\sqrt{-\frac{(1-q^{|n|})(1-t^{-|n|})(1-p^{|n|}q^{-|n|}t^{|n|})}{1-p^{|n|}}} \cdot \bar{a}_n.$$

- Expanding the previous equation by $\hbar$, we have

$$\lambda_n = \Big[\beta^{1/2}\hbar + \frac{n}{4}\frac{1+p^n}{1-p^n}(1-\beta)\beta^{1/2}\hbar^2$$

$$+\frac{n^2}{96}\left(4(2-3\beta+2\beta^2)\beta^{1/2} - 3\frac{(1+p^n)^2}{(1-p^n)^2}(1-\beta)^2\beta^{1/2}\right)\hbar^3 + O(\hbar^4)\Big]\cdot \bar{a}_n.$$

- Then the constant term $[\eta(z;p)]_1$ has the next $\hbar$-expansion :

$$[\eta(z;p)]_1 = 1 + \beta\sum_{n\geq 1}\bar{a}_{-n}\bar{a}_n\hbar^2 + \Big[\beta(1-\beta)\sum_{n\geq 1}\frac{n}{2}\frac{1+p^n}{1-p^n}\bar{a}_{-n}\bar{a}_n$$

$$+\frac{\beta^{3/2}}{2}\sum_{n,m\geq 1}\left(\bar{a}_{-n}\bar{a}_n\bar{a}_{n+m} + \bar{a}_{-n-m}\bar{a}_n\bar{a}_m\right)\Big]\hbar^3 + O(\hbar^4).$$

Remark. The coefficient of $\hbar^3$ in this expansion coincide with the operator $M(q, t_1, t_2)$ used by Okounkov and Pandharipande in their computation of the quantum cohomology of $\mathrm{Hilb}(\mathbb{A}^2)$.