7・19

☆講義アンケート.

§10.6 外積 つづき

V: 線形空間, $n \in \mathbb{N}$

$$\Lambda^n V := V^{\otimes n} / \langle v_1 \otimes \cdots \otimes v_n \mid v_1, \ldots, v_n \in V,\ 1 \leq {}^\exists i < {}^\exists j \leq n,\ v_i = v_j \rangle$$

n次外積空間

$$\overline{v_1 \otimes \cdots \otimes v_n} =: v_1 \wedge \cdots \wedge v_n$$

Eg.10.6.2. $n=2$. $\Lambda^2 V$において

- $(cv + c'v') \wedge w = c(v \wedge w) + c'(v' \wedge w)$

  $v \wedge (cw + c'w') = c(v \wedge w) + c'(v \wedge w')$
- $v \wedge v = 0$ $\quad \left(c, c' \in K,\ v, v', w, w' \in V \right)$
- $v \wedge w = -w \wedge v$

$\because$ $(v+w) \wedge (v+w) = 0$

$\overset{\shortparallel}{v \wedge v} + v \wedge w + w \wedge v + w \wedge w = v \wedge w + w \wedge v$ $\square$

Lem.10.6.3. $d := \dim V < \infty$, $v_1, \ldots, v_d$: Vの基底

$\Rightarrow \{ v_{i_1} \wedge \cdots \wedge v_{i_n} \mid 1 \leq i_1 < \cdots < i_n \leq d \}$

は $\Lambda^n V$ の基底.

特に $\dim \Lambda^n V = \binom{d}{n}$

$n > d+1$ なら $\Lambda^n V = \{0\}$. $\square$

§13. 復習

問 13.4.3

$$V := \{ f(x,y,z) \in \mathbb{R}[x,y,z] \mid \text{斉次3次式} \} : \text{実線形空間}$$

$$\cup$$

$$W := \left\{ f \in V \;\middle|\; \begin{array}{l} \text{対称式、つまり} \\ f(x,y,z) = f(x,z,y) = f(y,x,z) = f(y,z,x) \\ \qquad = f(z,x,y) = f(z,y,x) \end{array} \right\} \text{部分空間}$$

(1) $\dim_{\mathbb{R}} V = ?$

(3) $m_1 := xyz$, $m_2 := x^2y + xy^2 + x^2z + xz^2 + yz^2 + y^2z$

$m_3 := x^3 + y^3 + z^3$ が Wの基底である事を示せ

(4) $p_1 := x^3 + y^3 + z^3$, $p_2 := (x^2+y^2+z^2)(x+y+z)$, $p_3 := (x+y+z)^5$

がWの基底である事を示せ

(5) W上の実対称型式 $\langle \cdot, \cdot \rangle$ を

$$(\langle p_i, p_j \rangle)_{i,j=1}^{3} = \begin{pmatrix} 3 & & \\ & 2 & \\ & & 6 \end{pmatrix}$$

で定める. m_1, m_2, m_3 を Gram-Schmidt 直交化して得られる

正規直交基底 s_1, s_2, s_3 を求めよ.

(6) 以下を示せ.

$$s_1 = a_1/\Delta, \quad s_2 = a_2/\Delta, \quad s_3 = a_3/\Delta$$

$$a_1 := \begin{vmatrix} x^3 & y^3 & z^3 \\ x^2 & y^2 & z^2 \\ x & y & z \end{vmatrix}, \quad a_2 := \begin{vmatrix} x^4 & y^4 & z^4 \\ x^2 & y^2 & z^2 \\ 1 & 1 & 1 \end{vmatrix}, \quad a_3 := \begin{vmatrix} x^5 & y^5 & z^5 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$$

$$\Delta := \begin{vmatrix} x^2 & y^2 & z^2 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$$

(1) 単項式 $x^3, y^3, z^3, x^2y, xy^2, x^2z, xz^2, y^2z, yz^2, xyz$ がVの基底.

$\therefore \dim V = 10$ ($\{x, y, z\}$から重複を許して3つ選ぶ… $\binom{3+2}{2}$通り))

(3) $m_1, m_2, m_3 \in W$ と一次独立性は略.

$\forall f \in W$ が一次結合で書ける事を示す. (1)のVの基底を使って

$V \supset W \ni f = \sum_{i,j,k} a_{ijk} x^i y^j z^k$

$= a_{300}x^3 + a_{030}y^3 + a_{003}z^3 + a_{210}x^2y + \cdots + a_{111}xyz$

と書くと、$f \in W$より $a_{300} = a_{030} = a_{003}$ (=: c とおく)

$a_{210} = a_{120} = \cdots = a_{012}$ (=: b とおく)

$a := a_{111}$ とおくと $f = am_1 + bm_2 + cm_3$

(4) $p_1 = m_3$, $p_2 = m_3 + m_2$, $p_3 = m_3 + 3m_2 + 6m_1$ より

$(p_1\ p_2\ p_3) = (m_1\ m_2\ m_3)\begin{pmatrix} 0 & 0 & 6 \\ 0 & 1 & 3 \\ 1 & 1 & 1 \end{pmatrix}$ これは可逆行列.

(5) $m_1 = \frac{1}{3}p_1 - \frac{1}{2}p_2 + \frac{1}{6}p_3$, $m_2 = -p_1 + p_2$, $m_3 = p_1$

・$u_1 := m_1$ $\langle u_1, u_1 \rangle = \frac{1}{9}\cdot 3 + \frac{1}{4}\cdot 2 + \frac{1}{36}\cdot 6 = 1$

$\therefore s_1 = \frac{1}{\|u_1\|}u_1 = u_1 = m_1 = xyz$

・$u_2 := m_2 - \langle m_2, s_1 \rangle s_1$

$\langle m_2, s_1 \rangle = \langle -p_1 + p_2, \frac{1}{3}p_1 - \frac{1}{2}p_2 + \frac{1}{6}p_3 \rangle = \frac{-1}{3}\cdot 3 + \frac{-1}{2}\cdot 2 = -2$

$\therefore u_2 = m_2 + 2m_1 = \frac{-1}{3}p_1 + \frac{1}{3}p_3$

$\langle u_2, u_2 \rangle = \frac{1}{9}\cdot 3 + \frac{1}{9}\cdot 6 = 1$

$\therefore s_2 = \frac{1}{\|u_2\|}u_2 = u_2 = m_2 + 2m_1 = (x+y)(y+z)(z+x)$

・$u_3 := m_3 - \langle m_3, s_1 \rangle s_1 - \langle m_3, s_2 \rangle s_2$

$\langle m_3, s_1 \rangle = \langle p_1, \frac{1}{3}p_1 + \cdots \rangle = \frac{1}{3}\cdot 3 = 1$

$\langle m_3, s_2 \rangle = \langle p_2, \frac{-1}{3}p_1 + \cdots \rangle = \frac{-1}{3}\cdot 3 = -1$

$\therefore u_3 = m_3 - s_1 + s_2 = m_1 + m_2 + m_3 = \frac{1}{3}p_1 + \frac{1}{2}p_2 + \frac{1}{6}p_3$

$\langle u_3, u_3 \rangle = \frac{1}{9}\cdot 3 + \frac{1}{4}\cdot 2 + \frac{1}{36}\cdot 6 = 1$

$\therefore s_3 = u_3 = m_1 + m_2 + m_3 = \sum_{i+j+k=3} x^i y^j z^k$

(6) ・ $a_1 = xyz\Delta$　　∴ $a_1/\Delta = xyz = s_1$

・a_2は次の項で割り切れる：$(x-y)(y-z)(x-z) = \Delta$

$(x+y)(y+z)(z+x) = s_2$

a_2は6次式で x^4y^2 の係数は1.　∴ $a_2 = \Delta \cdot s_2$

・ $a_3 = xy(x^4-y^4) + yz(y^4-z^4) + zx(z^4-x^4)$

$= \quad〃 \quad + (y^5-x^5)z + (x-y)z^5$

$= (x-y)\left(xy(x+y)(x^2+y^2) - (x^4+x^3y+x^2y^2+xy^3+y^4)z + z^5\right)$

$= (x-y)(y-z)\left(x(x+y)(x^2+y^2) - z(y+z)(y^2+z^2)\right)$

$= (x-y)(y-z)(x-z)\cdot s_3$