§4. Derived Hall alg.

§4.1. Definition … slide

§4.2. Hereditary case

D : loc. fin. DG cat. over $\mathbb{F}_q$
$P(D)$: cat. of DG D-modules. $\exists$ model str., stable
$Ho(P(D))$: triang. cat.
Assume $\exists$ t-str. on $Ho(P(D))$. A: its heart

Fact. 4.2.1. $DH(D) \supset \langle 1_{[M]} \mid [M] \in Iso(A) \rangle_{alg.} \cong H(A)$

$DH(D) = (F(P(D)), \circ)$, $H(A) = (F(A), \circ)$

Assume A is hereditary & the natural triang. functor $D^b(A) \to Ho(P(D))$ is equiv.

Fact. 4.2.2 $DH(D) \cong$ the assoc. alg. w/ generators $\{ Z_x^{[n]} \mid x \in Iso(A), n \in \mathbb{Z} \}$
and rel.

$$Z_x^{[n]} \circ Z_y^{[n]} = \sum_{z \in Iso(A)} g_{xy}^z Z_z^{[n]} \qquad g_{xy}^z \text{ ; str. cst. in } H(A)$$

$$Z_x^{[n]} \circ Z_y^{[n+1]} = \sum_{z, w} \gamma_{xy}^{zw} q^{-\chi(w,z)} Z_z^{[n+1]} \circ Z_w^{[n]}$$

$$Z_x^{[m]} \circ Z_y^{[n]} = Z_y^{[n]} \circ Z_x^{[m]} \cdot q^{(-1)^{m-n} \chi(x,y)} \qquad m-n < -1$$

$$\gamma_{x,y}^{z,w} = \frac{\#\{ 0 \to z \to y \to x \to w \to 0 \text{ exact in } A \}}{a_x a_y}$$

$\{ \cdots \text{exact} \}$
$= \{ M^\bullet = (\overset{0}{y} \to x) \mid$
$H^0(M^\bullet) = z, H^1(M^\bullet) = w \}$

§4.3. Examples of hereditary case

(1) [Hernandez-Leclerc. "Quantum Grothendieck rings & derived Hall cat."]

$A = \mathrm{Rep}_{\mathbb{F}_q} Q_{ADE}$ $\quad D = \mathbb{F}_q Q$ (path alg./$\mathbb{F}_q$, category w/ trivial dg str.)

$\Rightarrow DT(D) \cong K_\nu$: $\nu = \sqrt{q}$
ν-deformation of Grothendieck ring of the monoidal cat. $\mathcal{C}$ of fin-dim. $U_q(L\mathfrak{g}_{ADE})$-modules

(2) $\mathcal{A} = \mathrm{Rep}_{\mathbb{F}_q} Q_{\mathrm{Jordan}}$ (recall §1)

$\chi(-,-) = 0$ (§ 2.1)

$DT(\mathcal{D})$ gen: $\{ Z_\lambda^{[n]} \mid \lambda \in \mathrm{Par}, n \in \mathbb{Z} \}$

$\mathrm{Hcl}[n]$ ‖ $\Leftrightarrow \langle Z_\lambda^{[n]} \mid \lambda \in \mathrm{Par} \rangle_{\mathrm{alg}} \cong \mathrm{Hcl}$

rel: $Z_\lambda^{[n]} Z_\mu^{[n]} = \sum_\nu g_{\lambda\mu}^\nu Z_\nu^{[n]}$

$Z_\lambda^{[n]} Z_\mu^{[m]} = Z_\mu^{[m]} Z_\lambda^{[n]}$ $\quad |m-n|>1$

$Z_\lambda^{[n]} Z_\mu^{[n+1]} = \cdots$ $\quad (\#)$

Prp. 4.3.1. $k \in \mathbb{Z}_{>0}$, $n \in \mathbb{Z}$

$$b_k^{[n]} := \sum_{|\lambda|=k} (q;q)_{\ell(\lambda)-1} \cdot Z_\lambda^{[n]}$$

$$b_{-k}^{[n]} := \sum_{|\lambda|=k} (q;q)_{\ell(\lambda)-1} \cdot Z_\lambda^{[n+1]}$$

$\Delta(P_k) = P_k \otimes 1 + 1 \otimes P_k$

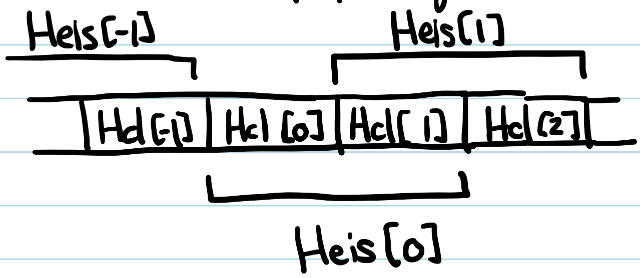
§1.4 $P_k = \sum (q;q)_{\ell(\lambda)-1} I_\lambda \in \mathrm{Hcl}$

$\mathrm{Hcl}[0] \ni b_k^{[0]} \mapsto P_k \in \mathrm{Hcl}$

$(q;q)_m := (1-q)(1-q^2)\cdots(1-q^m)$

$(\#) \Leftrightarrow b_k^{[n]} * b_\ell^{[n]} - b_\ell^{[n]} * b_k^{[n]} = \delta_{k+\ell,0} \frac{k}{q^k-1}$ Heisenberg alg

(c.f. Hopf pairing $\langle P_m, P_n \rangle = \delta_{m,n} \frac{n}{q^n-1}$ Fct. 1.4.6)



$\mathrm{Heis} = \langle P_\ell, \frac{k}{q^k-1} \frac{\partial}{\partial P_k} \ (k,\ell \in \mathbb{Z}_{>0}) \rangle_{\mathrm{alg}} \cong \mathrm{Heis}[n]$

= Differential operator ring on Λ

Λ ← ring of sym. func.

c.f. Hall-Littlewood sym. func. N.Jing's vertex operator

4 Derived Hall algebra

Toën introduced a version of Ringel-Hall algebra of complexes
using the model category of DG modules over a DG category.
Replace $\mathsf{A} = \mathsf{Rep}\, Q$ in Ringel-Hall algebra by the DG module category $\mathsf{M}(\mathsf{D})$.

4.1 DG category

A DG category over a commutative ring k is a category D whose morphism set is equipped with the structure of differential graded k-module and whose composition of morphisms is a homomorphism of differential graded k-modules.

$$\mathrm{Hom}_{\mathsf{D}}(M, N) = \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}^n_{\mathsf{D}}(M, N), \quad d\colon \mathrm{Hom}^n_{\mathsf{D}}(\cdot, \cdot) \longrightarrow \mathrm{Hom}^{n+1}_{\mathsf{D}}(\cdot, \cdot), \; d^2 = 0.$$

Example: The DG category $\mathsf{C}_{\mathsf{dg}}(\mathsf{A})$ of complexes over an additive category A.

- Recall that in the Ringel-Hall algebra $\mathbf{R}(\mathsf{A})$ for an abelian category A, the structure constant $g^N_{L,M}$ counts pairs $M \subset N$ of an object and its subobject.
- For a DG category D, we can do an analogous counting, using the model structure of the category $\mathsf{M}(\mathsf{D})$ of DG modules over D.
- Rough idea: instead of counting subobjects, count cofibrations up to homotopy.

Model structure

- A model structure on a category C consists of 3 classes of morphisms: fibrations, cofibrations, and weak equivalences, which are subject to certain axioms.
- It is designed to provide a natural setting of homotopy theory.
- Localization of C by weak equivalences gives the homotopy category $\mathrm{Ho}(\mathsf{C})$.

Examples:

1. $\mathsf{C}(k)$: the category of complexes of modules over a commutative ring k.
   It has a projective model structure with
   - A fibration is defined to be an epimorphism of complexes.
   - A weak equivalence is defined to be a quasi-isomorphism.
2. For a DG category D over k, a DG D-module is a DG functor $\mathsf{D}^{\mathsf{op}} \to \mathsf{C}_{\mathsf{dg}}(k)$.
   $\mathsf{M}(\mathsf{D})$: the category of DG D^{op}-modules.
   It has a model structure induced by the projective model structure of $\mathsf{C}(k)$.

4.2 The diagram of correspondence

Let D be a DG category over $k = \mathbb{F}_q$.

- $\mathsf{P}(\mathsf{D}) \subset \mathsf{M}(\mathsf{D})$: the full subcategory of perfect objects.
- $\mathsf{M}(\mathsf{D})^I := \mathsf{Fun}(I, \mathsf{M}(\mathsf{D}))$ with $I = \Delta^1$ the 1-simplex. It has the model structure induced levelwise by that of $\mathsf{M}(\mathsf{D})$.

We have a diagram (of left Quillen functors)

$$\begin{array}{ccc} \mathsf{M}(\mathsf{D})^I & \xrightarrow{\quad c \quad} & \mathsf{M}(\mathsf{D}) \\ {\scriptstyle p}\big\downarrow & & \\ \mathsf{M}(\mathsf{D}) \times \mathsf{M}(\mathsf{D}) & & \end{array} \qquad \begin{array}{ccc} (x \to y) & \longmapsto & y \\ \big\downarrow & & \\ (y \coprod_x 0, x) & & \end{array}$$

Restricting to the subcategories of cofibrant and perfect objects and of weak equivalences,

$$\begin{array}{ccc} w\big(\mathsf{P}(\mathsf{D})^I\big)^{\mathsf{cof}} & \xrightarrow{\quad c \quad} & w\mathsf{P}(\mathsf{D})^{\mathsf{cof}} \\ {\scriptstyle p}\big\downarrow & & \\ w\mathsf{P}(\mathsf{D})^{\mathsf{cof}} \times w\mathsf{P}(\mathsf{D})^{\mathsf{cof}} & & \end{array} \qquad \begin{array}{ccc} (x \rightarrowtail y) & \longmapsto & y \\ \big\downarrow & & \\ (y \coprod_x 0, y) & & \end{array}$$

Simplicial sets and the homotopy category of spaces

- Given a category C, the nerve construction yields a simplicial set $\mathrm{N}(\mathsf{C}) \in \mathsf{sSet}$.
- $\mathsf{sSet} := \mathsf{Fun}(\mathbf{\Delta}^{\mathsf{op}}, \mathsf{Set})$: the category of simplicial sets and simplicial maps.
  It has the Kan model structure where a fibration is a Kan fibration and a weak equivalence is a homotopy equivalence of geom. realizations.
- $\mathcal{H} := \mathrm{Ho}(\mathsf{sSet})$: the homotopy category of spaces. $[\cdot] : \mathsf{sSet} \to \mathcal{H}$.
  An object $X \in \mathrm{Ob}(\mathcal{H})$ is called a homotopy type.
  CG: the category of compactly generated Hausdorff spaces.
  The standard Quillen adjunction $|\ | : \mathsf{sSet} \rightleftarrows \mathsf{CG} : \mathrm{Sing}$ yields $\mathrm{Ho}(\mathsf{sSet}) \simeq \mathrm{Ho}(\mathsf{CG})$.

Define $X^{(0)}(\mathsf{D}), X^{(1)}(\mathsf{D}) \in \mathcal{H}$ by

$$X^{(0)}(\mathsf{D}) := \left[\mathrm{N}(w\mathsf{P}(\mathsf{D})^{\mathsf{cof}})\right], \quad X^{(1)}(\mathsf{D}) := \left[\mathrm{N}\big(w(\mathsf{P}(\mathsf{D})^{I})^{\mathsf{cof}}\big)\right].$$

Then we have the diagram of homotopy types

$$\begin{array}{ccc} X^{(1)}(\mathsf{D}) & \xrightarrow{\ c\ } & X^{0}(\mathsf{D}) \\ \Big\downarrow{\scriptstyle p} & & \\ X^{(0)}(\mathsf{D}) \times X^{(0)}(\mathsf{D}) & & \end{array}$$

Lemma. If the DG category D is locally finite, then

1. $p\colon X \to Y$ is proper (: $\iff$ for each $y \in \pi_0(Y)$, $|\{x \in \pi_0(X) \mid f(x) = y\}| < \infty$).
2. The homotopy types $X^{(i)}(\mathsf{D}) \in \mathcal{H}$ are locally finite.

Here we used:

Definition. A DG category D is called locally finite if the complex $\mathrm{Hom}_{\mathsf{D}}(M, N)$ is cohomologically bounded with finite-dimensional cohomology groups for any $M, N \in \mathsf{D}$.

Definition. A homotopy type $X \in \mathrm{Ob}(\mathcal{H})$ is called locally finite if for any $x \in X$ the group $\pi_i(X, x)$ is finite and there exists an $n \in \mathbb{N}$ such that $\pi_i(X, x)$ is trivial for $i > n$.
$\mathcal{H}^{\mathsf{lf}}$: the full subcategory of $\mathcal{H}$ spanned by locally finite objects

4.3 The definition of derived Hall algebra

For $X \in \mathcal{H}^{\mathrm{lf}}$, we denote $\mathrm{F}(X) := \{\alpha\colon \pi_0(X) \to \mathbb{C} \mid \text{having finite support}\}$.
For a proper morphism $f : X \to Y$ in $\mathcal{H}^{\mathrm{lf}}$, define $f^* : \mathrm{F}(Y) \to \mathrm{F}(X)$ by

$$f^*(\alpha)(x) := \alpha(f(x)) \quad (\alpha \in \mathrm{F}(Y),\ x \in \pi_0(X)).$$

Also, for a morphism $f : X \to Y$ in $\mathcal{H}^{\mathrm{lf}}$, define $f_! : \mathrm{F}(X) \to \mathrm{F}(Y)$ by

$$f_!(\alpha)(y) := \sum_{x \in \pi_0(X),\, f(x)=y} \alpha(x) \cdot \prod_{i>0} \Big(|\pi_i(X,x)|^{(-1)^i} \, |\pi_i(Y,y)|^{(-1)^{i+1}} \Big).$$

Theorem (Toën 2006)**.** Let D be a locally finite DG category over $\mathbb{F}_q$. Then

$$\mathbf{DH}(\mathsf{D}) = \mathrm{F}(X^{(0)}(\mathsf{D}))$$

has a structure of a unital associative algebra with the multiplication

$$\mu := c_! \circ p^* : \mathbf{DH}(\mathsf{D}) \otimes \mathbf{DH}(\mathsf{D}) \longrightarrow \mathbf{DH}(\mathsf{D}).$$

We call $\mathbf{DH}(\mathsf{D})$ the derived Hall algebra of D.