Yesterday $\mathcal{A}$: abelian cat., ess. small, finitary, $\mathbb{F}_q$-lin., gl.dim $< \infty$, enough projectives (*)

$\mathcal{P} \subset \mathcal{A}$: full subcat. of projectives $\quad C_2(\mathcal{P}) \subset C_2(\mathcal{A})$: cat. of 2-periodic cpx.

Dfn. 3.2.5. $BH(\mathcal{A}) := R(C_2(\mathcal{P}))[[M^\bullet]^{-1} \mid H^*(M^\bullet)=0]$: Bridgeland-Hall alg.

Lem. 3.2.2. $H^*(M^\bullet)=0 \Rightarrow M^\bullet \cong K_P \oplus K_Q^*$ $\quad K_P = (P \underset{0}{\overset{id}{\rightleftarrows}} P)$, $K_Q^* = (Q \underset{-id}{\overset{0}{\rightleftarrows}} Q)$

Claim/Dfn. $\exists$ 2 grp homs $K_0(\mathcal{A}) \rightrightarrows BH(\mathcal{A})$, $\alpha \mapsto K_\alpha, K_\alpha^*$

$\forall \alpha \in K_0(\mathcal{A})$ $\exists P, Q \in Ob(\mathcal{P})$ $\alpha = \overline{P} - \overline{Q}$ $\quad K_\alpha := [K_P]*[K_Q]^{-1}$

$K_\alpha^* := [K_P^*]*[K_Q^*]^{-1}$

§3.3 三角分解

Thm. 3.3.2. $\mathcal{A}$: abelian cat. s.t. (*) and hereditary. $\quad \nu := q^{1/2} \in \mathbb{C}$

$\tilde{R}(\mathcal{A}) = (\tilde{F}(\mathcal{A}), *)$: extended Ringel-Hall alg. $\quad \tilde{F}(\mathcal{A}) = F(\mathcal{A}) \otimes \mathbb{C}[K_0(\mathcal{A})]$

$\exists$ 2 alg. emb. $\tilde{R}(\mathcal{A}) \rightrightarrows DH(\mathcal{A})$, $\quad a_M \cdot [M] k_\alpha \mapsto E_M * K_\alpha,\ F_M * K_\alpha^*$

$\exists$ lin. isom. $\tilde{R}(\mathcal{A}) \otimes \tilde{R}(\mathcal{A}) \xrightarrow{\sim} DH(\mathcal{A})$, $\quad [M]k_\alpha \otimes [N]k_\beta \mapsto \frac{1}{a_M a_N} E_M * K_\alpha * F_N * K_\beta^*$

(E_M, F_M ... Dfn. 3.3.6)

$\forall M \in ob(\mathcal{A})$ $\exists$ proj. resol. $0 \to P \xrightarrow{f} Q \to M \to 0$ ($\because$ hereditary)

$\mathcal{A}$: Hom-finite $\Rightarrow$ Krull-Schmidt $\quad P = \oplus_i P_i \quad Q = \oplus_j Q_j$ fin. sum. decomp. into indecomposables

$\Rightarrow f = \oplus_{ij} f_{ij}$. $\quad f_{ij} : P_i \to Q_j$

Dfn. 3.3.3 The resolution is <u>minimal</u> if $\nexists f_{ij}$ isom.

Lem. 3.3.4. $\forall M \in ob(\mathcal{A})$ $\exists$ minimal proj. resol., unique up to isom.

$$0 \to P_M \xrightarrow{f_M} Q_M \to M \to 0$$

Dfn. 3.3.5. $C_M^\bullet := (\underset{1}{P_M} \underset{0}{\overset{f_M}{\rightleftarrows}} \underset{0}{Q_M})$ $\quad H^i(C_M^\bullet) = \begin{cases} M & i=0 \\ 0 & i=1 \end{cases}$

$*$: $C_2(\mathcal{P}) \to C_2(\mathcal{P})$ invol. $\quad M^\bullet = (M^1 \underset{d^0}{\rightleftarrows} M^0) \mapsto (M^0 \underset{-d^1}{\overset{-d^0}{\rightleftarrows}} M^1) = (M^\bullet)^*$

$\rightsquigarrow *$: $BH(\mathcal{A}) \to BH(\mathcal{A})$ invol. alg. hom

Dfn. 3.3.6. $\forall M \in ob(\mathcal{A})$ $\quad E_M := \frac{\nu^{\chi(P_M, M)}}{a_{C_M}} K_{-\overline{P_M}} * [C_M^\bullet]$, $\quad F_M := (E_M)^* \in BH(\mathcal{A})$

(Outline of proof of Thm. 3.3.2)

Prp. 3.3.9. $\exists$ alg. emb. $R(\mathcal{A}) \hookrightarrow BH(\mathcal{A})$, $\quad [\![M]\!] := a_M \cdot [M] \mapsto E_M$ ⇐ 証明はP3.

Cor. 3.3.10. $\exists$ 〃 $\quad \tilde{R}(\mathcal{A}) \hookrightarrow BH(\mathcal{A})$, $\quad [\![M]\!] k_\alpha \mapsto E_M * K_\alpha$

(prf.) 3.3.9. & 3.3.2. ($K_P * E_M$ etc.) $\square$

By the invol. $*$, $\exists \tilde{R}(\mathcal{A}) \hookrightarrow BH(\mathcal{A})$, $\quad [\![M]\!] k_\alpha \mapsto (E_M * K_\alpha)^* = F_M * K_\alpha^*$

The 1st half is finished.

The 2nd half of 3.3.2 is shown by

Lem. 3.3.11. $\forall M^\bullet \in \mathrm{Ob}(C_2(P))$ $\exists!\, A, B \in \mathrm{Ob}(A)$, $\exists!\, P, Q \in \mathrm{Ob}(P)$ (up to isom.)
$$M^\bullet \cong C_A^\bullet \oplus (C_B^\bullet)^* \oplus K_P \oplus K_Q^*$$
$(A = H^0(M^\bullet),\ B = H^1(M^\bullet))$ $\square$

Cor. 3.3.12 $\exists$ lin. isom $R(A) \otimes \mathbb{C}[K_0(A)] \otimes R(A) \xrightarrow{\sim} BH(A)$, $[M] \otimes K_\alpha \otimes [N] \mapsto E_M * K_\alpha * F_N$

§3.4 Relation to quantum groups

$\left\{\begin{array}{ll} \text{§3.2:} & E*K,\ F*K,\ K*K \\ \text{§3.3.} & E*E \quad F*F \\ \text{rest} & E*F \end{array}\right.$

Thm. 3.4.3. Q: quiver without loops. $\nu = q^{1/2} \in \mathbb{C}$
$\exists$ alg. emb. $U_\nu(\mathfrak{g}_Q) \hookrightarrow BH(\mathrm{Rep}_{\mathbb{F}_q}^{nil} Q)_{red}$
$$E_i \longmapsto \frac{1}{q-1} E_{S_i}$$
$$F_i \longmapsto \frac{-\nu}{\nu - \nu^{-1}} F_{S_i}$$
$$K_i \longmapsto K_{\overline{S_i}}$$
It is an isom. if Q is of fin type. $(\mathfrak{g}_Q = \mathfrak{g}_{ADE})$

Thm. 3.4.5. A: as in Thm 3.3.2. $(\Rightarrow \tilde{R}(A) \otimes \tilde{R}(A) \xrightarrow{\sim} BH(A)$ as lin. sp.$)$
$\Rightarrow BH(A)$ is the Drinfeld double of $\tilde{R}(A)$

Dfn. 3.4.4. $R = (R, *, \Delta, 1, \varepsilon)$: bialg. w/ Hopf pairing $\langle \cdot, \cdot \rangle$
$\Rightarrow \exists!$ assoc. alg. str $\circ$ on $R \otimes R$ s.t.
$$(a \otimes 1) \circ (1 \otimes b) = a \otimes b$$
$$\sum \langle a_{(2)}, b_{(1)} \rangle (a_{(1)} \otimes 1) \circ (1 \otimes b_{(2)}) = \sum \langle a_{(1)}, b_{(2)} \rangle (1 \otimes b_{(1)}) \circ (a_{(2)} \otimes 1)$$
$$\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$$
The unital assoc. alg. $(R \otimes R, \circ, 1)$: Drinfeld double
It has a str. of bialg, s.t. $R \rightrightarrows R \otimes R$ are bialg. emb.
If R is a Hopf alg. then it also has a str. of Hopf alg s.t. ... are emb

(Sketch of proof of 3.3.9) $\exists$ alg. emb. $R(A) \hookrightarrow BH(A)$, $[\![M]\!] = a_M \cdot [M] \mapsto E_M$
$$[\![M_1]\!] * [\![M_2]\!] = \nu^{\cdot} \sum_{[N]} g_{M_1 M_2}^{N} \cdot a_{M_1} a_{M_2} \cdot [N] = \nu^{\cdot} \sum_{[N]} e_{M_1 M_2}^{N} / a_N \cdot [\![N]\!]$$
$\quad$ $\tau |\{0 \to M_2 \to N \to M_1 \to 0\}|$

$\mathrm{Ext}_A^1(L,M)_N \subset \mathrm{Ext}_A^1(L,M)$
isom. cls. of exact seq. with middle $\cong N$
$$= \nu^{\cdot} \sum_{[N]} \frac{|\mathrm{Ext}_A^1(M_1, M_2)_N|}{|\mathrm{Hom}_A(M_1, M_2)|} [\![N]\!] \quad \text{Q.3.6}$$

$P_i \overset{f_i}{\hookrightarrow} Q_i \twoheadrightarrow M_i$ min. proj. resol.
$$E_{M_1} * E_{M_2} = \nu^{*} K_{-\overline{P_1}} * [\![C_{M_1}^\bullet]\!] * K_{-\overline{P_2}} * [\![C_{M_2}^\bullet]\!] \qquad [\![L^\bullet]\!] = \frac{[L^\bullet]}{a_{L^\bullet}}$$
$$= \nu^{*} K_{-\overline{P_1} - \overline{P_2}} * [\![C_{M_1}^\bullet]\!] * [\![C_{M_2}^\bullet]\!]$$
prp. 3.2.3

$$[\![C\dot{M}_1]\!] * [\![C\dot{M}_2]\!] = \sum_{[N^\cdot]} \frac{\# \mathrm{Ext}^1_{C_2(A)}(C\dot{M}_1, C\dot{M}_2)_{N^\cdot}}{\underbrace{\# \mathrm{Hom}_{C_2(A)}(C\dot{M}_1, C\dot{M}_2)}} [\![N^\cdot]\!]$$

$$0 \to C\dot{M}_1 \to N^\cdot \to C\dot{M}_2 \to 0$$

$$\begin{array}{ccccccccc} 0 & \to & P_1 & \to & P_1 \oplus P_2 & \to & P_2 & \to & 0 \\ & & f_1 \downarrow \uparrow 0 & & u \downarrow \uparrow v & & f_2 \downarrow \uparrow 0 & & \\ 0 & \to & Q_1 & \to & Q_1 \oplus Q_2 & \to & Q_2 & \to & 0 \\ & & \Downarrow & & \Downarrow & & \Downarrow & & \\ 0 & \to & M_1 & \to & N & \to & M_2 & \to & 0 \end{array}$$

f_1, f_2 : inj. $\Rightarrow u$: inj.
$u \circ v = 0 \Rightarrow v = 0$

$\rightsquigarrow$ $N^\cdot \cong C\dot{N}$, $\mathrm{Ext}^1_{C_2(A)}(C\dot{M}_1, C\dot{M}_2)_{N^\cdot} \cong \mathrm{Ext}^1_A(M_1, M_2)_N$

$\exists$ exact seq. $0 \to \mathrm{Hom}_A(Q_2, P_1) \to \mathrm{Hom}_{C_2(A)}(C\dot{M}_1, C\dot{M}_2) \to \mathrm{Hom}_A(M_1, M_2) \to 0$

$\uparrow \# = q^{\chi(Q_2, P_1)}$

$\therefore$ $\rightsquigarrow = \nu^{*} \cdot \# \mathrm{Ext}^1_A(M_1, M_2)_N / \# \mathrm{Hom}_A(M_1, M_2)$

$$\therefore\ E_{M_1} * E_{M_2} = \nu^{*} \sum_{[N] \in \mathrm{Iso}(A)} \frac{\# \mathrm{Ext}^1_A(M_1, M_2)_N}{\# \mathrm{Hom}_A(M_1, M_2)} \underbrace{K_{-P_1-P_2} * [\![C\dot{N}]\!]}_{\nu^{*} E_N} \qquad \square$$

(Relation of E & F) $\quad BH(A) \ni E_{S_i}$

Lem. 3.4.1. $M, N \in \mathrm{Ob}(A)$. $\mathrm{Hom}_A(M,N) = 0 = \mathrm{Hom}_A(N,M) \Rightarrow E_M * F_N = F_N * E_N$

(sketch of prf.) $\mathrm{Ext}^1_{C_2(A)}(C\dot{M}, (C\dot{N})^*) \underset{3.5.2.}{\cong} \mathrm{Hom}_{Ho_2(A)}(C\dot{M}, C\dot{N}) \cong \mathrm{Hom}_A(M,N)$

$\uparrow$ homotopy cat. of 2-periodic cpx.

It vanishes by assump.

$\therefore$ $[\![C\dot{M}]\!] * [\![(C\dot{N})^*]\!] = [\![C\dot{M} \oplus (C\dot{N})^*]\!] \cdot \nu^{*}$ $\quad \square$

Lem. 3.4.2. $M \in \mathrm{Ob}(A)$ $\mathrm{End}_A(M) = \mathbb{F}_q \Rightarrow E_M * F_M - F_M * E_M = (q-1)(K^*_{\hat{M}} - K_{\hat{M}})$

(sketch)

$\mathrm{Ext}^1_{C_2(A)}(C\dot{M}, (C\dot{M})^*) \cong \mathrm{Hom}_A(M,M) = \mathbb{F}_q$

$0 \to (C\dot{M})^* \to N^\cdot \to C\dot{M} \to 0$ non-split $\Rightarrow$ by long exact seq. $H^*(N^\cdot) = 0$

$\Rightarrow N^\cdot \cong K_P \oplus K^*_Q \quad P, Q \in \mathrm{Ob}(P)$

$\rightsquigarrow$ $[\![(C\dot{M})^*]\!] * [\![C\dot{M}]\!] = \nu^{*}([\![C\dot{M}^* \oplus C\dot{M}]\!] + (q-1)[\![K_P \oplus K^*_Q]\!])$ $\quad \square$

Lem. 3.5.2. $M^\cdot, N^\cdot \in \mathrm{Ob}(C_2(P))$

$\mathrm{Ext}^1_{C_2(A)}(N^\cdot, M^\cdot) \cong \mathrm{Hom}_{Ho_2(A)}(N^\cdot, (M^\cdot)^*)$

(prf) $0 \to M^\bullet \to P^\bullet \to N^\bullet \to 0$

$$\begin{tikzcd}
0 \arrow[r] & M^1 \arrow[r, "i"] \arrow[d, shift right, "f_M"'] & M^1 \oplus N^1 \arrow[r, "p"] \arrow[d, shift right, "f"'] & N^1 \arrow[r] \arrow[d, shift right, "f_N"'] & 0 \\
0 \arrow[r] & M^0 \arrow[r, "i"'] \arrow[u, shift right, "g_M"'] & M^0 \oplus N^0 \arrow[r, "p"'] \arrow[u, shift right, "g"'] & N^0 \arrow[r] \arrow[u, shift right, "g_N"'] & 0
\end{tikzcd}$$

$f = \begin{bmatrix} f_M & S^1 \\ 0 & g_M \end{bmatrix}$ $g = \begin{bmatrix} f_N & S^0 \\ 0 & g_N \end{bmatrix}$

$S^i : N^i \to M^{i+1}$

$fg = 0 = gf \Leftrightarrow S^\bullet : N^\bullet \to (M^\bullet)^*$ morphism of cpx

$\therefore$ extension $P^\bullet \leftrightsquigarrow$ morph. $S^\bullet$

$P^\bullet \sim \tilde{P}^\bullet \Leftrightarrow \exists k^\bullet : P^\bullet \xrightarrow{\sim} \tilde{P}^\bullet$

$\Leftrightarrow$ $k^1 = \begin{bmatrix} id_{M^1} & h^1 \\ 0 & id_{N^1} \end{bmatrix}$, $k^2 = \begin{bmatrix} id_{M^0} & h^0 \\ 0 & id_{N^0} \end{bmatrix}$

$h^i : N^i \to M^i$,

$k^\bullet$: morph. of two-periodic cpx.

$\Leftrightarrow$

$$\begin{tikzcd}
M^1 \oplus N^1 \arrow[r, "k^1"] \arrow[d, shift right, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & M^1 \oplus N^1 \arrow[d, shift right, "\tilde{f}"'] \\
M^0 \oplus N^0 \arrow[r, "k^0"'] \arrow[u, shift right, "g"'] & M^0 \oplus N^0 \arrow[u, shift right, "\tilde{g}"']
\end{tikzcd}$$

$\tilde{f} = \begin{bmatrix} f_M & \tilde{S}^1 \\ & g_M \end{bmatrix}$

$\Leftrightarrow$ $h^\bullet$ is a homotopy between $S^\bullet$ & $\tilde{S}^\bullet$