Yesterday -- $R(A) = (F(A), *, \Delta, 1, \varepsilon)$: Ringel-Hall bialg. w/ Hopf pairing $\langle \cdot, \cdot \rangle$

for cat. A s.t. (1) loc. small, abelian (2) finitary

(3) gl. dim $< \infty$, $|\mathrm{Ext}^i_A(\cdot,\cdot)| < \infty$

(4) fin. subobj. (5) hereditary

§2.4 Extended Ringel-Hall alg.

A : (1)–(5) $\qquad \mathbb{C}[K_0(A)] \ni k_\alpha \leftarrow\!\!| \; \alpha \in K_0(A)$ $\quad k_\alpha k_\beta = k_{\alpha+\beta}$

$\tilde{F}(A) = F(A) \otimes \mathbb{C}[K_0(A)] \ni [M] \otimes k_\alpha = [M] k_\alpha$

Prp 2.4.2 & 2.4.3. (prf.. Q.2.3, 2.4)

$[M] k_\alpha * [N] k_\beta := (\alpha, \bar{N})_m \cdot ([M] * [N]) k_{\alpha+\beta} \qquad (\alpha,\beta)_m := (\alpha,\beta)_m \langle \beta, \alpha \rangle_m$

$\Delta([N] k_\gamma) := \sum_{[L],[M]} \langle \bar{L}, \bar{M} \rangle_m \dfrac{e^N_{LM}}{a_N} ([L] k_{\gamma + \bar{M}}) \otimes ([M] k_\gamma)$

$1 := [0] k_0$, $\quad \varepsilon([M] k_\alpha) := \delta_{M,0}$, $\quad \langle [M] k_\alpha, [N] k_\beta \rangle = \delta_{M,N} (\alpha,\beta)_m / a_M$

$\Rightarrow \tilde{R}(A) := (\tilde{F}(A), *, \Delta, 1, \varepsilon)$: bialg. w/ Hopf pairing $\langle \cdot, \cdot \rangle$

: extended Ringel-Hall alg. (Fact 2.4.5 : Hopf alg)

§2.5. Relation to quantum groups

Q : quiver without loops, I : vertex set $\ni i, j$

$\underline{Q}$: underlying graph. $n_{ij} := \#\{\text{arrows } i \to j\}$ $\quad a_{ij} := 2\delta_{ij} - n_{ij}$

$\Rightarrow (a_{ij})_{i,j \in I}$: symmetric. generalized Cartan matrix

$\Rightarrow \mathfrak{g}_{\underline{Q}}$: KM Lie alg.

$\nu := q^{1/2} \in \mathbb{C}$

$U_\nu(\mathfrak{g}_{\underline{Q}})$: quantum env. alg. of $\mathfrak{g}_{\underline{Q}}$, specialized at ν

$= \langle E_i, F_i, K_i^{\pm 1} \mid i \in I) \mid (p.30) \rangle_{alg}$

$U_\nu(\mathfrak{b}_{\underline{Q}}) = \langle E_i, K_i^{\pm 1} \rangle_{alg}$: Borel subalg.

$\langle M, N \rangle_m = \nu^{\chi(M,N)}$ $\qquad \chi(M,N) = \sum_{i \geq 0} (-1)^i \mathrm{Ext}^i_A(M,N)$

Thm 2.5.3. (Ringel) $\exists \mathbb{C}$-alg. embedding $U_\nu(\mathfrak{b}_{\underline{Q}}) \hookrightarrow \tilde{R}(\mathrm{Rep}^{nil}_{\mathbb{F}_q} Q)$. $E_i \mapsto [S_i]$, $K_i \mapsto k_{\bar{S_i}}$

$S_i = (\oplus_{j \in I} \mathbb{F}_q \delta_{ij}, 0)$

It is an isom. $\Leftrightarrow$ Q is of fin. typ. ($\Leftrightarrow$ $\underline{Q}$ Dynkin ADE)

Thm. 2.5.4. (Green, Xiao) It is a Hopf alg. embedding.

Q.2.5. Check these statements for $Q = Q(A_2)$ $\bullet \to \bullet$

§3. Bridgeland-Hall alg.

$Q = Q(ADE)$ $\quad U_v(\mathfrak{b}_Q) \xrightarrow{\sim} \tilde{R}(\mathrm{Rep}_{\mathbb{F}_q} Q)$ $\quad E_i \mapsto [S_i],\ K_i \mapsto \tilde{k}_{S_i}$

$\mathrm{Rep}^{\mathrm{nil}}_{\mathbb{F}_q} Q$

$U_v(\mathfrak{b}_Q) \cap U_v(\mathfrak{g}_Q) \dashrightarrow\ ?$ $\leftarrow$ Bridgeland-Hall alg.

§3.1. two-periodic complexes

A: abel. cat. $\quad C_2(A)$: cat. of 2-periodic cpx.

ob: $M^\bullet = (M^1 \overset{d^1}{\underset{d^0}{\rightleftarrows}} M^0) = (\cdots \to M^0 \xrightarrow{d^0} M^1 \xrightarrow{d^1} M^0 \to \cdots)$, $\quad M_i \in \mathrm{Ob}(A)$, $d^0 d^1 = d^1 d^0 = 0$

$$\mathrm{Hom}_{C_2(A)}(M^\bullet, N^\bullet) \ni S^\bullet = \begin{array}{ccc} M^1 & \overset{d_M^1}{\rightleftarrows} & M^0 \\ {\scriptstyle s^1}\downarrow & & \downarrow{\scriptstyle s^0} \\ N^1 & \underset{d_N^0}{\rightleftarrows} & N^0 \end{array}, \quad s^{i+1} d_M^i = d_N^i s^i \quad (i \in \mathbb{Z}/2\mathbb{Z})$$

natural compos. $t^\bullet \circ s^\bullet$

$C_2(A)$ is abelian.

$0^\bullet = (0 \rightleftarrows 0)$, $M^\bullet \oplus N^\bullet$, $\mathrm{Ker}(s^\bullet) = (\mathrm{Ker}\, s^1 \rightleftarrows \mathrm{Ker}\, s^0)$

$\mathrm{Cok}(s^\bullet) = (\mathrm{Cok}\, s^1 \rightleftarrows \mathrm{Cok}\, s^0)$

exact seq. $0 \to L^\bullet \xrightarrow{s^\bullet} M^\bullet \xrightarrow{t^\bullet} N^\bullet \to 0$

$$\Leftrightarrow \quad \begin{array}{ccccccccc} 0 & \to & L^0 & \to & M^0 & \to & N^0 & \to & 0 \\ & & \downarrow\uparrow & & \downarrow\uparrow & & \downarrow\uparrow & & \\ 0 & \to & L^1 & \xrightarrow{s^1} & M^1 & \xrightarrow{t^1} & N^1 & \to & 0 \end{array} \quad \text{exact \& com.}$$

Lem 3.3.1. A satisfies (1) & (2) $\Rightarrow$ so does $C_2(A)$

(pf ... Q.3.1.)

$\rightsquigarrow$ $H(C_2(A)) = (F(C_2(A)), \circ)$ assoc. alg.

but not a desired one ...

Asmp 3.1. A is an abelian category s.t.

- ess. small, $|\mathrm{Hom}_A(\cdot,\cdot)| < \infty$
- $\mathbb{F}_q$-lin. $\leftarrow$ for simplicity
- having enough projectives & gl. dim $< \infty$

$P \subset A$: full subcat of projectives

$C_2(P) \subset C_2(A)$ full subcat of $P^\bullet = (P^1 \rightleftarrows P^0)$ w/ P^i projective

Lem. 3.3.2. $C_2(P)$ is closed under extension.

(pf ... Q.3.2)

$$F(C_2(A)) = \bigoplus_{[M^\bullet]\in \mathrm{Iso}(\ldots)} \mathbb{C}[M^\bullet]$$
$$\cup$$
$$F(C_2(P)) := \bigoplus_{[P^\bullet]\in \mathrm{Iso}(\ldots)} \mathbb{C}[P^\bullet] \quad \text{closed under } \circ \quad (\because \text{Lem. 3.3.2})$$

Lem. 3.1.3. $\nu = q^{1/2} \in \mathbb{C}$, $M^\bullet, N^\bullet \in \mathrm{Ob}(C_2(P))$

$$[M^\bullet] * [N^\bullet] := \nu^{\chi(M_0,N_0)+\chi(M_1,N_1)} [M^\bullet]\circ[N^\bullet]$$

$\Rightarrow$ $R(C_2(P)) := (F(C_2(P)), *, [0])$: unital assoc. $\mathbb{C}$-alg.

§3.2. Non-commutative localization

Lem. 3.2.2. $M^\bullet \in \mathrm{Ob}(C_2(P))$ acyclic ($\Leftrightarrow H^*(M^\bullet)=0$)

$M^1 \underset{d^0}{\overset{d^1}{\rightleftarrows}} M^0$

$\Rightarrow$ $M^\bullet \cong K_P \oplus K_Q^*$. $\exists P, Q \in \mathrm{Ob}(P)$

$$K_P = (P \underset{0}{\overset{\mathrm{id}}{\rightleftarrows}} P) \quad K_Q^* = (Q \underset{-\mathrm{id}}{\overset{0}{\rightleftarrows}} Q)$$

(prf.) $P := \mathrm{Im}\, d^0 = \mathrm{Ker}\, d^1$ $\quad Q = \mathrm{Ker}\, d^0 = \mathrm{Im}\, d^1$

$0 \to P \to M^1 \to Q \to 0$, $\quad 0 \to Q \to M^0 \to P \to 0$ exact

long exact seq. of $\mathrm{Hom}(-,-)$

$\rightsquigarrow$ $\mathrm{Ext}_A^i(P,-) \cong \mathrm{Ext}^{i+1}(Q,-) \cong \mathrm{Ext}^{i+2}(P,-)$ $\quad (i \geq 1)$

by gl.dim $A < \infty$, these vanishes $\therefore P, Q \in \mathrm{Ob}(P)$ $\quad \square$

Prp. 3.2.3. $\forall P \in \mathrm{Ob}(P)$, $\forall M^\bullet \in C_2(P)$
;Q

$$[K_P] * [M^\bullet] = \nu^{(\bar{P}, \bar{M}^1 - \bar{M}^0)_a} [M^\bullet] * [K_P]$$

$$[K_P^*] * [M^\bullet] = \nu^{-(\bar{P}, \bar{M}^1 - \bar{M}^0)_a} [M^\bullet] * [K_P^*]$$

$[K_P] * [K_Q] = [K_{P\oplus Q}]$ $\quad [K_P] * [K_Q^*] = [K_P \oplus K_Q^*]$, $\quad [K_P^*] * [K_Q^*] = [K_{P\oplus Q}^*]$

(prf ... §3.4)

$S := \{[M^\bullet] \mid M^\bullet \in \mathrm{Ob}(C_2(P)), \text{acyclic}\}$ is a multiplicative subset of $R(C_2(P))$ satisfying Ore condition

$\rightsquigarrow$ $R(C_2(P))[S^{-1}]$ assoc. alg.
localization

Dfn. 3.2.5. A satisfying Asmp 3.1.

$BH(A) := R(C_2(P))[\,[M^\bullet]^{-1} \mid \text{acyclic } M^\bullet\,]$ assoc. alg.

Dfn. 3.2.6. $BH(A)_{red} := BH(A) / (\,[K_P] * [K_{P^*}] = 1 \mid P \in ob(P)\,)$

↳ ideal.

§ 3.3 Relation to quantum groups (Tomorrow)

Thm. 3.3.8. Q : quiver without loops

$U_\nu(\mathfrak{g}_Q) \hookrightarrow BH(\mathrm{Rep}_{\mathbb{F}_q} Q)_{red}$ alg. emb.

$$\begin{aligned} E_i &\mapsto \boxed{[E_{S_i}]} \\ F_i &\mapsto \boxed{[E^*_{S_i}]} \\ K_i &\mapsto \boxed{K_{\overline{S_i}}} \end{aligned}$$

to be explained

It is an isom. if Q is of fin. type ($\forall$ conn. comp. of Q is ADE Dynkin)