§2 Ringel–Hall alg.

§2.1. Ring str.

Thm. 2.1.3. [Ringel] A : category (1) abelian, essentially small ($:\Leftrightarrow$ $\mathrm{Iso}(A)$ is a set)
(2) finitary i.e. $|\mathrm{Hom}_A(M,N)|, |\mathrm{Ext}^1_A(M,N)| < \infty$

$F(A) := \{ f : \mathrm{Iso}(A) \to \mathbb{C} \mid f([M]) = 0$ all but finitely many $[M] \in \mathrm{Iso}(A)\}$
: $\mathbb{C}$-lin. sp. w/ basis $\{1_{[M]} \mid [M] \in \mathrm{Iso}(A)\}$. $\underset{\substack{\shortparallel \\ 1_M}}{1_{[M]}}([N]) = \delta_{M,N}$

$\circ : F(A) \otimes F(A) \to F(A)$

$(f \circ g)([N]) = \sum_{M \subset N} f([N/M])\, g([M])$

$\Rightarrow H(A) := (F(A), \circ, 1_{[0]})$ is a unital assoc. alg.

Prop. 1.2.8. $L, M, N \in \mathrm{ob}(A)$ $G^N_{LM} := \{M' \in \mathrm{ob}(A) \mid M' \subset N,\ M' \cong M,\ N/M' \cong L\}$
$g^N_{LM} := |G^N_{LM}| < \infty$, $(1_L \circ 1_M)([N]) = g^N_{LM}$

(pf) $\cdot$ $\mathrm{Ex}^N_{L,M} := \{0 \to M \xrightarrow{f} N \xrightarrow{g} L \to 0 \mid$ exact in $A\} = \{(f,g) \mid \ldots\} \subset \mathrm{Hom}(M,N) \times \mathrm{Hom}(N,L)$ finite set by (2)

$A_{L,M} := \mathrm{Aut}(L) \times \mathrm{Aut}(M) \curvearrowright \mathrm{Ex}^N_{L,M}$ free.

$A_{L,M} \backslash \mathrm{Ex}^N_{L,M} = \{0 \to M' \xrightarrow{i} N \xrightarrow{q} N/M' \to 0 \mid M' \subset N,\ M' \cong M,\ N/M' \cong L\}$

i : embedding., q : proj. ← uniquely determined by M', N/M'

$= G^N_{LM}$

$\therefore\ g^N_{LM} = |\mathrm{Ex}^N_{L,M}| / |A_{L,M}| < \infty$

$(1_L \circ 1_M)([N]) = \sum_{M' \subset N} 1_L([N/M'])\, 1_M([M']) = \#\{M' \subset N \mid M' \cong M,\ N/M' \cong L\} = g^N_{LM}$ □

(well-def. of ∘) $1_L \circ 1_M = \sum_{[N] \in \mathrm{Iso}(A)} g^N_{LM} [N]$ enough to show $\#\{[N] \mid G^N_{LM} \neq \phi\} < \infty$

$\mathrm{Ext}^1_A(L,M) = \{0 \to M \to N \to L \mid$ exact in $A\}/\sim$ $= \#\{[N] \mid \mathrm{Ex}^N_{L,M} \neq \phi\}$

Fact. 2.0.1.

$\mathrm{Ext}^1_A(L,M) = (\bigcup_N \mathrm{Ex}^N_{L,M})/\sim$
fin. set by (2) ↗
$= \bigsqcup_{[N]} \mathrm{Ex}^N_{L,M}$

$(0 \to M \to N \to L) \sim (0 \to M \to N' \to L \to 0)$
$:\Leftrightarrow$
$$\begin{array}{ccccccccc} 0 & \to & M & \to & N & \to & L & \to & 0 \\ & & \| & & \downarrow f & & \| & & \\ 0 & \to & M & \to & N' & \to & L & \to & 0 \end{array}$$
$\exists f$. isom,

$\therefore \#\{[N] \mid \mathrm{Ex}^N_{L,M}\} < \infty$

(assoc. of ∘) $f \circ (g \circ h)([N]) = \sum_{M \subset N} f(N/M)\, g \circ h(M) = \sum_{L \subset M \subset N} f(N/M)\, g(M/L)\, h(L)$

$(f \circ g) \circ h\,(N) = \sum_{L \subset N} f \circ g(N/L)\, h(L) = \sum_{L \subset N,\ M' \subset N/L} f((N/L)/M')\, g(M')\, h(L)$

By the Isom. thm. $\{(L,M) \mid L \subset M \subset N\} \longrightarrow \{(L, M') \mid L \subset N,\ M' \subset N/L\}$
$(L,M) \overset{\text{bij}}{\longmapsto} (L, M/L)$

(1_0 is unit) Omit. □

Cor. 2.1.6. $H(A) \cong (\bigoplus_{[M] \in \mathrm{Iso}(A)} \mathbb{C}[M], \circ, [0])$ as $\mathbb{C}$-alg.

$1_M \mapsto [M]$, $[L] \circ [M] = \sum_{[N] \in \mathrm{Iso}(A)} g^N_{LM} [N]$,

$g^N_{LM} = e^N_{LM} / a_L a_M$. $a_M := |\mathrm{Aut}(M)|$, $e^N_{LM} = |\mathrm{Ex}^N_{LM}|$

Prp.2.1.7 $(H_{cl}, *, [0]) \cong H(\mathrm{Rep}^{nil}_{\mathbb{F}_q} Q)$

$K_0(A)$: Grothendieck 群
$:= \langle \overline{M}\ ([M]\in \mathrm{Iso}(A)) \mid \overline{L}-\overline{M}+\overline{N} \text{ for } 0\to L\to M\to N\to 0 \text{ exact in } A \rangle$
$\cup$
$K_0(A)_{\geq 0} := \langle \overline{M}\ ([M]\in \mathrm{Iso}(A)) \rangle_{\text{monoid}}$

Lem.2.1.10. $H(A)$ is $K_0(A)_{\geq 0}$-graded alg.
(pf) $g^N_{LM} = e^N_{LM}/a_L a_M$ and dfn. of $K_0(A)$ □

Prop 2.1.11. $K_0(\mathrm{Rep}^{nil}_F Q) \cong \mathbb{Z}$, $\overline{I_\lambda} \mapsto |\lambda|$
(sketch of pf) $\exists\ 0\to I_{(m)} \to I_{(n)} \to I_{(n-m)} \to 0$ exact. $\rightsquigarrow$ $\overline{I_{(n)}} = n\cdot \overline{I_{(1)}}$, $\overline{I_\lambda} = |\lambda|\cdot\overline{I_{(1)}}$ □

§2.2. twisted multiplication

can be omitted

Assume A satisfies (1), (2) &
(3) global dim. $<\infty$ & $\forall i\in\mathbb{N}\ \forall M,N\in \mathrm{ob}(A)\ |\mathrm{Ext}^i_A(M,N)|<\infty$

Lem.2.2.2. $\langle M,N\rangle_m := \prod_{i\geq 0} |\mathrm{Ext}^i_A(M,N)|^{(-1)^i/2}$.
$\Rightarrow \langle\cdot,\cdot\rangle_m : K_0(A)\times K_0(A) \to \mathbb{C}$: root of multiplicative Euler form
$\langle \alpha+\beta,\gamma\rangle_m = \langle\alpha,\gamma\rangle_m\langle\beta,\gamma\rangle_m$, $\langle\alpha,\beta+\gamma\rangle_m = \langle\alpha,\beta\rangle_m\langle\alpha,\gamma\rangle_m$
(pf ⋯ Q.2.2.)

Thm.2.2.3. [Ringel] $R(A) := (F(A), *, 1_{[0]})$ is unital assoc. $\mathbb{C}$-alg. w/ $K(A)_{\geq 0}$-grading
$f*g(N) := \sum_{M\subset N} \langle N/M, M\rangle_m f(N/M)g(M)$
(sketch of pf) use assoc. of $\circ$ and Lem.2.2.2 □

Cor. $R(A) \cong (\bigoplus \mathbb{C}[M], *, [0])$, $[M]*[N] = \langle M,N\rangle_m [M]\circ[N]$

Prp.2.2.4. $(H_{cl}, *, [0]) \cong R(\mathrm{Rep}^{nil}_F Q)$
(pf) enough to show $\langle M,N\rangle_m = 1\ \forall M,N \in \mathrm{ob}(\cdots)$
Since $\overline{I_\lambda} = |\lambda|\cdot\overline{I_1}$, enough to show $\langle I_1, I_1\rangle_m = 1$
$\mathrm{Ext}^i_A(I_1,I_1) = F\ (i=0,1),\ 0\ (i\geq 2)$ (Lem.2.2.5) □

§2.3. Green's comultiplication & Hopf pairing

Assume A satisfies (1), (3) & (4) $\forall M\in\mathrm{ob}(A)$. $\#\{\text{subobj. of } M\}<\infty$

Prp.2.3.1. $(F(A), \Delta, \varepsilon)$ is a counit. coassoc. coalg. w/
$\Delta([N]) := \sum_{[L],[M]} \langle L,M\rangle_m e^N_{LM}/a_N \cdot [L]\otimes[M]$, $\varepsilon([N]) = \delta_{N,0}$

(well-def. of Δ) $\#\{([L],[M]) \mid e^N_{LM} \neq 0\} \subset \#\{(L',M') \mid M' \subset N,\ L' \cong N/M'\} = \left|\prod_{M' \subset N} \mathrm{Aut}(N/M')\right| < \infty$ (4)

(coassoc.) $(\Delta \otimes \mathrm{id}) \circ \Delta([N]) \overset{?}{=} (\mathrm{id} \otimes \Delta) \circ \Delta([N])$

$\mathrm{LHS} = \sum_{[K],[L],[M]} C^N_{KLM}\, [K] \otimes [L] \otimes [M]$

$\mathrm{RHS} = \sum$ 〃 $\tilde{C}^N_{KLM}$ 〃

$C^N_{KLM} := \sum_{[J]} (J,K)_m \langle L,M \rangle_m\, e^N_{J,K}\, e^J_{L,M} / a_N a_J$

$\tilde{C}^N_{KLM} := \sum_{[J]} \langle L,J \rangle_m \langle M,K \rangle_m\, e^N_{L,J}\, e^J_{M,K} / a_N a_J$

$*$ is assoc. $\Leftrightarrow$ $C^N_{KLM} = \tilde{C}^N_{KLM}$ (Cor. 2.2.6.) $\square$

compl. w.r.t. $K_0(A)$

Rmk. If (4) is NOT assumed, then one can define $\Delta: F(A) \to F(A) \hat{\otimes} F(A)$

Thm. 2.3.5. (Green) Assume A satisfies (1),(3),(4) and (5) hereditary (gl. dim ≤ 1)
Then $\Delta([M]*[N]) = \Delta([M]) * \Delta([N])$
w/ $(x_1 \otimes x_2) * (y_1 \otimes y_2) := (x_2,y_1)_m\, (x_1 * y_1) \otimes (x_2 * y_2)$
$(x_2,y_1)_m := \langle x_2,y_1 \rangle_m \langle y_1,x_2 \rangle_m$

(pf... see the pdf.) $\Rightarrow$ Ringel-Hall bialg.
$R(A) = (F(A), *, \Delta, 1, \varepsilon)$

Thm. 2.3.9. (Green) For A satisfying (1),(3),(4),(5)
$([M],[N]) := \delta_{M,N} / a_M$ is a Hopf pairing of the bialg. $R(A)$

(pf) $([L]*[M],[N]) = \sum_{[J]} \langle L,M \rangle_m\, g^J_{LM}\, ([J],[N]) = \langle L,M \rangle_m\, g^N_{LM} / a_N$
$= \langle L,M \rangle_m\, e^N_{L,M} / a_L a_M a_N$
$([L] \otimes [M], \Delta([N])) = \left(-, \sum_{[J],[K]} (J,K)_m\, e^N_{J,K} / a_N \cdot [J] \otimes [K]\right)$
$= \langle L,M \rangle_m\, e^N_{L,M} / a_L a_M a_N$ $\square$

§1. $(H_{cl}, *, \Delta) = R(\mathrm{Rep}^{nil}_{\mathbb{F}_q} Q)$
$P_n = \sum_{\lambda \in \mathrm{Par},\, |\lambda| = n} (q;q)_{\ell(\lambda)-1} \cdot [I_\lambda]$ $\quad \Delta(P_n) = P_n \otimes 1 + 1 \otimes P_n$

Fact 1.4.6. $\langle P_n, P_m \rangle = \delta_{n,m} \cdot n / (q^n - 1)$

Ex. 1.4.7. $\langle P_1, P_1 \rangle = ([I_{(1)}], [I_{(1)}]) = 1/a_{(1)} = 1/(q-1)$
$P_2 = [I_{(2)}] + (1-q)[I_{(1^2)}]$
$\langle P_2, P_2 \rangle = 1/a_{(2)} + (1-q)^2 / a_{(1^2)} = 1/q(q-1) + (q-1)^2 / (q^2-1)(q^2-q)$
$= 1/q(q-1) + 1/q(q+1) = 2/(q^2-1)$

§2.4 Extended Ringel-Hall alg.

A satisfying (1)−(5) $\quad K_0(A) \ni \alpha \mapsto k_\alpha \in \mathbb{C}[K_0(A)]$
$F^e(A) := F(A) \otimes \mathbb{C}[K_0(A)]$ $\quad k_{\alpha+\beta} = k_\alpha k_\beta$
$\ni [M] \otimes k_\alpha = [M] k_\alpha$

Prp. 2.4.2. & 2.4.3.

$[M]k_\alpha * [N]k_\beta := (\alpha, \overline{N})_m \cdot ([M]*[N])(k_\alpha k_\beta)$, $1 = [0]\otimes k_0$

$\Delta([N]k_\gamma) := \sum_{[L],[M]} \langle L, M\rangle_m \frac{e^N_{LM}}{a_N} ([L]k_{M+\gamma}) \otimes ([M]k_\gamma)$, $\varepsilon([M]k_\alpha) = \delta_{M,0}$

$([M]k_\alpha, [N]k_\beta) := \delta_{M,N} \cdot (\alpha, \beta)_m / a_M$

$\Rightarrow \tilde{R}(A) = (\tilde{F}(A), *, \Delta, 1, \varepsilon)$ bialg. w/ Hopf pairing.

: extended Ringel-Hall bialg.

§2.5. Relation to quantum group.

Thm. 2.5.3. [Ringel]

Q : quiver w/o loop , I : vertex set $\ni i, j$

$\bar{Q}$: underlying graph. $n_{ij} = \#\{$edges connecting i and $j\}$ $a_{ij} := 2\delta_{i,j} - n_{i,j}$

$\Rightarrow (a_{ij})_{i,j\in I}$: sym. gen. Cartan matrix

$\Rightarrow \mathfrak{g}_{\bar{Q}}$: KM Lie alg.

$\nu = q^{1/2}$ $U_\nu(\mathfrak{g}_{\bar{Q}})$: quantum env. alg, specialized at ν

$= \langle E_i, F_i, K_i^{\pm 1}\rangle$ (p.30)

$U_\nu(\mathfrak{b}_{\bar{Q}}) := \langle E_i, K_i^{\pm 1}\rangle$ Borel subalg.

$\exists$ $\mathbb{C}$-alg. emb. $U_\nu(\mathfrak{b}_{\bar{Q}}) \hookrightarrow \tilde{R}(\mathrm{Rep}^{\mathrm{nilp}}_{\mathbb{F}_q} Q)$ ← satisfying (1)-(5)

$E_i \mapsto [S_i]$, $K_i \mapsto k_{\overline{S_i}}$ $S_i = (\oplus_{j\in I} \mathbb{C}\delta_{ij}, 0)$

The embedding is isom $\Leftrightarrow$ Q is of fin. typ. ($\Leftrightarrow$ $\bar{Q}$ Dynkin)

Fct. 2.5.2.

Thm. 2.5.4. [Green. Xiao] The embedding is a Hopf alg emb.