§1. Classical Hall algebra

§1.1. §1.2. Ring structure of classical Hall algebra

$F = \mathbb{F}_q$: fin. field

$Q =$ $\circlearrowleft$ Jordan quiver

$A = \mathrm{Rep}^{nil}_{\mathbb{F}_q} Q$: cat. of nilpotents reps. of Q over F

$Ob(A) \ni M = (V, x)$ $\qquad$ $V \overset{x}{\circlearrowleft}$

$\quad V$: fin. dim. F-lin. sp.

$\quad x \in \mathrm{End}_F(V,V)$

$\quad x^n = 0 \quad {}^\exists n \in \mathbb{Z}_{>0}$

$\mathrm{Hom}_A((V,x),(W,y)) \ni f$

$:\Leftrightarrow f \in \mathrm{Hom}_F(V,W)$ s.t.
$\begin{array}{ccc} V & \xrightarrow{f} & W \\ x\downarrow & \circlearrowright & \downarrow y \\ V & \xrightarrow{f} & W \end{array}$ $\quad y \circ f = f \circ x$

Lem. 1.1.3. $\mathrm{Iso}(A) := \left\{ \begin{array}{c} \text{isomorphic classes} \\ \text{of } Ob(A) \end{array} \right\} \ni [M]$

$= \{$Jordan normal form of nilpotent matrices$\}$

$= \{ [I_\lambda] \mid \lambda \in \mathrm{Par} \}$

$\mathrm{Par} := \{\text{partitions}\} \ni \lambda = (\lambda_1, \cdots, \lambda_r)$

$I_\lambda = (F^{|\lambda|}, J_\lambda) \in Ob(A)$ $\qquad \lambda_i \in \mathbb{N} = \mathbb{Z}_{\geq 0}$

$J_\lambda = J_{\lambda_1} \oplus \cdots \oplus J_{\lambda_r}$ $\qquad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$

$\quad \in \mathrm{End}(F^{|\lambda|})$ $\qquad (\lambda_1, \cdots, \lambda_r) = (\lambda_1, \cdots, \lambda_r, 0, 0, \cdots 0)$

$J_n = \underbrace{\begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}}_{n} \in \mathrm{End}(F^n)$ $\qquad \phi = (\ ) = (0) \in \mathrm{Par}, \ |\lambda| := \sum_{i=1}^{r} \lambda_i$

Thm. 1.2.2. $H_{cl} := \bigoplus_{\lambda \in \mathrm{Par}} \mathbb{C}[I_\lambda]$

$[I_\lambda] * [I_\mu] := \sum_{\nu \in \mathrm{Par}} g^\nu_{\lambda\mu} [I_\nu]$

$g^\nu_{\lambda\mu} := |G^\nu_{\lambda\mu}(\mathbb{F}_q)| < \infty$

$M = (V, x) \subset I_\nu = (F^{|\nu|}, y)$

subrep. $:\Leftrightarrow V \subset F^{|\nu|}$ subsp.

$\qquad x = y|_V$

$I_\nu / M = (F^{|\nu|}/V, \bar{y})$

quot. rep.

$G^\nu_{\lambda\mu}(F) := \{ M \in Ob(A) \mid M \subset I_\nu, M \cong I_\mu, I_\nu / M \cong I_\lambda \}$

$I_\phi = 0 = (\{0\}, 0) \in Ob(A)$

$\Rightarrow (H_{cl}, *, [0])$: unital assoc. $\mathbb{C}$-alg., $\mathbb{N}$-graded (classical Hall alg.)

$$[I_\lambda]*[I_\mu] := \sum_{\nu\in Par} g^\nu_{\lambda\mu}[I_\nu]$$

(pf.) (Well-definess: Lem. 1.10)

$g^\nu_{\lambda\mu}\neq 0 \Leftrightarrow G^\nu_{\lambda\mu}(F)\neq\phi$

$\ni$ $\quad M\subset I_\nu,\ M\cong I_\mu,\ I_\nu/M\cong I_\lambda$

$M=(V,x)$

$\Rightarrow V\subset F^{|\nu|}$, $\dim V=|\mu|$. $\dim(F^{|\nu|}/V)=|\lambda|$

$\Leftrightarrow |\nu|-|\mu|=|\lambda| \quad \Leftrightarrow |\nu|=|\lambda|+|\mu|$

$\therefore \#\{\nu\in Par \mid g^\nu_{\lambda\mu}\neq 0\}<\infty$

(associativity)

$$[I_\lambda]*([I_\mu]*[I_\nu]) \overset{?}{=} ([I_\lambda]*[I_\mu])*[I_\nu]$$

$$\text{LHS} = [I_\lambda]*\sum_{\alpha\in Par} g^\alpha_{\mu\nu}[I_\alpha] = \sum_{\alpha,\rho\in Par} g^\alpha_{\mu\nu}\, g^\rho_{\lambda\alpha}[I_\rho]$$

$$\sum_{\alpha\in Par} g^\alpha_{\mu\nu}g^\rho_{\lambda\alpha} = \#\coprod_{\alpha\in Par}(G^\alpha_{\mu\nu}\times G^\rho_{\lambda\alpha})$$

$$\coprod_{\alpha\in Par} G^\alpha_{\mu\nu}\times G^\rho_{\lambda\alpha} = \coprod_\alpha \{N\subset I_\alpha\mid N\cong I_\nu,\ I_\alpha/N\cong I_\mu\}\times\{A\subset I_\rho\mid A\cong I_\alpha,\ I_\rho/A\cong I_\lambda\}$$

$$=\{N\subset A\subset I_\rho \mid I_\rho/A\cong I_\lambda,\ A/N\cong I_\mu,\ N\cong I_\nu\}$$

$$=: G^\rho_{\lambda\mu\nu}$$

$$\text{LHS} =\sum_{\rho\in Par}|G^\rho_{\lambda\mu\nu}|\cdot[I_\rho]$$

Similarly, $\text{RHS} =\sum_{\rho\in Par}|\tilde G^\rho_{\lambda\mu\nu}|\cdot[I_\rho]$

$$\tilde G^\rho_{\lambda\mu\nu} = \coprod_\beta G^\beta_{\lambda\mu}\times G^\rho_{\beta\nu} = \coprod_\beta\{M\subset I_\beta\mid M\cong I_\mu,\ I_\beta/M\cong I_\lambda\}\times\{N\subset I_\rho\mid N\cong I_\nu,\ I_\rho/N\cong I_\beta\}$$

$$=\{(N,M)\mid N\subset I_\rho,\ N\cong I_\nu,\ M\subset I_\rho/N,\ M\cong I_\mu,\ (I_\rho/N)/M\cong I_\lambda\}$$

$G^\rho_{\lambda\mu\nu}\xrightarrow[\text{bij}]{}\tilde G^\rho_{\lambda\mu\nu} \quad (N,A)\longmapsto(N,A/N)$

by the isom. theorem. $(I_\rho/N)/(A/N)\cong I_\rho/A$

([0] is the unit) omit

($\mathbb{N}$-graded) $\deg[I_\lambda]:=|\lambda| \quad g^\nu_{\lambda\mu}\neq0\Rightarrow|\nu|=|\lambda|+|\mu|$ □

Cor. 1.26. $[I_{\lambda^{(1)}}]*[I_{\lambda^{(2)}}]*\cdots*[I_{\lambda^{(r)}}]$

$$=\sum_{\mu\in Par}|G^\mu_{\lambda^{(1)},\dots,\lambda^{(r)}}|\cdot[I_\mu]$$

$$G^\mu_{\lambda^{(1)},\dots,\lambda^{(r)}} := \{I_\mu=M_1\supset M_2\supset\cdots\supset M_r\supset M_{r+1}=0\mid M_i/M_{i+1}\cong I_{\lambda^{(i)}}\}$$

Prp. 1.2.7. $\mathcal{H}_{\mathrm{cl}}$ is commutative. $\quad$ $\mathrm{Hom}_{\mathbb{F}}(V,\mathbb{F})$

(sketch of prf.) $\quad G_{\lambda\mu}^{\nu} \xrightarrow{\text{bij}} G_{\mu\lambda}^{\nu}$ $\quad$ $\parallel$

$$M=(V,x) \mapsto M^* = (V^*, {}^tx) \mapsto M^\perp := \{\xi \in I_\nu^* \mid \xi(M)=0\}$$

$M \subset I_\nu,\ M \cong I_\mu,$ $\quad$ Q.1.1. $\left\{ \begin{array}{l} M^\perp \subset I_\nu^* \cong I_\nu \\ M^\perp \cong I_\lambda,\ I_\nu/M^\perp \cong I_\mu \end{array}\right.$ $\square$

$I_\nu/M \cong I_\lambda$

Thm. 1.2.8. $\mathcal{H}_{\mathrm{cl}} \cong \mathbb{C}[\,[I_{(1)}], [I_{(1^2)}], \dots]$ as $\mathbb{C}$-alg.

$$(1^n) = (\underbrace{1,\dots,1}_{n}) \in \mathrm{Par}$$

(sketch of prf.)

$$\mathrm{Par} \ni \lambda = (n^{m_n}, \dots, 2^{m_2}, 1^{m_1})$$

$$\mathcal{H}_{\mathrm{cl}} \ni X_\lambda := [I_{(1^{m_n})}] * [I_{(1^{m_{n-1}+m_n})}] * \dots * [I_{(1^{m_1+m_2+\dots+m_n})}]$$

can show $\quad X_\lambda = [I_\lambda] + \sum_{\mu<\lambda} a_{\lambda\mu}[I_\mu] \qquad a_{\lambda\mu} \in \mathbb{Z}$

$$\mu \le \lambda \overset{\cdot}{\Leftrightarrow} |\mu| = |\lambda|,\ \forall j = 1,2,\dots\ \textstyle\sum_{i=1}^{j} \mu_i \le \sum_{i=1}^{j} \lambda_i$$

(dominance order).

then $\quad [I_\lambda] = X_\lambda + \sum_{\mu<\lambda} a^{\lambda\mu} X_\mu \qquad (a^{\lambda\mu}) = (a_{\lambda\mu})^{-1} \in \mathrm{Mat}(|\lambda|, \mathbb{Z})$

$\in$ RHS. $\square$

Explicit formulas of $g_{\lambda\mu}^{\nu}$

Lem. 1.2.10. (Lem. 1.1.9 & Q.1.3)

$$g_{(n-m),(m)}^{(n)} = 1,$$

$$g_{(1^{n-m}),(1^m)}^{(1^n)} = |\mathrm{Gr}(m, \mathbb{F}_q^n)| = \begin{bmatrix} n \\ m \end{bmatrix}_q' := \frac{(q;q)_n}{(q;q)_m (q;q)_{n-m}}$$

$$(q;q)_n := (1-q)(1-q^2)\cdots(1-q^n)$$

Eg. 1.2.12. $e_1 := [I_{(1)}] \in \mathcal{H}_{\mathrm{cl}}$

$$\begin{aligned} e_1 * e_1 &= g_{(1),(1)}^{(2)}[I_{(2)}] + g_{(1),(1)}^{(1^2)}[I_{(1^2)}] \\ &= 1 \cdot [I_{(2)}] + \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q' \cdot [I_{(1^2)}] \\ &= [I_{(2)}] + (q+1)[I_{(1^2)}] \end{aligned}$$

§1.3. Coalgebra structure

Thm. 1.3.2 $\Delta_{\mathcal{H}_{\mathrm{cl}}} : \mathcal{H}_{\mathrm{cl}} \to \mathcal{H}_{\mathrm{cl}} \otimes_{\mathbb{C}} \mathcal{H}_{\mathrm{cl}}$

$$\Delta([I_\nu]) := \sum_{\lambda,\mu \in \mathrm{Par}} \frac{a_\lambda a_\mu}{a_\nu} g_{\lambda\mu}^{\nu} [I_\lambda] \otimes [I_\mu]$$

$$a_\lambda := |\mathrm{Aut}(I_\lambda)| \in \mathbb{Z}_{>0}$$

$\varepsilon : \mathcal{H}_{\mathrm{cl}} \to \mathbb{C} \qquad \varepsilon([I_\lambda]) = \delta_{\lambda,\phi} \qquad \deg([I_\lambda]) = |\lambda|$

$\Rightarrow (\mathcal{H}_{\mathrm{cl}}, \Delta, \varepsilon)$: cocomm. coassoc. coalg. w/ $\mathbb{N}$-grading.

Prp. 1.3.6. $\Delta([I_{c}(1^n)]) = \sum_{r=0}^{n} q^{-r(n-r)} [I_{c}(1^{n-r})] \otimes [I_{c}(1^r)]$

(sketch of pf) use $g^{(1^n)}_{(1^m)(1^{n-m})} = \begin{bmatrix} n \\ m \end{bmatrix}_q$

$a_{(1^n)} = |\mathrm{Aut}(I_c(1^n))| = |\mathrm{Aut}(I_{c(1)}^{\oplus n})|$
$= |GL(n, \mathbb{F}_q)| = (q^n-1)(q^n-q)\cdots(q^n-q^{n-1})$

Dfn. A primitive element of H_{cl} is $p \in H_{cl}$ s.t. $\Delta(p) = p\otimes 1 + 1 \otimes p$ $\quad 1 = [0]$

Fact. 1.3.7. A homogeneous primitive elem. of H_{cl} is a scalar multiple of

$$p_n := \sum_{\lambda \in \mathrm{Par}, |\lambda| = n} (q;q)_{\ell(\lambda)-1} [I_\lambda] \qquad (n \in \mathbb{Z}_{>0})$$

$\ell(\lambda) :=$ length of $\lambda = (\lambda_1, \ldots, \lambda_r)$
$= \#\{i \mid \lambda_i > 0\}$

eg 1.3.8. $p_1 = [I_{c}(1)] \qquad \Delta(p_1) = p_1 \otimes 1 + 1 \otimes p_1$

$p_2 = 1\cdot[I_c(2)] + (1-q)\cdot[I_c(1^2)]$

$\Delta(I_c(1^2)) = [I_c(1^2)]\otimes 1 + q^{-1} p_1 \otimes p_1 + 1 \otimes [I_c(1^2)]$

$\Delta([I_c(2)]) = [I_c(2)] \otimes 1 + \underbrace{\frac{a_{(1)}a_{(1)}}{a_{(2)}} g^{(2)}_{(1)(1)}}_{\substack{\| \\ \frac{(q-1)^2}{q^2-q} \times 1 \; = \; q^{-1}(q-1)}} [I_c(1)] \otimes [I_c(1)] + 1 \otimes [I_c(2)]$

$\therefore \Delta(p_2) = p_2 \otimes 1 + 1 \otimes p_2$

§1.4. Bialgebra str. & Hopf pairing.

Thm. 1.4.2. $(H_{cl}, *, \Delta, [0], \varepsilon)$ is a unital & counital bialg.
In particular, $\Delta : H_{cl} \to H_{cl} \otimes H_{cl}$ is an alg. hom.

i.e. $\Delta(x * y) = \Delta(x) * \Delta(y) \qquad x, y \in H_{cl}$

$:= \sum_{i,j} (x_i^1 * y_j^1) \otimes (x_i^2 * y_j^2)$

(pf. — §2)

$\Delta(x) = \sum_i x_i^1 \otimes x_i^2$

$\Delta(y) = \sum_j y_j^1 \otimes y_j^2$

Prp. 1.4.4. $\langle \cdot, \cdot \rangle : H_{cl} \otimes H_{cl} \to \mathbb{C}$

$\langle [I_\lambda], [I_\mu] \rangle := \delta_{\lambda,\mu} / a_\lambda$

is a Hopf pairing of the bialg. $(H_{cl}, *, \Delta)$

i.e. $\langle x * y, z\rangle = \langle x \otimes y, \Delta(z)\rangle$ $\quad \Delta(z) = \sum_i z_i^1 \otimes z_i^2$

$:= \sum_i \langle x, z_i^1\rangle \langle y, z_i^2\rangle$

(pf ... §2)