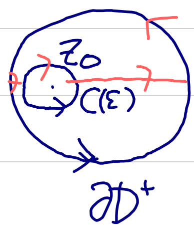
§7 留数定理

§7.1. 定理7.1.1 $D \subset \mathbb{C}$: 開円板. $z_0 \in D$

f : $\overline{D}$ を含む開集合上、z_0 を除いた部分で正則.

z_0 を極に持つ.

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{2\pi i}\int_{\partial D^+} f(z)\,dz$$



$C(\varepsilon)$: z_0 中心、D 内部にある正向きの円.

$$\int_{\partial D^+} - \int_{C(\varepsilon)} = \int + \int = 0+0=0$$

(∵ Cauchy の積分定理)

$$f(z) = \sum_{k=1}^{n} a_{-k}(z-z_0)^{-k} + G(z)$$

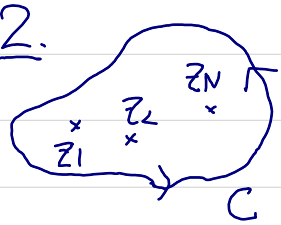
: Laurent 展開.　　(G : 正則)

$$\frac{1}{2\pi i}\int_{C(\varepsilon)} (z-z_0)^{-k}\,dz = \delta_{k,1}$$

$$\frac{1}{2\pi i}\int_{\partial D^+} f(z)\,dz = \sum_{k=1}^{n} \delta_{k,1}\cdot a_{-k} + \int_{\partial D^+} G(z)\,dz$$

$$= a_{-1} + 0 = \operatorname*{Res}_{z=z_0} f(z) \quad //$$

系7.1.2.



C : 単純閉曲線 (正向き), $z_1, \dots, z_N$: C 内部の点

f : C と内部を含む開集合上、

$z_1, \dots, z_N$ を除いた所で正則.

$z_1, \dots, z_N$ は極.

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{N} \operatorname*{Res}_{z=z_k} f(z)$$

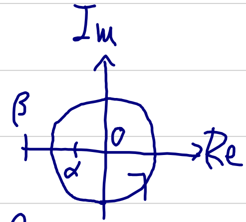
例7.1.3.(問7.1.3)

$$I = \int_0^{2\pi} \frac{d\theta}{a+\cos\theta} \qquad (a \in \mathbb{R}_{>1})$$

$z = \exp(i\theta)$. $\quad d\theta = dz/iz \quad \cos\theta = \frac{1}{2}(z+z^{-1})$

$$I = \int_{|z|=1} \frac{1}{a+\frac{1}{2}(z+z^{-1})} \frac{dz}{iz}$$

$$= \int_{|z|=1} \frac{-2i}{z^2+2az+1} dz$$



極 : $\alpha = -a+\sqrt{a^2-1}$, $\beta = -a-\sqrt{a^2-1}$

$\alpha, \beta \in \mathbb{R} \qquad |\alpha| = -\alpha < 1 < |\beta| = -\beta$

$$\therefore I = 2\pi i \operatorname*{Res}_{z=\alpha} \frac{-2i}{z^2+2az+1} = 4\pi \operatorname*{Res}_{z=\alpha} \frac{1}{(z-\alpha)(z-\beta)}$$

$$= 4\pi g(\alpha)$$

$$= 4\pi/(\alpha-\beta) = \frac{2\pi}{\sqrt{a^2-1}}$$

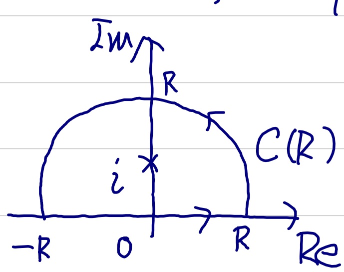
$\uparrow$

$\frac{1}{(z-\alpha)} \times g(z)$

g : $z=\alpha$のまわりで正則

例7.1.4 (問7.1.5)

$$I = \int_{-\infty}^{\infty} \frac{dx}{1+x^2}$$



$f(z) := \frac{1}{1+z^2}$

$$\int_{-R}^{R} f(z)dz + \int_{C(R)} f(z)dz$$

$$= 2\pi i \operatorname*{Res}_{z=i} f(z) \qquad (R>1)$$

$$= 2\pi i \operatorname*{Res}_{z=i} \frac{1}{(z+i)(z-i)} = 2\pi i \cdot \frac{1}{2i} = \pi$$

$R \to \infty$で $\int_{-R}^{R} \to \int_{-\infty}^{\infty}$, $\int_{C(R)} \to 0$

なので $\quad I = \pi$.

$$J := \int_{C(R)} \frac{1}{1+z^2}\,dz \longrightarrow 0 \quad (R\to\infty)$$

∵ $C(R):\ z = Re^{i\theta} \quad (0\le\theta\le\pi)$

$$J = \int_0^{\pi} \frac{1}{1+R^2e^{2i\theta}} \cdot Rie^{i\theta}\,d\theta$$

$|J| \le \cdots$

$$\left|\frac{Rie^{i\theta}}{1+R^2e^{2i\theta}}\right| \le \frac{R}{R^2-1} \quad \left(|1+R^2e^{2i\theta}| \ge R^2-1\right)$$

$$\therefore\ |J| \le \pi\times\frac{R}{R^2-1} \xrightarrow[R\to\infty]{} 0 \quad //$$

§7.2. 偏角の原理

<u>定理7.2.1</u> C: 単純閉曲線. 正向き

f: Cとその内部を含む開集合上の有理型関数

C上に極も零点もない.

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = \sum_{C\text{内部の零点}z} \mathrm{ord}_f(z) - \sum_{C\text{内部の極}z} \mathrm{ord}_f(z)$$

($\mathrm{ord}_f(z)$: 位数) //

∵ $(\text{左辺}) = \displaystyle\sum_{w:\frac{f'}{f}\text{の極}} \operatorname{Res}_{z=w} \frac{f'(z)}{f(z)}$ (∵ 留数定理)

w: fのn位の零点 $\Rightarrow f(z) = (z-w)^n g(z)$

$f'/f = \frac{n}{z-w} + g'/g \quad \operatorname{Res}_{z=w}(f'/f) = n$

w: 〃 極 $\Rightarrow f(z) = (z-w)^{-n} g(z)$

$\Rightarrow \operatorname{Res}(f'/f) = -n$. //

Rouchéの定理7.2.2.

C:円　f, g: Cと内部を含む開集合上の正則関数.

$\forall z \in C$　$|f(z)| > |g(z)|$ $\Rightarrow$ $\sum \mathrm{ord}_{f+g}(z) = \sum \mathrm{ord}_f(z)$

　　　　z: C内部の零点　　　　z: C内部の零点.

☺ $0 \leq t \leq 1$　$f_t(z) := f(z) + t g(z)$

$N(t) := \sum \mathrm{ord}_{f_t}(z)$.

がtの連続関数であることを偏角の原理から示す. //

開写像の定理7.2.4　$U \subset \mathbb{C}$:開

$f: U \to \mathbb{C}$: 正則. 定数でない $\Rightarrow$ fは開写像

($\forall V \subset U$ 開. $f(V)$:開)

☺ Rouchéの定理から. //

定理7.2.5.(最大値の原理)

$U \subset \mathbb{C}$:開　$f: U \to \mathbb{C}$: 正則, 定数でない

$\Rightarrow$ $|f|$は最大値を持たない.

☺ $z_0 \in U$で$|f|$が最大値を持つと仮定.

$z_0 \in D \subset U$:開円板. $f(D) \subset \mathbb{C}$は開集合 ($\because$ 7.2.4)

$\therefore \exists w = f(z) \in f(D)$　$|f(z)| > |f(z_0)|$. 矛盾. //

§8 函数の表示 (大域的記述)

§8.1. 有理型函数の部分分数展開

定理8.1.2. f: $\mathbb{C}$ 上有理型. 極 $\{a_n\}_{n=1}^{\infty}$ ($a_m \neq a_n$ $\forall m \neq n$) は全て位数1.
必要なら $\{a_n\}$ を並びかえて

$$|a_1| \leq |a_2| \leq \cdots$$

とした時、$a_1 \neq 0$ かつ $\lim_{n\to\infty} |a_n| = +\infty$ だと仮定.
また $\exists \{R_m\}_{m=1}^{\infty} \subset \mathbb{R}_{>0}$, $\lim_{m\to\infty} R_m = \infty$ s.t.

- $|z| = R_m$ 上に極 a_n 達はない
- $\exists M \in \mathbb{R}_{>0}$. $\forall m$. $\sup_{|z|=R_m} |f(z)| < M$

$$f(z) = f(0) + \lim_{m\to\infty} \sum_{|a_n| \leq R_m} \left[\operatorname{Res}_{z=a_n} f(z)\right] \times \left(\frac{1}{z-a_n} + \frac{1}{a_n}\right)$$

例:

命題8.1.3.
$$\frac{1}{\sin z} - \frac{1}{z} = \sum_{n \in \mathbb{Z}\setminus\{0\}} \frac{(-1)^n}{z - n\pi} \quad \left(\mathbb{C}\setminus\mathbb{Z}\pi \text{ で広義一様収束}\right)$$

$\left(a_n = n\pi,\ R_n = (n+\tfrac{1}{2})\pi : \operatorname{Res}_{z=n\pi} = (-1)^n\right)$

命題8.1.4.
$$\cot z - \frac{1}{z} = \sum_{n \in \mathbb{Z}\setminus\{0\}} \frac{1}{z - n\pi} \quad (〃)$$

8.1.4を微分して

命題8.1.5
$$\frac{1}{\sin^2 z} = \sum_{n \in \mathbb{Z}} \frac{1}{(z - n\pi)^2} \quad (〃)$$

$\sin z = z(1 + O(z^2)) \Rightarrow \frac{1}{\sin z} = \frac{1}{z} \times (\text{正則})$. $\operatorname{Res}_{z=0} = 1$

$\frac{1}{\sin z} = \frac{1}{\sin(z - 2n\pi)} = \frac{1}{z - 2n\pi} \times (\text{正則})$ $\therefore \operatorname{Res}_{z=2n\pi} = 1$

$\frac{1}{\sin z} = \frac{-1}{\sin(z - (2n+1)\pi)} = \frac{-1}{z - (2n+1)\pi} \times (\text{正則})$ $\therefore \operatorname{Res}_{z=(2n+1)\pi} = -1$

§8.2. 整函数の無限積表示

命題8.2.3.
$$\sin z = z \times \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2\pi^2}\right) \quad (\mathbb{C}\text{で広義一様収束})$$

注　両辺を対数微分すると

$f(z) \longmapsto \frac{d}{dz}\log f(z) = \frac{f'(z)}{f(z)}$

$$\begin{aligned}\cot z &= \frac{1}{z} + \sum_{n=1}^{\infty}\frac{-2z/n^2\pi^2}{1 - z^2/n^2\pi^2}\\ &= \frac{1}{z} + \sum_{n=1}^{\infty}\frac{2z}{z^2 - n^2\pi^2}\\ &= \frac{1}{z} + \sum_{n\neq 0}\frac{1}{z-n\pi} \quad (8.1.4)\end{aligned}$$

//