Moduli spaces of Bridgeland stable objects on abelian surfaces

Shintarou Yanagida

京都大学数理解析研究所　談話会

07/11/2012

§0 Abstract and Contents

代数曲面上の安定層のモジュライ空間の研究において、Fourier-Mukai 変換の下で Gieseker 安定性が保たれるかという問題は重要であった。

2000 年代には Bridgeland により安定性の概念が導入されたが、上記の問題はこの Bridgeland 安定性で考えた方がより良く扱える事が分かりつつある。

今回の講演ではアーベル曲面の場合に Bridgeland 安定性条件の空間の構造の説明をする。ある場合には、壁越え現象が半等質層を用いた安定層のモジュライ空間の具体的な記述と密接に関係することを解説する。

Contents

§1 Statements of two main theorems
§2 Main Thm. 1 [YY1]
§3 Main Thm. 2 [YY2]
§4 Conclusion

[YY1] Yoshioka, Y.,
"Semi-homogeneous sheaves, Fourier-Mukai transforms and moduli of stable sheaves on abelian surfaces" (arXiv:0906.4603)

[YY2] Yoshioka, Y.,
"Bridgeland's stabilities on abelian surfaces" (arXiv:1203.0884)

§1 Main theorems

- X: abelian surface over $\mathbb{C}$, H: ample divisor on X
- A torsion free coherent sheaf E is H-(semi-)stable

$$\overset{\text{def}}{\Longleftrightarrow} \forall F \subset E,\ \chi(F(nH))/\operatorname{rk}(F) \underset{(=)}{<} \chi(E(nH))/\operatorname{rk}(E) \quad \text{for } n \gg 0.$$

- $M_X^H(v)$:=moduli scheme of H-stable sheaves E with $v(E) = v$.
- Chern character of $E \in \operatorname{Coh}(X)$:

$$v(E) := \operatorname{ch}(E) = (\operatorname{rk}(E), c_1(E), \chi(E)) \in H^{ev}(X, \mathbb{Z})_{alg} := H^0(X, \mathbb{Z}) \oplus NS(X) \oplus H^4(X, \mathbb{Z})$$

Fact (1) [Mukai (1984)]

$M_X^H(v)$ is non-singular, $\dim = 2 + \langle v^2 \rangle$ and symplectic (if non-empty).

$\langle (r, \xi, a), (s, \eta, b) \rangle := (\xi, \eta) - as - rb$.

(2) [Yoshioka (2003)]

If $v = (r, \xi, a)$ is positive

($\overset{\text{def}}{\iff}$ $r > 0$ or "$r = 0$ and ξ: effective"
or "$r = 0$ and $\xi = 0$ and $a > 0$"),

then $M_X^H(v) \neq \emptyset \iff \langle v^2 \rangle \geq 0$. □

Main Thm. 1 [conjectured by Mukai (1980)]

Assume

- X: principally polarized and $NS(X) = \mathbb{Z}H$,
- v: positive,
  $\ell := \langle v^2 \rangle /2$ (always $\in \mathbb{Z}$) is not a square number,
  $\#\{\text{quadratic form with discriminant} = \ell\}/\underset{GL(2,\mathbb{Z})}{\sim} = 1$.

Then

$$M_X^H(v) \overset{\text{birat.}}{\dashrightarrow} X \times \mathrm{Hilb}^{\ell}(X)$$

□

Key tools of proof

- Fourier-Mukai transform (FMT)
- Semi-homogeneous (SH) sheaves

Main Thm. 2 [stated by Mukai (1979,1980)]

Assume conditions on X, v as in Main Thm.1.
For $\ell = 1, 2, 3$ (satisfying conditions in Thm.1),

$$M_X^H(v) \xrightarrow{\sim} X \times \mathrm{Hilb}^\ell(X)$$

□

Key tool of proof

- Wall structure of space of Bridgeland stability conditions

§2 Birational morphism of moduli spaces

§2.1 SH sheaves and FMTs on abelian surfaces

Def. (Semi-homogeneous sheaves)

For $E \in \mathbf{Coh}(X)$,

$$S(E) := \{(x, y) \in X \times \widehat{X} \mid T_x^* E \otimes P|_{X \times \{y\}} \cong E\}$$

($\widehat{X} := \mathrm{Pic}^0 X$: Picard variety, T_x: translation of X, P: Poincaré bundle)

$S(E)$ is always abelian subvariety and $\dim \leq 2$.

$$E \text{ is SH} \overset{\text{def}}{\iff} \dim S(E) = 2. \qquad \square$$

Rmk. For SH vector bundle E,

$$\forall x \in X \ \ \exists L \in \mathrm{Pic}^0 Y \ \ T_x^* E \cong L \otimes E.$$

$\square$

Fact (1) [Mukai (1978)]

(classification of 2-dim moduli of stable sheaves)

H: ample divisor on X, $\quad E \in \mathrm{Coh}(X), \langle v(E)^2 \rangle = 0$.

(a) E: H-semi-stable $\Longrightarrow$ E: SH.

(b) E: H-stable $\Longleftrightarrow$ $v(E)$ is primitive.

(2) [Orlov (2002)]

(classification of FMTs on abelian surfaces)

X, Y abelian surf., $\Phi : D(X) := D^b\,\mathrm{Coh}(X) \xrightarrow{\sim} D(Y)$

(a) $\exists v$, $\langle v^2 \rangle = 0$, positive, $Y \cong M_X^H(v)$.

(b) $\exists \mathcal{E}$: universal family on $Y \times X$, $\exists k \in \mathbb{Z}$

$$\Phi \cong \Phi_{X \to Y}^{\mathcal{E}[k]} := \mathbb{R}p_{Y*}(p_X^*(-) \overset{\mathbb{L}}{\otimes} \mathcal{E}[k]).$$

□

§2.2 SH presentation

Def. SH presentation of $E \in \mathrm{Coh}(X)$ is an exact sequence

$$0 \to E \to E_1 \to E_2 \to 0$$

or

$$0 \to E_1 \to E_2 \to E \to 0$$

with E_1, E_2 SH s.t.

$$(\ell_1 - 1)(\ell_2 - 1) = 0, \quad \langle v_1, v_2 \rangle = -1$$

($\mathrm{ch}(E_i) = \ell_i v_i, \quad \ell_i \in \mathbb{Z}$, v_i: primitive) $\square$

Rmk. 1

(1) The conditions implies

- $\mathbf{E}_1$ or $\mathbf{E}_2$: stable
- $\exists$ FMT $\Phi : \mathbf{D}(\mathbf{X}) \xrightarrow{\sim} \mathbf{D}(\mathbf{Y_i})$, $\mathbf{Y_i} := \mathbf{M}^{\mathbf{H}}_{\mathbf{X}}(\mathbf{v_i})$.

(2) In the case

$$\mathbf{0} \to \mathbf{E}_1 \to \mathbf{E}_2 \to \mathbf{E} \to \mathbf{0}$$

with $\ell_1 = 1$, then by $\Phi := \Phi^{\mathcal{E}[-1]}_{\mathbf{X} \to \mathbf{Y}_1}$,

$$\Phi(\mathbf{E}) \cong \mathbf{L} \otimes \mathcal{I}_{\mathbf{Z}}$$

with $\mathbf{L} \in \mathbf{Pic}^0(\mathbf{Y}_1)$, $\mathbf{Z} \in \mathbf{Hilb}^{\langle \mathbf{v} \rangle^2/2}(\mathbf{Y}_1)$.

(The other 3 cases have similar properties.) □

Def. The numerical equation for $\mathbf{v}$:

$$(\#) \qquad \mathbf{v} = \ell_1 \mathbf{v}_1 - \ell_2 \mathbf{v}_2$$

$$\text{with} \begin{cases} \ell_i \in \mathbb{Z}_{>0}, & \mathbf{v}_i : \text{positive and primitive}, \\ (\ell_1 - 1)(\ell_2 - 1) = 0, \\ \langle \mathbf{v}_i^2 \rangle = 0, \quad \langle \mathbf{v}_1, \mathbf{v}_2 \rangle = -1. \end{cases}$$

□

Prop. 1 Assume $\mathbf{NS(X)} = \mathbb{Z}\mathbf{H}$, $\mathbf{v}$: positive, $\langle \mathbf{v}^2 \rangle > 0$.

If (#) has a solution $(\mathbf{v}_1, \mathbf{v}_2, \ell_1, \ell_2)$,
then a general member of $\mathbf{M}_{\mathbf{X}}^{\mathbf{H}}(\mathbf{v})$ has a SH presentation. □

Sketch of Prf. of Main Thm. 1

- Assume $\mathbf{NS(X)} = \mathbb{Z}\mathbf{H}$. Put $\mathbf{v} = (\mathbf{r}, \mathbf{dH}, \mathbf{a})$, $\mathbf{n} := \frac{(\mathbf{H}^2)}{2}$.

  Then by Rmk. 1 and Prop. 1,

  if $\exists$ sol. $(\mathbf{x}, \mathbf{y}) \in \mathbb{Z}^2$ of $-\mathbf{rx}^2 + 2\mathbf{ndxy} - \mathbf{ay}^2 = \pm 1$ $(\star)$, then

$$\exists \text{FMT} : \mathbf{M}_{\mathbf{X}}^{\mathbf{H}}(\mathbf{v}) \overset{\text{birat}}{\dashrightarrow} \mathbf{Pic}^{\mathbf{0}}(\mathbf{Y}) \times \mathbf{Hilb}^{\ell}(\mathbf{Y}), \quad \mathbf{Y} = \mathbf{M}_{\mathbf{X}}^{\mathbf{H}}(\mathbf{v_i}).$$

- If $\mathbf{X}$ is principally polarized ($\mathbf{n} = 1$), then

  $\#\{\text{quadratic form, discr.} = \ell\}/\sim = 1 \Longrightarrow (\star)$,

  $\mathbf{Y} \cong \widehat{\mathbf{Y}} \cong \mathbf{X}$. $\square$

§3 Isomorphisms of moduli spaces

§3.1 Bridgeland stability

Def. A stability condition $\sigma = (\mathcal{A}, Z)$ on a triangl. cat. $\mathcal{T}$

is a pair $\begin{cases} \text{t-structure on } \mathcal{T} \text{ (with } \mathcal{A} \text{ its core)} \\ \text{group homomorphism } Z : K(\mathcal{A}) \to \mathbb{C} \end{cases}$

s.t. for any $0 \neq E \in \mathcal{A}$,

(1) $Z(E) \in \mathbb{H}' := \{re^{\sqrt{-1}\pi\phi} \mid r \in \mathbb{R}_{>0}, \phi \in (0, 1]\}$

(2) $\exists$ filtr. $0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$

in $\mathcal{A}$ s.t. for $F_j := E_j/E_{j-1}$,

$$\phi(F_1) > \cdots > \phi(F_n)$$

and F_j is semi-stable (i.e., $0 \neq \forall F' \subset F_j$, $\phi(F') \leq \phi(F_j)$).

Prop. $\mathbf{X}$: abelian surface

For each $\beta \in \mathbf{NS(X)}_{\mathbb{R}}$ and $\omega \in \mathbf{Amp(X)}_{\mathbb{R}}$,

$\exists$ stab. cond. $\sigma_{(\beta,\omega)} := (\mathcal{A}_{(\beta,\omega)}, Z_{(\beta,\omega)})$ on $\mathbf{D(X)}$ with

$$Z_{(\beta,\omega)}(\mathbf{E}) := \left\langle \exp(\beta + \sqrt{-1}\omega), \mathbf{v(E)} \right\rangle,$$

$\mathcal{A}_{(\beta,\omega)}$: tilting of the torsion pair $(\mathcal{T}_{(\beta,\omega)}, \mathcal{F}_{(\beta,\omega)})$

$$\mathcal{T}_{(\beta,\omega)} := \left\langle \beta\text{-twisted stable sheaf } \mathbf{E}, Z_{(\beta,\omega)}(\mathbf{E}) \in \mathbb{H}' \right\rangle$$

$$\mathcal{F}_{(\beta,\omega)} := \left\langle \beta\text{-twisted stable sheaf } \mathbf{E}, -Z_{(\beta,\omega)}(\mathbf{E}) \in \mathbb{H}' \right\rangle \qquad \square$$

Proved in [Minamide-Yoshioka-Y.]

"Fourier-Mukai transforms and the wall-crossing behavior for Bridgeland's stability conditions" (arXiv:1106.5217)

Def. (1) ($\sigma_{(\beta,\omega)}$-semi-stability)

$\mathbf{E} \in \mathbf{D(X)}$ is $\sigma_{(\beta,\omega)}$-semi-stable

$\overset{\text{def}}{\Longleftrightarrow}$ $\exists \mathbf{n} \in \mathbb{Z}$ $\mathbf{E[-n]}$ is semi-stable object of $\mathcal{A}_{(\beta,\omega)}$.

(2) (moduli space of Bridgeland semi-stable objects)

$\mathbf{M_{(\beta,\omega)}(v)}$:= moduli scheme of S-equivalence classes of semi-stable objects $\mathbf{E}$ of $\mathcal{A}_{(\beta,\omega)}$ with $\mathbf{v(E) = v}$. □

Prop. 2 (1) (large volume limit)

For $\mathbf{E} \in \mathbf{D(X)}$ with $\mathbf{rk(E)} \geq 0$ and $\langle \mathbf{v(E)}^2 \rangle \ll (\omega^2)$,

$\mathbf{E}$: $\sigma_{(\beta,\omega)}$-semi-stable $\Longleftrightarrow$

$\mathbf{E} \in \mathbf{Coh(X)}$ and β-twisted semi-stable

(2) (preservation of stability)

Any FMT Φ on $D(X)$ preserves stability, i.e.,

$$E : \sigma_{(\beta,\omega)}\text{-semi-stable} \Longrightarrow \Phi(E) : \sigma_{(\beta',\omega')}\text{-semi-stable}$$

(β', ω') is determined by $\Phi(e^{\beta+\sqrt{-1}\omega}) = r'e^{\beta'+\sqrt{-1}\omega'}$ $\square$

§3.2 Chamber-wall structure

Def. (Wall and chamber)

Fix $\mathbf{v}$. Consider $\mathbf{v}_1$ with $\mathbf{v}_1 \notin \mathbb{Q}\mathbf{v}$,

$$\langle \mathbf{v}_1^2 \rangle \geq 0, \langle (\mathbf{v}-\mathbf{v}_1)^2 \rangle \geq 0, \langle \mathbf{v}_1, \mathbf{v}-\mathbf{v}_1 \rangle > 0.$$

(1) A wall for $\mathbf{v}$ (of type $\mathbf{v}_1$)

$$W_{\mathbf{v}_1,\mathbf{v}} := \{(\beta,\omega) \mid \mathbb{R}Z_{(\beta,\omega)}(\mathbf{v}) = \mathbb{R}Z_{(\beta,\omega)}(\mathbf{v}_1)\}.$$

(2) A chamber of $\mathbf{v}$:= a connected component of $\mathrm{NS}(X)_{\mathbb{R}} \times \mathrm{Amp}(X)_{\mathbb{R}} \setminus \cup_{\mathbf{v}_1} W_{\mathbf{v}_1,\mathbf{v}}$. □

Lem. For $\mathbf{v}$ with $\langle \mathbf{v}^2 \rangle > 0$ and C a chamber for $\mathbf{v}$,

$M_{(\beta,\omega)}(\mathbf{v})$ is independent of $(\beta,\omega) \in C$. □

From now on, assume $\mathbf{NS(X) = \mathbb{Z}H}$.

Lem. (1) A wall $W_{v_1,v}$ for v is of the form

$$W_a := \{(sH, tH) \mid (s-a)^2 + t^2 = (p-a)^2 - q\}$$

or

$$W_\infty := \{(sH, tH) \mid s = p\}$$

with $p \in \mathbb{Q}$, $q \in \mathbb{Q}_{>0}$ determined by v,
and $a \in \mathbb{Q}$ determined by v_1, v.

(2) $W_a \cap W_{a'} = \emptyset$ for $a \neq a'$. □

Def. (codimension of wall)

For W a wall for v,

$$\mathrm{codim} W := \min_{v=\sum_i v_i} \left\{ 1 + \sum_{i<j} \langle v_i, v_j \rangle - \sum_i (\dim \mathcal{M}_X^{H,\beta'}(v_i)^{ss} - \langle v_i^2 \rangle) \right\}$$

In the summation, considering $v = \sum_i v_i$ s.t.

$$\phi_{(\beta,\omega)}(v) = \phi_{(\beta,\omega)}(v_i) \quad \forall\, i,$$
$$\phi_{(\beta',\omega')}(v_i) > \phi_{(\beta',\omega')}(v_j) \quad \forall\, i < j.$$

with (β', ω') in the adjacent chamber of $W \ni (\beta, \omega)$. $\square$.

Lem. (codimension 0 = numerical solution)

$$\{W_{v_1,v} \mid \text{codimension 0 wall for } v\} \overset{1:1}{\longleftrightarrow}$$
$$\{\text{solution } (v_1, v_2, \ell_1, \ell_2) \text{ of the numerical equation}(\#)\}$$

□

§3.2 Walls for $(1, 0, -\ell)$

Assume $\mathbf{NS(X)} = \mathbb{Z}\mathbf{H}$. $\mathbf{n} := (\mathbf{H}^2)/2$.

Lem.

$$\{\text{codim 0 wall for } (1, 0, -\ell)\} = \{\mathbf{W}^0_{\mathbf{m}} \mid \mathbf{m} \in \mathbb{Z}\}$$

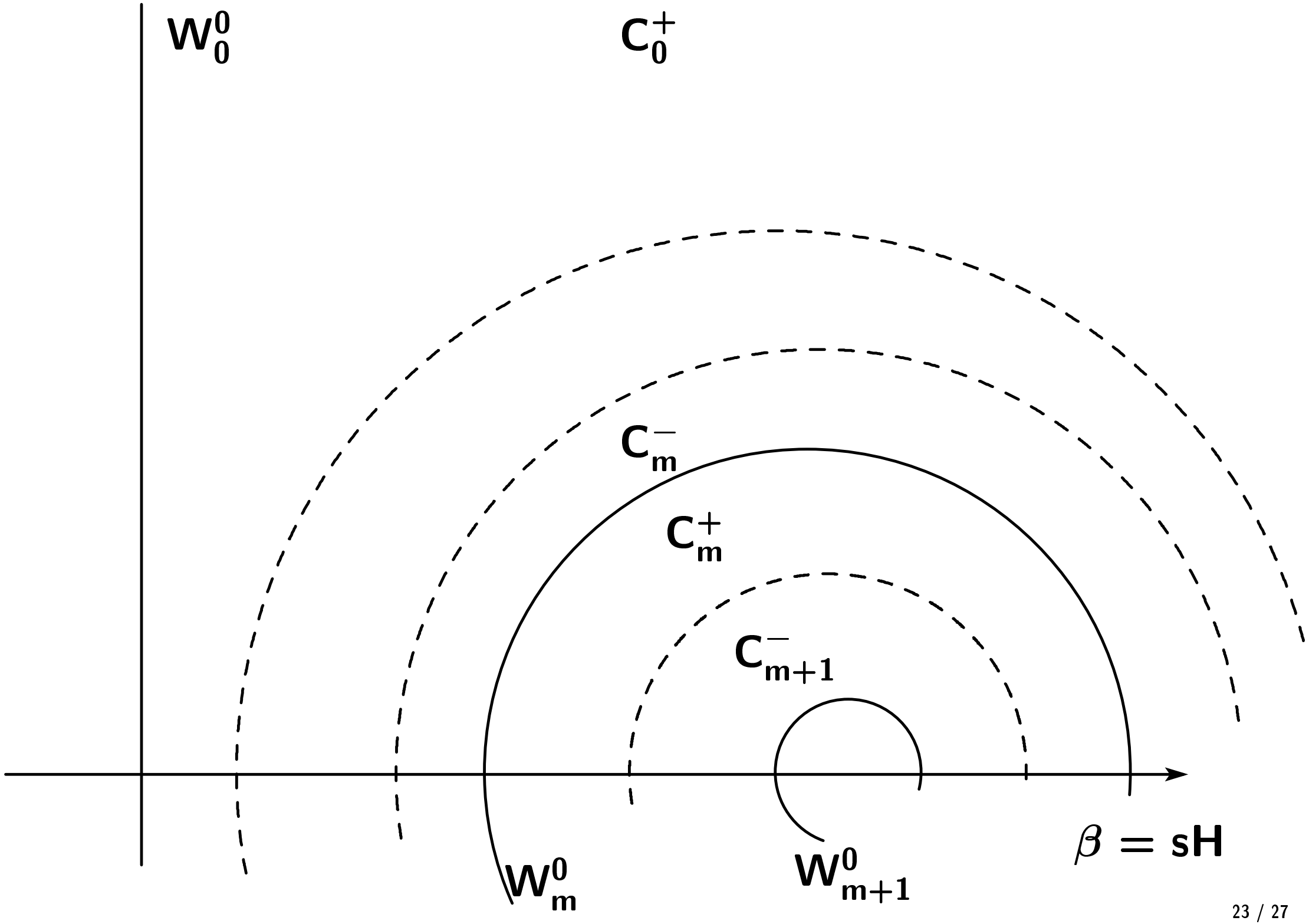
with

$$\mathbf{W}^0_{\mathbf{m}} := \{(\mathbf{sH}, \mathbf{tH}) \mid (\mathbf{s} - \tfrac{1}{\sqrt{\mathbf{n}}}\tfrac{\mathbf{b_m}}{\mathbf{a_m}})(\mathbf{s} - \tfrac{1}{\sqrt{\mathbf{n}}}\tfrac{\ell\mathbf{b_m}}{\mathbf{a_m}}) + \mathbf{t}^2 = 0\},$$

$$\mathbf{W}^0_0 := \{(\mathbf{sH}, \mathbf{tH}) \mid \mathbf{s} = 0\},$$

$$\begin{pmatrix} \mathbf{b_m} & \ell\mathbf{a_m} \\ \mathbf{a_m} & \mathbf{b_m} \end{pmatrix} = \begin{pmatrix} \mathbf{q} & \ell\mathbf{p} \\ \mathbf{p} & \mathbf{q} \end{pmatrix}^{\mathbf{m}}, \ \mathbf{p}, \mathbf{q} \in \mathbb{Z}_{>0}, \ \mathbf{q}^2 - \ell\mathbf{p}^2 = \pm 1.$$

□


W_0^0
C_0^+
C_m^-
C_m^+
C_{m+1}^-
W_m^0
W_{m+1}^0
$\beta = sH$

Set $M_m := M_{C_m^-}(1,0,-\ell) \cap M_{C_{m-1}^+}(1,0,-\ell)$

Prop. 3 (1) (By Prop. 2 (1) Large volume limit)

$$M_0 \cong \widehat{X} \times \mathrm{Hilb}^{\ell}(X).$$

(2) $M_m \neq \emptyset$, $M_m \overset{\text{birat.}}{\dashrightarrow} M_{C_{m-1}^+}, M_{C_m^-}$

(3) $\exists$ isomorphism of schemes

$$\cdots \to M_{-2} \overset{\Psi_{-2}}{\to} M_{-1} \overset{\Psi_{-1}}{\to} M_0 \overset{\Psi_0}{\to} M_1 \overset{\Psi_1}{\to} M_2 \to \cdots$$

(Ψ_m is of the form $\Phi^{\mathcal{E}}_{Y\to X}\mathcal{D}_Y\Phi^{\mathcal{E}^\vee}_{X\to Y}[-]$) □

Sketch of Prf. of Main Thm. 2 ($\ell = 2$)

- Condition $\#\{\text{quadratic form, discr.} = \ell\}/\sim = 1$ and Prop. 2 (2) [preservation for stability],

  $\Longrightarrow \exists$ FMT inducing $\mathbf{M}_X^H(\mathbf{v}) \xrightarrow{\sim} \mathbf{M}_{(\beta,\omega)}(1, 0, -\ell)$ with some (β, ω) in a chamber $\mathbf{C}$ for $(1, 0, -\ell)$.

- $\not\exists$ wall for $(1, 0, -\ell)$ of codimension ≥ 1

  $\Longrightarrow \mathbf{C} = \mathbf{C}^+_{m-1} = \mathbf{C}^-_m$

- In the end we have

$$\begin{array}{ccccc} \mathbf{M}_X^H(\mathbf{v}) & \overset{\text{FMT}}{\cong} & \mathbf{M}_{(\beta,\omega)}(1, 0, -\ell) = \mathbf{M}_m & & \\ \overset{\text{Prop3 (3)}}{\cong} & \mathbf{M}_0 & \overset{\text{Prop3 (1)}}{\cong} & \mathbf{X} \times \mathbf{Hilb}^\ell(\mathbf{X}). & \end{array}$$

§4. Conclusion

- By utilizing FMTs, one can find many birational morphisms between moduli spaces of stable sheaves.

- By analyzing chamber-wall structure of Bridgeland stability conditions, one can check the above birational morphisms are isomorphisms in some cases.

Remaining Conjecture [Mukai (1980)]

Assume X is principally polarized and $NS(X) \cong \mathbb{Z}$. Then

$$\#\{\text{quadratic form with discriminant} = \ell\}/\underset{GL(2,\mathbb{Z})}{\sim}$$

$$\overset{?}{=} \#\{M_X^H(v) \mid v : \text{positive}, \langle v^2 \rangle = 2\ell\}/\underset{\text{birat.}}{\sim}$$

□

Rmk. By the method of Main Thm. 1, we have

$$(\text{LHS}) \geq (\text{RHS})$$

□