Summary and Problems of Lecture 4 [*1]

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The assignmentsare Exercises 4.1–4.6. The deadline of the report is **October 29th (Monday)**.

Correction of §2.2 Definition and Exercise 2.4.

The power of t in $A_I(x;t)$ was wrong.

$$D_x^{(r)} = \sum_{I \subset \{1,\dots,n\},\, |I|=r} A_I(x;t) T_{q,x}^I, \quad A_I(x;t) := \undertilde{t^{\binom{r}{2}}} \prod_{i \in I, j \notin I} \frac{tx_i - x_j}{x_i - x_j}.$$

4 Koornwinder polynomials

4.1 Macdonald polynomials for $R = \mathrm{BC}_n$

We consider the Macdonald polynomial of the admissible pair $(R, S) = (\mathrm{BC}_n, \mathrm{B}_n)$. Let us use the description of BC_n given in §3.1.

$$\begin{aligned}
V &:= \mathbb{R}^n = \oplus_{i=1}^n \mathbb{R}\varepsilon_i, \quad L := \oplus_{i=1}^n \mathbb{Z}\varepsilon_i, \\
R &:= \{v \in V \mid (v,v) = 1, 2 \text{ or } 4\} \cap L = R(\mathrm{B}_n) \cup R(\mathrm{C}_n) \\
&= \{\pm\varepsilon_i \mid 1 \le i \le n\} \cup \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \le i < j \le n\} \cup \{\pm 2\varepsilon_i \mid 1 \le i \le n\}, \\
W &\simeq S_n \ltimes (\{\pm 1\})^n, \quad \text{acting on } V \text{ by permuting } \varepsilon_i\text{'s and } \varepsilon_i \mapsto -\varepsilon_i.
\end{aligned}$$

The weight lattice P, the root lattice Q, and etc. of R are given by

$$\begin{aligned}
P &= Q = \textstyle\sum_{i=1}^n \mathbb{Z}\varepsilon_i. \\
A &:= \mathbb{C}[P] = \mathbb{C}\text{-span of } \{e^\lambda \mid \lambda \in P\}. \\
R^+ &:= \{\varepsilon_i\} \cup \{\varepsilon_i \pm \varepsilon_j \mid i < j\} \cup \{2\varepsilon_i\}. \\
P^+ &= \{m_1\varepsilon_1 + \cdots + m_n\varepsilon_n \mid m_i \in \mathbb{N},\ m_1 \ge m_2 \ge \cdots \ge m_n\}. \\
Q^+ &= \{m_1(\varepsilon_1 - \varepsilon_2) + \cdots + m_{n-1}(\varepsilon_{n-1} - \varepsilon_n) + m_n\varepsilon_n \mid m_i \in \mathbb{N}\}.
\end{aligned}$$

Recall that the dominance order on P is given by $\lambda \ge \mu \iff \lambda - \mu \in Q^+$.

To define the weight function for BC_n we set

$$R_1^+ := \{2\varepsilon_i \mid 1 \le i \le n\}, \quad R_2^+ := \{\varepsilon_i \pm \varepsilon_j \mid 1 \le i < j \le n\}.$$

Definition. Fix $q \in (0,1)$ and $a, b, c, d, t \in \mathbb{C}$. The weight function $\Delta(x)$ on V is defined to be

$$\Delta(x) := \Delta^+(x)\overline{\Delta^+(x)}, \quad \Delta^+ := \prod_{\alpha \in R_1^+} \frac{(e^\alpha; q)_\infty}{(ae^{\alpha/2}, be^{\alpha/2}, ce^{\alpha/2}, de^{\alpha/2}; q)_\infty} \prod_{\alpha \in R_2^+} \frac{(e^\alpha; q)_\infty}{(te^\alpha; q)_\infty}.$$

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The pairing $\langle\cdot,\cdot\rangle$ on A^W to be

$$\langle f, g\rangle \;:=\; \int_T f(\dot{x})g(\dot{x})\Delta(\dot{x})\,d\dot{x},$$

where $T = V/2\pi Q^\vee = \prod_{i=1}^n(\mathbb{R}\varepsilon_i/2\pi\mathbb{Z}\varepsilon_i)$, and $d\dot{x}$ is the normalized Haar measure on T.

Recall that for $\lambda \in P$ we defined

$$m_\lambda \;=\; \sum_{\mu\in W.\lambda} e^\mu \;\in\; A^W.$$

Theorem 4.1. Assume a, b, c, d satisfy the following three conditions.

- $a, b, c, d \in \mathbb{R}$, or $a, b, c, d \in \mathbb{C}$ appearing in complex conjugate pairs,
- $|a|, |b|, |c|, |d| \le 1$,
- none of pairwise products of a, b, c, d is ≥ 1.

Then for each $\lambda \in P^+$ there exists a unique $P_\lambda \in A^W$ satisfying the following two conditions.

(i) $P_\lambda = m_\lambda + \sum_{\mu<\lambda} c_{\lambda,\mu} m_\mu$ with some $c_{\lambda,\mu} \in \mathbb{C}$.

(ii) $\langle P_\lambda, m_\mu\rangle = 0$ for $\mu \in P^+$ with $\mu < \lambda$.

This P_λ is called the **Koornwinder polynomial**.

4.2 BC_1 case = Askey-Wilson polynomial

Let us apply the argument of the previous §4.1 to the case BC_1. We have

$$\begin{aligned} &R \;=\; \{\pm\varepsilon_1, \pm 2\varepsilon_1\} \;\subset\; V \;=\; \mathbb{R}\varepsilon_1 \curvearrowleft W \;=\; \{\pm 1\}, \quad \varepsilon_1 \mapsto \pm\varepsilon_1, \\ &P \;=\; \mathbb{Z}\varepsilon_1 \;\supset\; P^+ \;=\; \mathbb{N}\varepsilon_1, \quad A \;=\; \mathbb{C}[e^{\pm\varepsilon_1}]. \end{aligned}$$

The monomial function m_λ for $\lambda = l\varepsilon_1$ is given by

$$m_{l\varepsilon 1}(x) \;:=\; \begin{cases} e^{ilx} + e^{-ilx} & (l = 1, 2, \dots) \\ 1 & (l = 0) \end{cases}$$

with $x \in \mathbb{R} = V$. The weight function $\Delta^+(x)$ is given by

$$\Delta^+(x) \;=\; \frac{(e^{2ix}; q)_\infty}{(ae^{ix}, be^{ix}, ce^{ix}, de^{ix}; q)_\infty}.$$

Definition. The **basic hypergeometric series** is defined to be

$${}_r\phi_s\left[\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, z\right] \;:=\; \sum_{n=0}^\infty \frac{(a_1, \dots, a_r; q)_n}{(b_1, \dots, b_s; q)_n}\left((-1)^n q^{\binom{n}{2}}\right)^{1+s-r} \frac{z^n}{(q;q)_n}. \tag{4.1}$$

We always assume that no zeros appear in the denominatorof (4.1).

Exercise 4.1 ($**$). Show the following convergence condition of the basic hypergeometric series: If $r \ge s$, the series (4.1) converges for any z if $|q| < 1$. If $r = s - 1$, then it converges for $|z| < 1$.

Definition. For $l \in \mathbb{N}$, the **Askey-Wilson polynomial** is defined to be

$$P_l(y; a, b, c, d \,|\, q) \;:=\; a^{-l}\frac{(ab, ac, ad; q)_l}{(abcd; q)_l} \cdot {}_4\phi_3\left[\begin{matrix} q^{-l}, q^{l-1}abcd, az, az^{-1} \\ ab, ac, ad \end{matrix}; q, q\right],$$

where we set $y = (z + z^{-1})/2$.

Exercise 4.2 (∗)**.** Check that $P_l(y; a, b, c, d \,|\, q)$ is a polynomial of degree l in terms of y.

Theorem. The Koornwinder polynomial of BC_1 is equal to the Askey-Wilson polynomial. Precisely speaking, setting $y = (e^{ix} + e^{-ix})/2$, we have

$$P_{l\varepsilon_1}(x) \;=\; P_l(y; a, b, c, d \,|\, q).$$

4.3 Wilson and Jacobi polynomials

Definition. The (generalized) hypergeometric series is defined to be

$${}_rF_s\Big[\begin{matrix}\alpha_1, \alpha_2, \ldots, \alpha_r \\ \beta_1, \ldots, \beta_s\end{matrix}; x\Big] \;:=\; \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_r)_n}{(\beta_1)_n \cdots (\beta_s)_n} \frac{x^n}{n!},$$

where we used

$$(\alpha)_0 \;:=\; 1, \quad (\alpha)_n \;:=\; \alpha(\alpha+1)\cdots(\alpha+n-1).$$

The case $(r, s) = (2, 1)$ is the **Gauss hypergeometric function**(Gauss HGF in short).

Exercise 4.3 (∗)**.** Assume $\alpha, \beta, \gamma \in \mathbb{C}$ with $\gamma \notin \{-1, -2, -3, \ldots\}$. Show that the series

$${}_2F_1(a, b; c; x) \;=\; \sum_{n \geq 0} \frac{(a)_n (b)_n}{(1)_n (c)_n} x^n \;=\; 1 + \frac{ab}{c} x + \frac{a(a+1)b(b+1)}{2c(c+1)} x^2 + \cdots$$

converges for $|x| < 1$, and that it is a solution of the **Gauss hypergeometric differential equation**:

$$x(1-x)\frac{d^2y}{dx^2} + (c - (a+b+1)x)\frac{dy}{dx} - aby \;=\; 0.$$

Exercise 4.4 (∗)**.** Assume $|x| < 1$. Show that the limit of ${}_2\phi_1$ under $q \to 1$ is the Gauss HGF:

$$\lim_{q \to 1} {}_2\phi_1\Big[\begin{matrix}q^\alpha, q^\beta \\ q^\gamma\end{matrix}; q, x\Big] \;=\; {}_2F_1(\alpha, \beta; \gamma; x).$$

Remark. ${}_2\phi_1$ is called **Heine's q-hypergeometric series** after the work of E. Heine in 1840s.

Proposition. Set $a = q^\alpha$, $b = q^\beta$, $c = q^\gamma$, $d = q^\delta$ and $y = z + z^{-1} = q^{ix} + q^{-ix}$. Then we have

$$\lim_{q \to 1} P_n(y; a, b, c, d \,|\, q) \;=\; (\alpha+\beta)_n(\alpha+\gamma)_n(\alpha+\delta)_n \cdot {}_4F_3\Big[\begin{matrix}-n, \alpha+\beta+\gamma+\delta+n-1, \alpha+ix, \alpha-ix \\ \alpha+\beta, \alpha+\gamma, \alpha+\delta\end{matrix}; 1\Big].$$

The right hand side is called the **Wilson polynomial**.

Definition. The **Jacobi polynomial** $P_n^{(\alpha,\beta)}(z)$ is defined as a specialization of Gauss HGF.

$$P_n^{(\alpha,\beta)}(z) \;:=\; \frac{(\alpha+1)_n}{(1)_n} {}_2F_1\big(-n, \alpha+\beta+n+1; \alpha+1; \tfrac{1}{2}(1-z)\big).$$

Remark. The Jacobi polynomial $P_n^{(\alpha,\beta)}(z)$ can be obtained from the Wilson polynomial $p_n(x; \alpha, \beta, \gamma, \delta)$ by setting $\gamma = \overline{\delta} := b + iN$, $x := iNz$, then taking the limit $N \to \infty$, and replacing parameters $(\alpha+\beta, b) \mapsto (\alpha, \beta)$.

Since $(-n)_k = 0$ for $k > n$, $P_n^{(\alpha,\beta)}(z)$ is a terminate series, in other words a polynomial, of degree n.

$$\begin{aligned} P_n^{(\alpha,\beta)}(z) &= \frac{(\alpha+1)_n}{(1)_n} \sum_{k=0}^{n} \frac{(-n)_k(\alpha+\beta+n+1)_k}{(1)_k(\alpha+1)_k} \Big(\frac{1-z}{2}\Big)^k \\ &= \frac{(\alpha+1)_n}{(1)_n} \sum_{k=0}^{n} \binom{n}{k} \frac{(\alpha+\beta+n+1)_k}{(\alpha+1)_k} \Big(\frac{z-1}{2}\Big)^k. \end{aligned} \tag{4.2}$$

Exercise 4.5 $(*)$**.** Check the following **Rodrigues formula**.

$$P_n^{(\alpha,\beta)}(z) = \frac{(-1)^n}{2^n(1)_n}(1-z)^{-\alpha}(1+z)^{-\beta}\frac{d^n}{dz^n}\Big((1-z)^\alpha(1+z)^\beta(1-z^2)^n\Big). \tag{4.3}$$

Remark. Setting $\alpha = \beta = 0$ in the Rodrigues formula (4.3), we see that the Jacobi polynomial reduces to the **Legendre polynomial**.

$$P_n^{(0,0)}(z) = \frac{1}{2^n n!}\frac{d^n}{dz^n}\Big((1-z^2)^n\Big).$$

Proposition 4.2. The Jacobi polynomials enjoy the following orthogonality condition.

$$\int_{-1}^{1} (1-x)^\alpha(1+x)^\beta P_m^{(\alpha,\beta)}(x)P_n^{(\alpha,\beta)}(x)\,dx = \delta_{m,n}I_n, \tag{4.4}$$

where

$$I_n := \frac{2^{\alpha+\beta+1}}{\alpha+\beta+2n+1}\frac{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{\Gamma(n+1)\Gamma(\alpha+\beta+n+1)}.$$

Exercise 4.6 $(**)$**.** Show Proposition 4.2 directly by taking the following steps.

(1) Using the Rodrigues formula (4.3) and integration by parts, show that for any polynomial $Q(x)$ of degree $m \le n$, we have

$$\int_{-1}^{1} (1-x)^\alpha(1+x)^\beta P_n^{(\alpha,\beta)}(x)Q(x)\,dx = \frac{1}{2^n n!}\int_{-1}^{1} Q^{(n)}(x)\,(1-x)^{\alpha+n}(1+x)^{\beta+n}\,dx.$$

(2) Set $I_{m,n} :=$ (LHS of (4.4)). Shows that $I_{m,n} = 0$ for $m \ne n$.

(3) In the case $m = n$, the calculation in (1) will give

$$\int_{-1}^{1} (1-x)^\alpha(1+x)^\beta \big(P_n^{(\alpha,\beta)}(x)\big)^2\,dx = \frac{1}{2^n n!}\int_{-1}^{1}(1-x)^{\alpha+n}(1+x)^{\beta+n}\frac{d^n}{dx^n}\big(P_n^{(\alpha,\beta)}(x)\big)\,dx.$$

Since $P_n^{(\alpha,\beta)}(x)$ is a polynomial of degree n, it is enough to calculate its top term $k_n x^n$. Using the expression (4.2), check that

$$k_n = \frac{1}{2^n}\frac{\Gamma(\alpha+\beta+2n+1)}{\Gamma(n+1)\Gamma(\alpha+\beta+n+1)}.$$

(4) Using the beta integral

$$B(a,b) := \int_0^1 x^{a-1}(1-x)^{b-1}\,dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},$$

show that $I_{n,n} = I_n$.