Lecture 4: Koornwinder polynomials *1

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4 Koornwinder polynomials

4.1 Macdonald polynomials for $R = \mathrm{BC}_n$

This subsection follows [K92, §5].

Recall that for an admissible pair (R, S) of root systems one can construct the Macdonald polynomials P_λ for $\lambda \in P =$ the weight lattice of R. Let us study the Macdonald polynomial in the case R being irreducible but not reduced. By the classification of admissible pairs in §3.2, we have $R = \mathrm{BC}_n$.

Let us use the description of BC_n given in §3.1.

$$\begin{aligned}
V &:= \mathbb{R}^n = \oplus_{i=1}^n \mathbb{R}\varepsilon_i, \quad L := \oplus_{i=1}^n \mathbb{Z}\varepsilon_i, \\
R &:= \{v \in V \mid (v,v) = 1, 2 \text{ or } 4\} \cap L = R(\mathrm{B}_n) \cup R(\mathrm{C}_n) \\
&= \{\pm\varepsilon_i \mid 1 \le i \le n\} \cup \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \le i < j \le n\} \cup \{\pm 2\varepsilon_i \mid 1 \le i \le n\}, \\
W &\simeq S_n \ltimes (\{\pm 1\})^n, \quad \text{acting on } V \text{ by permuting } \varepsilon_i\text{'s and } \varepsilon_i \mapsto -\varepsilon_i.
\end{aligned}$$

The weight lattice P and the root lattice Q of R are given by

$$P = Q = \textstyle\sum_{i=1}^n \mathbb{Z}\varepsilon_i,$$

and we have

$$A := \mathbb{C}[P] = \mathbb{C}\text{-span of } \{e^\lambda \mid \lambda \in P\}.$$

As for the set of positive roots, we put

$$R^+ := \{\varepsilon_i\} \cup \{\varepsilon_i \pm \varepsilon_j \mid i < j\} \cup \{2\varepsilon_i\}.$$

Then we have

$$\begin{aligned}
P^+ &= \{m_1\varepsilon_1 + \cdots + m_n\varepsilon_n \mid m_i \in \mathbb{N},\ m_1 \ge m_2 \ge \cdots \ge m_n\}, \\
Q^+ &= \{m_1(\varepsilon_1 - \varepsilon_2) + \cdots + m_{n-1}(\varepsilon_{n-1} - \varepsilon_n) + m_n\varepsilon_n \mid m_i \in \mathbb{N}\}.
\end{aligned}$$

Recall that the dominance order on P is given by $\lambda \ge \mu \iff \lambda - \mu \in Q^+$.

To define the weight function for BC_n we set

$$R_1^+ := \{2\varepsilon_i \mid 1 \le i \le n\}, \quad R_2^+ := \{\varepsilon_i \pm \varepsilon_j \mid 1 \le i < j \le n\}.$$

Definition. Fix $q \in (0,1)$ and $a, b, c, d, t \in \mathbb{C}$. The weight function $\Delta(x)$ on V is defined to be

$$\Delta(x) := \Delta^+(x)\overline{\Delta^+(x)}, \quad \Delta^+ := \prod_{\alpha \in R_1^+} \frac{(e^\alpha; q)_\infty}{(ae^{\alpha/2}, be^{\alpha/2}, ce^{\alpha/2}, de^{\alpha/2}; q)_\infty} \prod_{\alpha \in R_2^+} \frac{(e^\alpha; q)_\infty}{(te^\alpha; q)_\infty}.$$

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The pairing $\langle \cdot, \cdot \rangle$ on A^W to be

$$\langle f, g \rangle := \int_T f(\dot{x}) g(\dot{x}) \Delta(\dot{x})\, d\dot{x},$$

where $T = V/2\pi Q^\vee = \prod_{i=1}^n (\mathbb{R}\varepsilon_i / 2\pi\mathbb{Z}\varepsilon_i)$, and $d\dot{x}$ is the normalized Haar measure on T.

Recall that for $\lambda \in P$ we defined

$$m_\lambda = \sum_{\mu \in W.\lambda} e^\mu \in A^W.$$

Theorem 4.1 ([K92, §5]). Assume a, b, c, d satisfy the following three conditions.

- $a, b, c, d \in \mathbb{R}$, or $a, b, c, d \in \mathbb{C}$ appearing in complex conjugate pairs,
- $|a|, |b|, |c|, |d| \le 1$,
- none of pairwise products of a, b, c, d is ≥ 1.

Then for each $\lambda \in P^+$ there exists a unique $P_\lambda \in A^W$ satisfying the following two conditions.

(i) $P_\lambda = m_\lambda + \sum_{\mu < \lambda} c_{\lambda,\mu} m_\mu$ with some $c_{\lambda,\mu} \in \mathbb{C}$.

(ii) $\langle P_\lambda, m_\mu \rangle = 0$ for $\mu \in P^+$ with $\mu < \lambda$.

This P_λ is called the **Koornwinder polynomial**.

Theorem 4.2 ([K92, §5]). The Koornwinder polynomials $\{P_\lambda \mid \lambda \in P^+\}$ form an orthogonal system with respect to the pairing $\langle \cdot, \cdot, \rangle$.

$$\langle P_\lambda, P_\mu \rangle = 0 \quad \lambda \ne \mu.$$

Remark. For the more detailed discussion, see [M03, Chap.5].

4.2 BC_1 case = Askey-Wilson polynomial

Let us apply the argument of the previous §4.1 to the case BC_1. We have

$$\begin{aligned} &R = \{\pm\varepsilon_1, \pm 2\varepsilon_1\} \subset V = \mathbb{R}\varepsilon_1 \curvearrowleft W = \{\pm 1\}, \quad \varepsilon_1 \mapsto \pm\varepsilon_1, \\ &P = \mathbb{Z}\varepsilon_1 \supset P^+ = \mathbb{N}\varepsilon_1, \quad A = \mathbb{C}[e^{\pm\varepsilon_1}]. \end{aligned}$$

The monomial function m_λ for $\lambda = l\varepsilon_1$ is given by

$$m_{l\varepsilon_1}(x) := \begin{cases} e^{ilx} + e^{-ilx} & (l = 1, 2, \ldots) \\ 1 & (l = 0) \end{cases}$$

with $x \in \mathbb{R} = V$. The weight function $\Delta^+(x)$ is given by

$$\Delta^+(x) = \frac{(e^{2ix}; q)_\infty}{(ae^{ix}, be^{ix}, ce^{ix}, de^{ix}; q)_\infty}.$$

An explicit expression of the Koornwinder polynomial $P_{l\varepsilon_1}(x)$ is known. To explain it, we prepare

Definition. The **basic hypergeometric series**[*2] is defined to be

$$ {}_r\phi_s \left[\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} ; q, z \right] := \sum_{n=0}^\infty \frac{(a_1, \ldots, a_r; q)_n}{(b_1, \ldots, b_s; q)_n} \left((-1)^n q^{\binom{n}{2}} \right)^{1+s-r} \frac{z^n}{(q;q)_n}. \tag{4.1}$$

[*2] hypergeometric series は超幾何級数. basic の訳語は特に無く, かわりに q 超幾何級数と言ったりする.

We always assume that no zeros appear in the denominator*3 of (4.1).

Exercise 4.1 (∗∗)**.** Show the following convergence condition of the basic hypergeometric series: If $r \geq s$, the series (4.1) converges for any z if $|q| < 1$. If $r = s - 1$, then it converges for $|z| < 1$.

Definition ([AW85, (1.15)])**.** For $l \in \mathbb{N}$, the **Askey-Wilson polynomial** is defined to be

$$P_l(y; a, b, c, d \,|\, q) \;:=\; a^{-l} \frac{(ab, ac, ad; q)_l}{(abcd; q)_l} \cdot {}_4\phi_3 \left[\begin{matrix} q^{-l}, q^{l-1}abcd, az, az^{-1} \\ ab, ac, ad \end{matrix} ; q, q \right],$$

where we set $y = (z + z^{-1})/2$.

Remark. We slightly changed normalization from [AW85] so that the condition (i) of Theorem 4.1 is satisfied.

Exercise 4.2 (∗)**.** Check that $P_l(y; a, b, c, d \,|\, q)$ is a polynomial of degree l in terms of y.

Theorem. The Koornwinder polynomial of BC_1 is equal to the Askey-Wilson polynomial. Precisely speaking, setting $y = (e^{ix} + e^{-ix})/2$, we have

$$P_{l\varepsilon_1}(x) \;=\; P_l(y; a, b, c, d \,|\, q).$$

The orthogonality of the Koornwinder polynomial (Theorem 4.2) implies

Proposition 4.3 ([AW85, Theorem 2.2])**.** Assume the same conditions on a, b, c, d as in Theorem 4.1. Then we have

$$\int_{-1}^{1} P_m(y; a, b, c, d \,|\, q)\, P_n(y; a, b, c, d \,|\, q) \frac{w(y)}{2\pi\sqrt{1 - y^2}}\, dy \;=\; 0, \quad m \neq n,$$

where

$$w(y) \;:=\; \frac{\prod_{k=0}^{\infty}(1 - (2y^2 - 1)q^k + q^{2k})}{h(y, a)h(y, b)h(y, c)h(y, d)}, \quad h(y, a) \;:=\; \prod_{k=0}^{\infty}(1 - 2ayq^k + a^2q^{2k}).$$

Remark. For the more detailed discussion, see [M03, §§6.4–6.5].

4.3 Wilson and Jacobi polynomials

Finally we treat the "Schur limit" of the Askey-Wilson polynomial. It gives rise to the classical orthogonal polynomial called **Jacobi polynomial**.

Definition. The (generalized) hypergeometric series is defined to be

$${}_rF_s \left[\begin{matrix} \alpha_1, \alpha_2, \ldots, \alpha_r \\ \beta_1, \ldots, \beta_s \end{matrix} ; x \right] \;:=\; \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_r)_n}{(\beta_1)_n \cdots (\beta_s)_n} \frac{x^n}{n!},$$

where we used

$$(\alpha)_0 \;:=\; 1, \quad (\alpha)_n \;:=\; \alpha(\alpha + 1) \cdots (\alpha + n - 1).$$

The case $(r, s) = (2, 1)$ is the **Gauss hypergeometric function***4(Gauss HGF in short).

*3 分母

*4 Gauss の超幾何関数

Exercise 4.3 ($*$)**.** Assume $\alpha, \beta, \gamma \in \mathbb{C}$ with $\gamma \notin \{-1, -2, -3, \ldots\}$. Show that the series

$$ {}_2F_1(a, b; c; x) \;=\; \sum_{n \geq 0} \frac{(a)_n (b)_n}{(1)_n (c)_n} x^n \;=\; 1 + \frac{ab}{c} x + \frac{a(a+1)b(b+1)}{2c(c+1)} x^2 + \cdots $$

converges for $|x| < 1$, and that it is a solution of the **Gauss hypergeometric differential equation** [*5]:

$$ x(1-x)\frac{d^2y}{dx^2} + (c - (a+b+1)x)\frac{dy}{dx} - aby \;=\; 0. $$

The following exercise shows that the basic hypergeometric series is a q**-analogue**[*6] of the hypergeometric series.

Exercise 4.4 ($*$)**.** Assume $|x| < 1$. Show that the limit of ${}_2\phi_1$ under $q \to 1$ is the Gauss HGF:

$$ \lim_{q \to 1} {}_2\phi_1 \begin{bmatrix} q^\alpha, q^\beta \\ q^\gamma \end{bmatrix} ; q, x \Big] \;=\; {}_2F_1(\alpha, \beta; \gamma; x). $$

Remark. ${}_2\phi_1$ is called **Heine's q-hypergeometric series** after the work of E. Heine in 1840s. For more details about basic hypergeometric series, see [GR04].

Let us calculate the $q \to 1$ limit of the Askey-Wilson polynomial $P_n(y; a, b, c, d \,|\, q)$.

Proposition. Set $a = q^\alpha$, $b = q^\beta$, $c = q^\gamma$, $d = q^\delta$ and $y = z + z^{-1} = q^{ix} + q^{-ix}$. Then we have

$$ \lim_{q \to 1} P_n(y; a, b, c, d \,|\, q) \;=\; (\alpha+\beta)_n (\alpha+\gamma)_n (\alpha+\delta)_n \cdot {}_4F_3 \left[\begin{matrix} -n, \alpha+\beta+\gamma+\delta+n-1, \alpha+ix, \alpha-ix \\ \alpha+\beta, \alpha+\gamma, \alpha+\delta \end{matrix} ; 1 \right]. $$

The right hand side[*7] is called the **Wilson polynomial**.

Here we treat a degenerate version of Wilson polynomial.

Definition. The **Jacobi polynomial** $P_n^{(\alpha,\beta)}(z)$ is defined as a specialization of Gauss HGF.

$$ P_n^{(\alpha,\beta)}(z) \;:=\; \frac{(\alpha+1)_n}{(1)_n} {}_2F_1\big(-n, \alpha+\beta+n+1; \alpha+1; \tfrac{1}{2}(1-z)\big). $$

Remark. The Jacobi polynomial $P_n^{(\alpha,\beta)}(z)$ can be obtained from the Wilson polynomial $p_n(x; \alpha, \beta, \gamma, \delta)$ by setting $\gamma = \overline{\delta} := b + iN$, $x := iNz$, then taking the limit $N \to \infty$, and replacing parameters $(\alpha+\beta, b) \mapsto (\alpha, \beta)$.

Since $(-n)_k = 0$ for $k > n$, $P_n^{(\alpha,\beta)}(z)$ is a terminate series, in other words a polynomial, of degree n.

$$ \begin{aligned} P_n^{(\alpha,\beta)}(z) \;&=\; \frac{(\alpha+1)_n}{(1)_n} \sum_{k=0}^{n} \frac{(-n)_k (\alpha+\beta+n+1)_k}{(1)_k (\alpha+1)_k} \Big(\frac{1-z}{2}\Big)^k \\ &=\; \frac{(\alpha+1)_n}{(1)_n} \sum_{k=0}^{n} \binom{n}{k} \frac{(\alpha+\beta+n+1)_k}{(\alpha+1)_k} \Big(\frac{z-1}{2}\Big)^k. \end{aligned} \tag{4.2} $$

[*5] 超幾何微分方程式
[*6] q 類似
[*7] 右辺

Exercise 4.5 (∗)**.** Check the following **Rodrigues formula**.

$$P_n^{(\alpha,\beta)}(z) \;=\; \frac{(-1)^n}{2^n(1)_n}(1-z)^{-\alpha}(1+z)^{-\beta}\frac{d^n}{dz^n}\Big((1-z)^{\alpha}(1+z)^{\beta}(1-z^2)^n\Big). \tag{4.3}$$

Remark. Setting $\alpha=\beta=0$ in the Rodrigues formula (4.3), we see that the Jacobi polynomial reduces to the **Legendre polynomial**.

$$P_n^{(0,0)}(z) \;=\; \frac{1}{2^n n!}\frac{d^n}{dz^n}\Big((1-z^2)^n\Big).$$

The orthogonality of the Askey-Wilson polynomials (Proposition 4.3) implies

Proposition 4.4. The Jacobi polynomials enjoy the following orthogonality condition.

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}P_m^{(\alpha,\beta)}(x)P_n^{(\alpha,\beta)}(x)\,dx \;=\; \delta_{m,n}I_n, \tag{4.4}$$

where

$$I_n \;:=\; \frac{2^{\alpha+\beta+1}}{\alpha+\beta+2n+1}\frac{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{\Gamma(n+1)\Gamma(\alpha+\beta+n+1)}.$$

Exercise 4.6 (∗∗)**.** Show Proposition 4.4 directly by taking the following steps.

(1) Using the Rodrigues formula (4.3) and integration by parts, show that for any polynomial $Q(x)$ of degree $m\le n$, we have

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}P_n^{(\alpha,\beta)}(x)Q(x)\,dx \;=\; \frac{1}{2^n n!}\int_{-1}^{1}Q^{(n)}(x)\,(1-x)^{\alpha+n}(1+x)^{\beta+n}\,dx.$$

(2) Set $I_{m,n} :=$ (LHS[*8]of (4.4)). Shows that $I_{m,n}=0$ for $m\neq n$.

(3) In the case $m=n$, the calculation in (1) will give

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}\big(P_n^{(\alpha,\beta)}(x)\big)^2\,dx \;=\; \frac{1}{2^n n!}\int_{-1}^{1}(1-x)^{\alpha+n}(1+x)^{\beta+n}\,\frac{d^n}{dx^n}\big(P_n^{(\alpha,\beta)}(x)\big)\,dx.$$

Since $P_n^{(\alpha,\beta)}(x)$ is a polynomial of degree n, it is enough to calculate its top term k_nx^n. Using the expression (4.2), check that

$$k_n \;=\; \frac{1}{2^n}\frac{\Gamma(\alpha+\beta+2n+1)}{\Gamma(n+1)\Gamma(\alpha+\beta+n+1)}.$$

(4) Using the beta integral

$$B(a,b) \;:=\; \int_0^1 x^{a-1}(1-x)^{b-1}\,dx \;=\; \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},$$

show that $I_{n,n}=I_n$.

4.4 Askey scheme

We close Part I lectures by mentioning the **Askey scheme** of hypergeometric orthogonal polynomials. In the last subsection we treated two orthogonal polynomials: Wilson and Jacobi polynomials. They are placed in the following degenerate diagram of orthogonal polynomials which can be expressed as hypergeometric series. There also exists a q-version of the Askey scheme. See [K94] for the detail.

[*8] left hand side = 左辺.

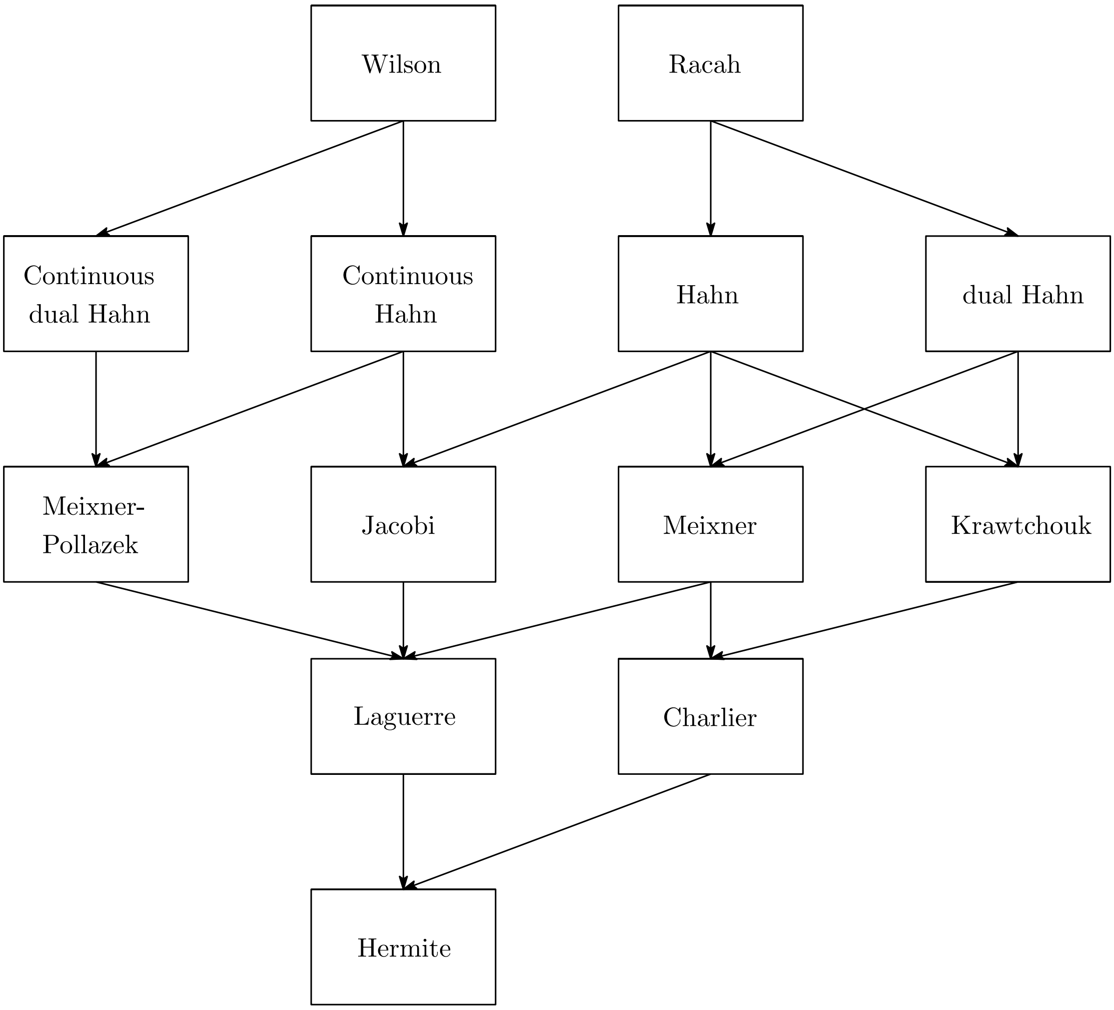

Figure 1 Askey scheme

References

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