Summary and Problems of Lecture 3 [*1]

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The assignments are Exercises 3.1–3.6 below. The deadline of the report is **October 29th (Monday)**.

3 Macdonald polynomials for general root systems

3.1 Root system

Theorem 3.2. Irreducible and reduced root systems are classified into the following types:

$$A_n\ (n \geq 1),\ \ B_n\ (n \geq 2),\ \ C_n\ (n \geq 3),\ \ D_n\ (n \geq 4),\ \ E_6,\ E_7,\ E_8,\ \ F_4,\ \ G_2.$$

The root system of type A_n is described as

type A_n $(n \geq 1)$: $V := \{\sum_{i=1}^{n+1} a_i \varepsilon_i \mid \sum_{i=1}^{n+1} a_i = 0\} \subset \mathbb{R}^{n+1} = \oplus_{i=1}^{n+1} \mathbb{R}\varepsilon_i, \quad L := \oplus_{i=1}^{n+1} \mathbb{Z}\varepsilon_i,$

$R := \{v \in V \mid (v, v) = 2\} \cap L = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i \neq j \leq n+1\}$, $|R| = (n+1)n$.

Simple roots: $\{\alpha_1 := \varepsilon_1 - \varepsilon_2,\ \alpha_2 := \varepsilon_2 - \varepsilon_3,\ \ldots,\ \alpha_n := \varepsilon_n - \varepsilon_{n+1}\}$.

$W \simeq S_{n+1}$, acting on V by permuting indices i of ε_i's.

Dynkin diagram: $\underset{\alpha_1}{\circ} - \underset{\alpha_2}{\circ} - \cdots - \underset{\alpha_{n-1}}{\circ} - \underset{\alpha_n}{\circ}$

Theorem 3.3. For each $n \in \mathbb{Z}_{\geq 1}$, there exists, up to isomorphism, a unique irreducible non-reduced root system of rank n. This root system is called of type BC_n.

3.2 Admissible pair

Definition. A pair (R, S) of root systems in a Euclidean space V is called **admissible** if it satisfies

(AP1) R is irreducible (but not necessarily reduced). S is irreducible and reduced.

(AP2) The sets of lines $\{\mathbb{R}a \mid a \in R\}$ and $\{\mathbb{R}b \mid b \in S\}$ coincide.

(AP3) $W(R) = W(S)$.

Proposition 3.4. Every admissible pair (R, S) is either of the following three classes.

- $(R, S) = (S, S)$ with S listed in Theorem 3.2.
- $(R, S) = (S, S^\vee)$ with S listed in Theorem 3.2.
- $(R, S) = (\mathrm{BC}_n, \mathrm{B}_n)$ or $(\mathrm{BC}_n, \mathrm{C}_n)$.

Proposition 3.5. There exists a decomposition $R = R^+ \cup (-R^+)$ for any admissible pair (R, S).

Proof. We only deal with the case where R is reduced. Choose the set $\{\alpha_i \mid i \in I\} \subset R$ of simple roots, Then every $\alpha \in R$ can be expanded as $\alpha = \sum_{i \in I} c_i \alpha_i$ with all the signs of c_i's being equal. Thus we can set $R^+ := \{\alpha \in R \mid c_i \geq 0\}$. □

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Definition 3.6. Choose a decomposition $R = R^+ \cup (-R^+)$.

(1) The **root lattice** $Q \subset R$ and the **positive cone** $Q^+ \subset Q$ of positive roots in R are defined to be

$$Q := \mathbb{Z}R = \mathbb{Z}\text{-span of } R, \quad Q^+ := \mathbb{N}R^+.$$

(2) The **weight lattice** P and the cone P^+ of **dominant weights** of R are defined to be

$$P := \{\lambda \in V \mid (\lambda, \alpha^\vee) \in \mathbb{Z} \ \ \forall \alpha \in R\}, \quad P^+ := \{\lambda \in V \mid (\lambda, \alpha^\vee) \in \mathbb{N} \ \ \forall \alpha \in R^+\}.$$

(3) The **dominance order** is a partial order on P defined by

$$\lambda \geq \mu \iff \lambda - \mu \in Q^+.$$

Exercise 3.1 $(*)$**.** Consider the root system R of type A_n described in Theorem 3.2. Let $R = R^+ \cup (-R^+)$ be the decomposition of R given in Proof of Proposition 3.5. Check that P^+ is then given by

$$P^+ = \textstyle\sum_{i=1}^n \mathbb{N}\,\omega_i = \{\lambda = \lambda^1\omega_1 + \cdots + \lambda^n\omega_n \mid \lambda^i \in \mathbb{N}\},$$

where ω_i is the **fundamental weight** given by

$$\omega_i = \frac{n+1-i}{n+1}(\varepsilon_1 + \cdots + \varepsilon_i) - \frac{i}{n+1}(\varepsilon_{i+1} + \cdots + \varepsilon_{n+1}).$$

3.3 Macdonald polynomials for general root systems

Fix an admissible pair (R, S) of root systems in a Euclidean space V. $W := W(R) = W(S)$. Fix a decomposition $R = R^+ \cup (-R^+)$. We use the symbols Q, Q^+, P, P^+ in Definition 3.6, and

$$A := \mathbb{C}[P] = \mathbb{C}\text{-span of } \{e^\lambda \mid \lambda \in P\}, \quad A^W := \{f \in A \mid wf = f \ \ \forall w \in W\}.$$

Proposition 3.7. For each $\alpha \in R$, there exists a unique $u_\alpha > 0$ such that $\alpha_* := \alpha/u_\alpha \in S$.

Remark 3.8. In particular, in the case $R = \mathrm{A}_n, \mathrm{D}_n, \mathrm{E}_n$ the value t_α is independent of $\alpha \in R$. We denote it by $t := t_\alpha$. Since $u_\alpha = 1$ for any $\alpha \in R$ by Proposition 3.7, we denote $k := k_\alpha$.

Recall the q**-shifted factorial**:

$$(a;q)_\infty := \prod_{j=0}^{\infty}(1 - aq^j), \quad (a_1, \ldots, a_r; q)_\infty := \prod_{i=1}^{r}(a_i; q)_\infty.$$

Definition. Assume R is reduced.

(1) The **weight function** $\Delta(v)$ on $v \in V$ is defined to be

$$\Delta(v) := \prod_{\alpha \in R} \frac{(t_{2\alpha}^{1/2} e^\alpha(v); q_\alpha)_\infty}{(t_\alpha t_{2\alpha}^{1/2} e^\alpha(v); q_\alpha)_\infty}.$$

Here we set $t_a := 1$ for $a \in V \setminus R$.

(2) A pairing $\langle \cdot, \cdot \rangle$ on A^W is defined to be

$$\langle f, g \rangle := |W|^{-1} \int_T f(\dot{v})\overline{g(\dot{v})}\Delta(\dot{v})\, d\dot{v}, \quad f, g \in A^W, \tag{3.1}$$

where $d\dot{v}$ denotes the Haar measure on T with the normalization $\int_T d\dot{v} = 1$.

Exercise 3.2 ($*$). (1) Check that $\langle \cdot, \cdot \rangle$ is a Hermitian pairing, i.e., $\overline{\langle f, g \rangle} = \langle g, f \rangle$.

(2) Assume $R = S = \mathrm{A}_n$ as in Exercise 3.1, and put $x_i := e^{\varepsilon_i}$. Recalling Remark 3.8, show that the weight function Δ is given by

$$\Delta = \prod_{1 \le i \ne j \le n+1} \frac{(x_i/x_j; q)_\infty}{(tx_i/x_j; q)_\infty}.$$

Theorem 3.9. Assume R is reduced. There exists a unique family $\{P_\lambda \mid \lambda \in P^+\} \subset A^W$ such that

(i) $P_\lambda = m_\lambda + \sum_{\mu < \lambda} c_{\lambda,\mu} m_\mu$ with some $c_{\lambda,\mu} \in \mathbb{C}$.

(ii) $\langle P_\lambda, m_\mu \rangle = 0$ if $\mu < \lambda$, where $<$ denotes the dominance order in Definition 3.6 (3).

Each $P_\lambda \in A^W$ is called the **Macdonald polynomial** of the admissible pair (R, S).

Theorem 3.10. Assume R is reduced. The family $\{P_\lambda \mid \lambda \in P^+\}$ satisfies

$$\langle P_\lambda, P_\mu \rangle = 0 \quad \lambda \ne \mu.$$

Theorem 3.11. Assume R is reduced. We have

$$\langle P_\lambda, P_\lambda \rangle = \prod_{\alpha \in R^+} \frac{(q^{1+(\lambda+\rho_k, \alpha^\vee)}; q)_\infty}{(tq^{1+(\lambda+\rho_k, \alpha^\vee)}; q)_\infty} \frac{(q^{(\lambda+\rho_k, \alpha^\vee)}; q)_\infty}{(tq^{(\lambda+\rho_k, \alpha^\vee)}; q)_\infty},$$

where ρ_k is defined to be

$$\rho_k := \frac{1}{2} \sum_{\alpha \in R^+} k_\alpha \alpha. \tag{3.2}$$

3.4 Macdonald difference operators for general root systems

We continue to use the same symbols as in the previous subsections. Define the q-shift operator $T_{q,v} = T_v$ on the functions $f : V \to \mathbb{C}$ to be

$$(T_v f)(x) := f(x - i \log q \cdot v) \quad (x, v \in V).$$

We have $T_v : A \to A$. Recall that by Proposition 3.7 we have $\alpha_* = \alpha/u_\alpha \in S$ for any $\alpha \in R$.

Proposition 3.12. For an admissible pair (R, S) with S not being of type E_8, F_4 nor G_2, there is a $\sigma \in V$ such that $\{(\sigma, \alpha_*) \mid \alpha \in R^+\} \subset \{0, 1\}$.

Exercise 3.3 ($*$). Assume $R = S = \mathrm{A}_n$ as in Exercise 3.2 (2). As for σ in Proposition 3.12, show that we can put $\sigma = \omega_r$ for each $r = 1, \ldots, n$.

Definition. We set

$$\Delta^+ := \prod_{\alpha \in R^+} \frac{(t_{2\alpha}^{1/2} e^\alpha; q_\alpha)_\infty}{(t_\alpha t_{2\alpha}^{1/2} e^\alpha; q_\alpha)_\infty}.$$

Definition 3.13. Using σ in the previous Proposition 3.12, we set

$$\Phi_\sigma := \frac{T_\sigma \Delta^+}{\Delta^+}, \quad D_\sigma f := |W_\sigma|^{-1} \sum_{w \in W} w(\Phi_\sigma (T_\sigma f - f)).$$

The operator $D_\sigma : A^W \to A^W$ is called the **Macdonald difference operator**.

Exercise 3.4 (∗)**.** Assume $R = S = \mathrm{A}_n$ as in Exercise 3.3, and take $\sigma = \omega_r$. Using the notation $x_i = e^{\varepsilon_i}$, check the following formula.

$$\frac{T_{\omega_r}\Delta^+}{\Delta^+} = \prod_{j=r+1}^{n+1} \frac{1 - tx_r/x_j}{1 - x_r/x_j}.$$

Theorem 3.14. Assume R is reduced. The operator D_σ preserves A^W, and the Macdonald polynomial P_λ is an eigenfunction of D_σ with eigenvalue

$$q^{(\sigma,\rho_k)}(\widetilde{m}_\sigma(\lambda+\rho_k) - \widetilde{m}_\sigma(\rho_k)).$$

where ρ_k is given in (3.2) and $\widetilde{m}_\sigma(\lambda) := |W_\sigma|^{-1}\sum_{w\in W} q^{(w\sigma,\lambda)}$.

3.5 GL_n case

Recall the setting for $R = S = \mathrm{A}_{n-1}$:

$\mathbb{R}^n = \sum_{i=1}^n \mathbb{R}\varepsilon_i \supset V = \{\sum_{i=1}^n c_i\varepsilon_i \mid \sum_{i=1}^n c_i = 0\}$,
$R = \{\varepsilon_i - \varepsilon_j \mid 1 \le i \ne j \le n\} \supset R^+ = \{\varepsilon_i - \varepsilon_j \mid 1 \le i < j \le n\}$, $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ $(1 \le i \le n-1)$,
$Q = \mathbb{Z}[R] = \sum_{i=1}^{n-1}\mathbb{Z}\alpha_i \supset Q^+ = \mathbb{N}[R^+] = \sum_{i=1}^{n-1}\mathbb{N}\alpha_i$, $W = S_n$.

Definition. We set

$$\widetilde{P} := \sum_{i=1}^n \mathbb{Z}\varepsilon_i, \quad \widetilde{\omega}_i := \varepsilon_1 + \cdots + \varepsilon_i \ (1 \le i \le n), \quad \widetilde{A} := \mathbb{C}[\widetilde{P}] = \mathbb{C}[e^{\pm\varepsilon_1}, \ldots, e^{\pm\varepsilon_n}].$$

Proposition 3.15. For $v \in V$, consider the operator

$$E_v f := |W_v|^{-1} \sum_{w\in W} w(\Phi_v T_v f).$$

Then for $r = 1, \ldots, n$, we have

$$E_{\widetilde{\omega}_r} = \sum_{I\subset\{1,\ldots,n\},\, |I|=r} \widetilde{A}_I(x;t) T^I_{q,x}, \quad \widetilde{A}_I := \prod_{i\in I,\, j\notin I} \frac{tx_i - x_j}{x_i - x_j}.$$

Here we used the notation $x_i = e^{\varepsilon_i}$. Thus we have $t^{\binom{n}{r}} E_{\widetilde{\omega}_r} = D^{(r)}$.

Exercise 3.5 (∗)**.** Show Proposition 3.15.

Since $E_{\widetilde{\omega}_r}$ is essentially the same as the Macdonald difference operators in §2.2, the Macdonald symmetric function

$$P_\lambda(x;q,t) \in \mathbb{C}[x]^W \subset \mathbb{C}[x^{\pm1}]^W = \widetilde{A}^{S_n}$$

is the simultaneous eigenfunction of $E_{\widetilde{\omega}_r}$'s. As for the eigenvalue, we have

Proposition 3.16. Under the notation in Theorem 3.14, the eigenvalue of $P_\lambda(x;q,t)$ with respect to $E_{\widetilde{\omega}_r}$ is given by

$$q^{(\widetilde{\omega}_r,\rho_k)}\widetilde{m}_{\widetilde{\omega}_r}(\lambda+\rho_k).$$

Exercise 3.6 (∗)**.** Check that Proposition 3.16 is consistent with Theorem 2.6, which says that the eigenvalue of $P_\lambda(x;q,t)$ with respect to $D^{(r)}$ is $e_r(q^\lambda t^\delta)$.