Lecture 3: Macdonald polynomials for general root systems [*1]

Shintaro Yanagida (office: A441)
yanagida [at] math.nagoya-u.ac.jp
https://www.math.nagoya-u.ac.jp/~yanagida

3 Macdonald polynomials for general root systems

3.1 Root system

This subsection is the recollection of root systems. See [B68, Chap. 6] for the full account and proofs.

Let V be a Euclidean space, i.e., a finite dimensional real vector space with inner product $(\cdot,\cdot)$. We denote the norm of $v \in V$ by $|v| := (v,v)^{1/2}$. For $a \in V \setminus \{0\}$, we denote $a^\vee := 2a/|a|^2$, and we define the **reflection**[*2] $s_a : V \to V$ to be the linear map given by

$$s_a(v) \,:=\, v - 2\frac{(v,a)}{(a,a)}a \,=\, v - (v,a)a^\vee.$$

Definition. A finite subset $R \subset V \setminus \{0\}$ is called a **root system**[*3] in V if the following three conditions are satisfied for any $\alpha,\beta \in R$.
(RS1) R spans V. (RS2) $s_\alpha(\beta) \in R$. (RS3) $(\alpha^\vee,\beta) \in \mathbb{Z}$.
Each element of a root system S is called a **root**, and $\dim V$ is called the **rank** of R.

Obviously we have the notion of product of root system. To simplify the argument, we introduce

Definition. A root system R is called **irreducible**[*4] if there exists no partition of R into two non-empty subsets R_1 and R_2 such that $(\alpha_1,\alpha_2) = 0$ for any $\alpha_i \in S_i$ $(i = 1,2)$.

Hereafter we fix a root system R in V.

Proposition. If $\alpha,\beta \in R$ are proportional, then the factor of proportionality is ± 1, $\pm 1/2$ or ± 2.

Definition. (1) A root $\alpha \in R$ is called an indivisible root if $\alpha/2 \notin R$.
(2) R is called **reduced**[*5] if every $\alpha \in R$ is indivisible.

We denote by $\mathrm{GL}(V)$ the general linear group of V, i. e., the group consisting of linear automorphisms of V. We also denote by $\mathrm{O}(V)$ the **orthogonal group**[*6] of V. We have

$$\mathrm{O}(V) \,=\, \{A \in \mathrm{GL}(V) \mid (Av,Aw) = (v,w) \text{ for any } v,w \in V\}.$$

Definition. The **Weyl group** $W(R)$ of a root system R is the subgroup of $\mathrm{O}(V)$ generated by $\{s_\alpha \mid \alpha \in R\}$.

[*1] 2018/10/14, ver. 0.2.
[*2] 鏡映
[*3] ルート系
[*4] 既約
[*5] 被約
[*6] 直交群

$W(R)$ has a good presentation in terms of the simple roots.

Definition. A subset $\{\alpha_i \mid i \in I\} \subset R$ is called the set of **simple roots** if the following two conditions are satisfied.
(S1) $\{\alpha_i \mid i \in I\}$ is a basis V.
(S2) If we expand $\alpha = \sum_{i \in I} c_i \alpha_i$ for each $\alpha \in R$, then all the signs of non-zero c_i's coincide.

Proposition 3.1. If R is irreducible and reduced, then R has a set of simple roots.

Proposition. Let R be an irreducible and reduced root system, and $\{\alpha_i \mid i \in I\} \subset R$ be a set of simple roots. Then the Weyl group $W = W(R)$ has the following presentation[*7].

$$W = \langle s_i \ (i \in I) \mid (s_i s_j)^{m(i,j)} = 1 \ (i, j \in I) \rangle,$$

where $s_i := s_{\alpha_i} \in W$ for $i \in I$, and $m(i,j)$ denotes the order of the element $s_i s_j \in W$.

Note that $m(\alpha, \alpha) = 1$ since $(s_\alpha)^2 = \mathrm{id}$.

Definition. (1) The **Coxeter graph** of R is the graph defined by the following rule: the set of vertices is given by I, the set of simple roots, and the vertices i and j with $i \neq j$ are joined by $(\alpha_i, \alpha_j) \cdot (\alpha_j, \alpha_i)$ edges.
(2) The **Dynkin diagram** of R is the figure obtained by adding an arrow to the shorter of the two roots in the Coxeter graph of R.

Now we can state the fundamental classification theorem of finite root systems.

Theorem 3.2. Irreducible and reduced root systems are classified into the following types:

$A_n \ (n \geq 1), \ B_n \ (n \geq 2), \ C_n \ (n \geq 3), \ D_n \ (n \geq 4), \ E_6, E_7, E_8, \ F_4, \ G_2.$

Each class is described as below.

type A_n $(n \geq 1)$: $V := \{\sum_{i=1}^{n+1} a_i \varepsilon_i \mid \sum_{i=1}^{n+1} a_i = 0\} \subset \mathbb{R}^{n+1} = \oplus_{i=1}^{n+1} \mathbb{R}\varepsilon_i, \quad L := \oplus_{i=1}^{n+1} \mathbb{Z}\varepsilon_i,$
$R := \{v \in V \mid (v,v) = 2\} \cap L = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i \neq j \leq n+1\},$
$|R| = (n+1)n.$
Simple roots: $\{\alpha_1 := \varepsilon_1 - \varepsilon_2, \, \alpha_2 := \varepsilon_2 - \varepsilon_3, \, \ldots, \, \alpha_n := \varepsilon_n - \varepsilon_{n+1}\}.$
$W \simeq S_{n+1}$, acting on V by permuting indices i of ε_i's.
Dynkin diagram: $\underset{\alpha_1}{\circ} \!-\!\!-\!\!-\! \underset{\alpha_2}{\circ} \!-\!\!-\!\!-\! \cdots \!-\!\!-\!\!-\! \underset{\alpha_{n-1}}{\circ} \!-\!\!-\!\!-\! \underset{\alpha_n}{\circ}$

type B_n $(n \geq 2)$: $V := \mathbb{R}^n = \oplus_{i=1}^n \mathbb{R}\varepsilon_i, \ L := \oplus_{i=1}^n \mathbb{Z}\varepsilon_i,$
$R := \{v \in V \mid (v,v) = 1 \text{ or } 2\} \cap L = \{\pm\varepsilon_i \mid 1 \leq i \leq n\} \cup \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq n\},$
$|R| = 2n + 2n(n-1) = 2n^2.$
Simple roots: $\{\alpha_1 := \varepsilon_1 - \varepsilon_2, \, \alpha_2 := \varepsilon_2 - \varepsilon_3, \, \ldots, \, \alpha_{n-1} := \varepsilon_{n-1} - \varepsilon_n, \, \alpha_n := \varepsilon_n\}.$
$W \simeq S_n \ltimes (\{\pm 1\})^n$, acting on V by permuting ε_i's and $\varepsilon_i \mapsto -\varepsilon_i$.
Dynkin diagram: $\underset{\alpha_1}{\circ} \!-\!\!-\!\!-\! \underset{\alpha_2}{\circ} \!-\!\!-\!\!-\! \cdots \!-\!\!-\!\!-\! \underset{\alpha_{n-1}}{\circ} \Longrightarrow \underset{\alpha_n}{\circ}$

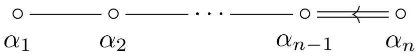
[*7] (群の) 表示

type C_n $(n \geq 3)$: $V := \mathbb{R}^n = \oplus_{i=1}^n \mathbb{R}\varepsilon_i$, $\quad L := \oplus_{i=1}^n \mathbb{Z}\varepsilon_i$,

$R := \{v \in V \mid (v,v) \in \{2,4\}\} \cap L = \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq n\} \cup \{\pm 2\varepsilon_i \mid 1 \leq i \leq n\}$, $\quad |R| = 2n^2$.

Simple roots: $\{\alpha_1 := \varepsilon_1 - \varepsilon_2,\, \alpha_2 := \varepsilon_2 - \varepsilon_3,\, \ldots,\, \alpha_{n-1} := \varepsilon_{n-1} - \varepsilon_n,\, \alpha_n := 2\varepsilon_n\}$.

$W \simeq S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$, acting on V in the same way as B_n.

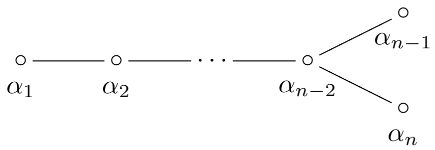
Dynkin diagram: 

type D_n $(n \geq 4)$: $V := \mathbb{R}^n = \oplus_{i=1}^n \mathbb{R}\varepsilon_i$, $\quad L := \oplus_{i=1}^n \mathbb{Z}\varepsilon_i$,

$R := \{v \in V \mid (v,v) = 2\} \cap L = \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq n\}$, $\quad |R| = 2n(n-1)$,

Simple roots: $\{\alpha_1 := \varepsilon_1 - \varepsilon_2,\, \alpha_2 := \varepsilon_2 - \varepsilon_3,\, \ldots,\, \alpha_{n-1} := \varepsilon_{n-1} - \varepsilon_n,\, \alpha_n := \varepsilon_{n-1} + \varepsilon_n\}$.

$W \simeq S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^{n-1}$, acting on V by permuting ε_i's and by even number of sign changes $\varepsilon_i \mapsto -\varepsilon_i$.

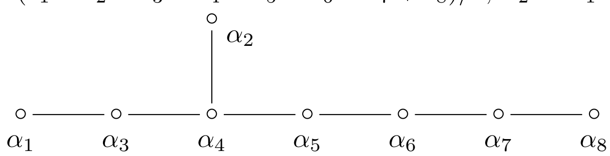
Dynkin diagram: 

type E_8: $V := \mathbb{R}^8 = \oplus_{i=1}^8 \mathbb{R}\varepsilon_i$, $\quad L := \{\sum_{i=1}^8 c_i\varepsilon_i \mid c_i \in \mathbb{Z},\ \sum_{i=1}^8 \varepsilon_i \in 2\mathbb{Z}\} + \mathbb{Z}(\sum_{i=1}^8 \varepsilon_i/2)$,

$R := \{v \in L \mid (v,v) = 2\} = \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq 8\} \cup \{\sum_{i=1}^8 \pm \varepsilon_i/2 \mid \text{number of } - \text{ is even}\}$,

$|R| = 28 \cdot 4 + 2^8/2 = 72 + 128 = 240$.

Simple roots: $\{\alpha_1 := (\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4 - \varepsilon_5 - \varepsilon_6 - \varepsilon_7 + \varepsilon_8)/2,\, \alpha_2 := \varepsilon_1 + \varepsilon_2,\, \alpha_i := \varepsilon_{i-1} - \varepsilon_{i-2}\}$.

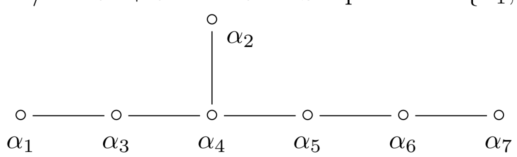
Dynkin diagram: 

type E_7: Denote the simple roots of E_8 as $\{\alpha_1, \ldots, \alpha_8\} \subset R(\mathrm{E}_8) \subset \mathbb{R}^8$. $\quad V := \mathbb{R}\{\alpha_1, \ldots, \alpha_7\} \subset \mathbb{R}^8$,

$R := R(\mathrm{E}_8) \cap V$

$= \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq 6\} \cup \{\pm(\varepsilon_7 - \varepsilon_8)\} \cup \{\pm(\varepsilon_7 - \varepsilon_8 + \sum_{i=1}^6 \pm \varepsilon_i)/2 \mid \text{number of } - \text{ is odd}\}$,

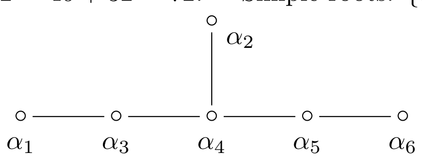
$|R| = 4 \cdot 15 + 2 + 2 \cdot 2^6/2 = 62 + 64 = 126$. $\quad$ Simple roots: $\{\alpha_1, \ldots, \alpha_7\}$.

Dynkin diagram: 

type E_6: Use again $R(\mathrm{E}_8) \subset \mathbb{R}^8$. $\quad V := \mathbb{R}\{\alpha_1, \ldots, \alpha_6\} \subset \mathbb{R}^8$,

$R := R(\mathrm{E}_8) \cap V = \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq 5\} \cup \{\pm(\varepsilon_8 - \varepsilon_7 - \varepsilon_6 + \sum_{i=1}^5 \pm \varepsilon_i)/2 \mid \text{number of } - \text{ is odd}\}$,

$|R| = 4 \cdot 10 + 2 \cdot 2^5/2 = 40 + 32 = 72$. $\quad$ Simple roots: $\{\alpha_1, \ldots, \alpha_6\}$.

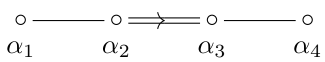
Dynkin diagram: 

type F_4: $V := \mathbb{R}^4 = \sum_{i=1}^4 \mathbb{R}\varepsilon_i$, $\quad L := \oplus_{i=1}^4 \mathbb{Z}\varepsilon_i + \mathbb{Z}(\sum_{i=1}^4 \varepsilon_i/2)$,

$R := \{v \in L \mid (v,v) = 1 \text{ or } 2\}$

$= \{\pm\varepsilon_i \mid 1 \leq i \leq 4\} \cup \{(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)/2\} \cup \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq 4\}$, $\quad |R| = 24 + 24 = 48$.

Simple roots: $\{\alpha_1 := \varepsilon_2 - \varepsilon_3,\, \alpha_2 := \varepsilon_3 - \varepsilon_4,\, \alpha_3 := \varepsilon_4,\, \alpha_4 := (\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)/2\}$.

Dynkin diagram: 

type G_2: $V := \{\sum_{i=1}^{3} a_i\varepsilon_i \mid \sum_{i=1}^{3} a_i = 0\} \subset \mathbb{R}^3$, $\quad L := \oplus_{i=1}^{3} \mathbb{Z}\varepsilon_i$,

$$R := \{v \in V \mid (v,v) = 2 \text{ or } 6\} \cap L$$
$$= \{\pm(\varepsilon_i - \varepsilon_j) \mid 1 \le i < j \le 3\} \cup \{\pm(2\varepsilon_i - \varepsilon_j - \varepsilon_k) \mid \{i,j,k\} = \{1,2,3\}\}, \quad |R| = 6 + 6 = 12.$$

Simple roots: $\{\alpha_1 := \varepsilon_1 - \varepsilon_2,\ \alpha_2 := -2\varepsilon_1 + \varepsilon_2 + \varepsilon_3\}$.

$W \simeq$ the dihedral group[*8] of order 12.

Dynkin diagram: $\quad \underset{\alpha_1}{\circ} \Lleftarrow \underset{\alpha_2}{\circ}$

Next we recall

Proposition. For a root system R in V, the set $R^\vee := \{a^\vee \mid a \in R\}$ is again a root system in V.

Definition. The root system $R^\vee$ is called the **dual root system**.

By the description in Theorem 3.2, we have

$$R^\vee = R \quad (R = \mathrm{A}_n, \mathrm{D}_n, \mathrm{E}_n), \quad (\mathrm{B}_n)^\vee = \mathrm{C}_n, \quad (\mathrm{C}_n)^\vee = \mathrm{B}_n.$$

As for non-reduced root systems, we have

Theorem 3.3. For each $n \in \mathbb{Z}_{\ge 1}$, there exists, up to isomorphism, a unique irreducible non-reduced root system of rank n. This root system is called of type BC_n, and described as follows.

type BC_n $(n \ge 1)$: $V := \mathbb{R}^n = \oplus_{i=1}^{n} \mathbb{R}\varepsilon_i$, $\quad L := \oplus_{i=1}^{n} \mathbb{Z}\varepsilon_i$,

$$R := \{v \in V \mid (v,v) = 1, 2 \text{ or } 4\} \cap L = R(\mathrm{B}_n) \cup R(\mathrm{C}_n)$$
$$= \{\pm\varepsilon_i \mid 1 \le i \le n\} \cup \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \le i < j \le n\} \cup \{\pm 2\varepsilon_i \mid 1 \le i \le n\}, \quad |R| = 2(n+1)n.$$

$W = W(\mathrm{B}_n) \simeq S_n \ltimes (\{\pm 1\})^n$, acting on V in the same way as B_n.

3.2 Admissible pair

Subsections §§3.2–3.4 follow [K92, §2] and [M03, §2]. Let V be a Euclidean space.

Definition. A pair (R, S) of root systems in V is called admissible if it satisfies

(AP1) R is irreducible (but not necessarily reduced). S is irreducible and reduced.

(AP2) The sets of lines $\{\mathbb{R}a \mid a \in R\}$ and $\{\mathbb{R}b \mid b \in S\}$ coincide.

(AP3) $W(R) = W(S)$.

By Theorems 3.2 and 3.3, admissible pairs are classified as

Proposition 3.4. Every admissible pair (R, S) is either of the following three classes.

- $(R, S) = (S, S)$ with S listed in Theorem 3.2.
- $(R, S) = (S, S^\vee)$ with S listed in Theorem 3.2.
- $(R, S) = (\mathrm{BC}_n, \mathrm{B}_n)$ or $(\mathrm{BC}_n, \mathrm{C}_n)$.

Now recall the notion of positive roots of root systems. We introduce a restricted definition which is enough in our context.

[*8] 二面体群

Definition. For an admissible pair (R, S), it we can decompose $R = R^+ \cup (-R^+)$, then we call elements of R^+ the **positive roots** of R.

Proposition 3.5. There exists a (not unique) decomposition $R = R^+ \cup (-R^+)$ for any admissible pair (R, S).

Proof. First assume that R is reduced. Then by Proposition 3.1 we can choose the set $\{\alpha_i \mid i \in I\} \subset R$ of simple roots, Then every $\alpha \in R$ can be expanded as $\alpha = \sum_{i \in I} c_i \alpha_i$ with all the signs of c_i's being equal. Thus we can set $R^+ := \{\alpha \in R \mid c_i \geq 0\}$.

Next assume that R is non-reduced, so $R = \mathrm{BC}_n$.

Using the notation in Theorem 3.3, take the set of simple roots as $\{\alpha_i = \varepsilon_i - \varepsilon_{i+1} \mid 1 \leq i \leq n-1\} \cup \{\alpha_n = \varepsilon_n\}$. Then each $\alpha \in R$ is expanded as $a = \sum_{i=1}^{n} c_i \alpha_i$ with all the signs of c_i's being equal. Thus we can set $R^{\pm}$ as in the reduced case. □

Definition 3.6. Choose a decomposition $R = R^+ \cup (-R^+)$.

(1) The **root lattice** $Q \subset R$ and the **positive cone** $Q^+ \subset Q$ of positive roots in R are defined to be

$$Q := \mathbb{Z}R = \mathbb{Z}\text{-span of } R, \quad Q^+ := \mathbb{N}R^+.$$

(2) The **weight lattice**[*9] P and the cone P^+ of **dominant weights**[*10]of R are defined to be

$$P := \{\lambda \in V \mid (\lambda, \alpha^\vee) \in \mathbb{Z} \ \ \forall \alpha \in R\}, \quad P^+ := \{\lambda \in V \mid (\lambda, \alpha^\vee) \in \mathbb{N} \ \ \forall \alpha \in R^+\}.$$

(3) The **dominance order** is a partial order on P defined by

$$\lambda \geq \mu \iff \lambda - \mu \in Q^+.$$

Exercise 3.1 $(*)$**.** Consider the root system R of type A_n with the set of simple roots described in Theorem 3.2. Let $R = R^+ \cup (-R^+)$ be the decomposition of R given in Proof of Proposition 3.5. Check that P^+ is then given by

$$P^+ = \sum_{i=1}^{n} \mathbb{N}\,\omega_i = \{\lambda = \lambda^1 \omega_1 + \cdots + \lambda^n \omega_n \mid \lambda^i \in \mathbb{N}\},$$

where ω_i is the **fundamental weight**[*11] given by

$$\omega_i = \frac{n+1-i}{n+1}(\varepsilon_1 + \cdots + \varepsilon_i) - \frac{i}{n+1}(\varepsilon_{i+1} + \cdots + \varepsilon_{n+1}).$$

3.3 Macdonald polynomials for general root systems

In this and the next subsections, we fix an admissible pair (R, S) of root systems in a Euclidean space V. We denote $W := W(R) = W(S)$. We also fix a decomposition $R = R^+ \cup (-R^+)$ and use the symbols Q, Q^+, P, P^+ given in Definition 3.6.

[*9] ウェイト格子

[*10] 優整ウェイト

[*11] 基本ウェイト

For $\lambda \in P$, let e^λ be the function on V defined by

$$e^\lambda(v) := \exp\bigl(i(\lambda, v)\bigr), \quad v \in V.$$

Extend this function holomorphically to $V + iV$. For a function f on V we also define

$$(wf)(v) := f(w^{-1}v), \quad w \in W,\ v \in V.$$

Thus we have $we^\lambda = e^{w\lambda}$.

Definition. We set

$$A := \mathbb{C}[P] = \mathbb{C}\text{-span of } \{e^\lambda \mid \lambda \in P\}, \quad A^W := \{f \in A \mid wf = f \ \ \forall\, w \in W\}.$$

Proposition. A basis of A^W is given by $\{m_\lambda \mid \lambda \in P^+\}$ with

$$m_\lambda := |W_\lambda|^{-1} \sum_{w \in W} e^{w\lambda} = \sum_{\mu \in W\lambda} e^\mu,$$

where W_λ denotes the stabilizer of $\lambda \in P^+$ in W.

Remark. Note that $\overline{m_\lambda(v)} = m_\lambda(-v)$ for $\lambda \in P^+$ and $v \in V$, where $\bar{c}$ denotes the complex conjugate of $c \in \mathbb{C}$. If $-\operatorname{id} \in W$, then $f(v) = f(-v)$ for $f \in A^W$. Thus, if $-\operatorname{id} \in W$, then m_λ is real-valued on V.

Next we want to introduce a Hermitian pairing on A^W via integration with certain weight function. For this purpose, consider the dual root lattice $Q^\vee$, i.e.,

$$Q^\vee := \mathbb{Z}R^\vee = \mathbb{Z}\text{-span of } R^\vee.$$

Then

$$T := V/(2\pi Q^\vee)$$

is a torus, in other words, $T \simeq (\mathbb{R}/\mathbb{Z})^n$ as a group ($n := \dim V$). Below we denote by $\dot{v}$ the image in T of $v \in V$. For $\lambda \in P$, we define the function e^λ on T by

$$e^\lambda(\dot{v}) := \exp(i(\lambda, v)).$$

$e^\lambda(\dot{v})$ is obviously well-defined.

Proposition 3.7. For each $\alpha \in R$, there exists a unique $u_\alpha > 0$ such that $\alpha_* := \alpha/u_\alpha \in S$.

Proof. Recalling Proposition 3.4, we first assume $R = S$. Then we have $u_\alpha = 1$ for any $\alpha \in R$.

Next assume $S = R^\vee$. In the case $R = \mathrm{A}_n, \mathrm{D}_n, \mathrm{E}_n$, we have $u_\alpha = 1$ for any $\alpha \in R$. In the case $R = \mathrm{B}_n$, we have $u_\alpha = 1$ or $1/2$. In the case $R = \mathrm{C}_n$ or F_n, we have $u_\alpha = 1$ or 2. In the case $R = \mathrm{G}_n$, we have $u_\alpha = 1$ or 3.

Assume $R = \mathrm{BC}_n$ and $S = \mathrm{B}_n$. Then we have $u_\alpha = 1$ or 2. Finally assume $R = \mathrm{BC}_n$ and $S = \mathrm{C}_n$. Then we hvae $u_\alpha = 1$ or $1/2$. $\square$

Hereafter we fix a real number q such that $0 < q < 1$. We also choose a W-invariant function

$$\alpha \longmapsto t_\alpha$$

on R taking values in $(0, 1)$. Then we have

Proposition. For $\alpha \in R$, the value t_α depends only on $|\alpha|$.

Definition. For each $\alpha \in R$, define $q_\alpha := q^{u_\alpha}$ and $k_\alpha \geq 0$ by $q_\alpha^{k_\alpha} = t_\alpha$.

Remark 3.8. In particular, in the case $R = \mathrm{A}_n, \mathrm{D}_n, \mathrm{E}_n$ the value t_α is independent of $\alpha \in R$. We denote it by $t := t_\alpha$. Since $u_\alpha = 1$ for any $\alpha \in R$ by Proof of Proposition 3.7, we denote $k := k_\alpha$.

Recall the **q-shifted factorial**:

$$(a;q)_\infty := \prod_{j=0}^{\infty}(1-aq^j), \quad (a_1,\ldots,a_r;q)_\infty := \prod_{i=1}^{r}(a_i;q)_\infty$$

for $a, a_1, \ldots, a_r \in \mathbb{C}$. Using q-shifted factorials, we introduce

Definition. Assume R is reduced.

(1) The **weight function** $\Delta(v)$ on $v \in V$ is defined to be

$$\Delta(v) := \prod_{\alpha \in R} \frac{(t_{2\alpha}^{1/2} e^\alpha(v); q_\alpha)_\infty}{(t_\alpha t_{2\alpha}^{1/2} e^\alpha(v); q_\alpha)_\infty}.$$

Here we set $t_a := 1$ for $a \in V \setminus R$.

(2) A pairing $\langle \cdot, \cdot \rangle$ on A^W is defined to be

$$\langle f, g \rangle := |W|^{-1} \int_T f(\dot{v})\overline{g(\dot{v})}\Delta(\dot{v})\, d\dot{v}, \quad f, g \in A^W, \tag{3.1}$$

where $d\dot{v}$ denotes the Haar measure on T with the normalization $\int_T d\dot{v} = 1$.

Proposition. The pairing $\langle \cdot, \cdot \rangle$ is a Hermitian inner product on A^W.

Exercise 3.2 ($*$). (1) Check that $\langle \cdot, \cdot \rangle$ is a Hermitian pairing, i.e., $\overline{\langle f, g \rangle} = \langle g, f \rangle$.

(2) Assume $R = S = \mathrm{A}_n$ as in Exercise 3.1, and put $x_i := e^{\varepsilon_i}$. Recalling Remark 3.8, show that the weight function Δ is given by

$$\Delta = \prod_{1 \leq i \neq j \leq n+1} \frac{(x_i/x_j; q)_\infty}{(tx_i/x_j; q)_\infty}.$$

Now we can state

Theorem 3.9. Assume R is reduced. There exists a unique family $\{P_\lambda \mid \lambda \in P^+\} \subset A^W$ such that

(i) $P_\lambda = m_\lambda + \sum_{\mu < \lambda} c_{\lambda,\mu} m_\mu$ with some $c_{\lambda,\mu} \in \mathbb{C}$.

(ii) $\langle P_\lambda, m_\mu \rangle = 0$ if $\mu < \lambda$, where $<$ denotes the dominance order in Definition 3.6 (3).

Each $P_\lambda \in A^W$ is called the **Macdonald polynomial** of the admissible pair (R, S).

The Macdonald polynomial P_λ is an orthogonal polynomial in the following sense.

Theorem 3.10. Assume R is reduced. The family $\{P_\lambda \mid \lambda \in P^+\}$ satisfies

$$\langle P_\lambda, P_\mu \rangle = 0 \quad \lambda \neq \mu.$$

The following theorem on the norm of P_λ was conjectured by Macdonald and named Macdonald's evaluation conjecture. It was solved in full generality by Cherednik [C95].

Theorem 3.11. Assume R is reduced. We have

$$\langle P_\lambda, P_\lambda \rangle \;=\; \prod_{\alpha \in R^+} \frac{(q^{1+(\lambda+\rho_k,\alpha^\vee)};q)_\infty}{(tq^{1+(\lambda+\rho_k,\alpha^\vee)};q)_\infty} \frac{(q^{(\lambda+\rho_k,\alpha^\vee)};q)_\infty}{(tq^{(\lambda+\rho_k,\alpha^\vee)};q)_\infty},$$

where ρ_k is defined to be

$$\rho_k \;:=\; \frac{1}{2}\sum_{\alpha\in R^+} k_\alpha \alpha. \tag{3.2}$$

3.4 Macdonald difference operators for general root systems

We continue to use the same symbols as in the previous subsections. Define the q-shift operator $T_{q,v} = T_v$ on the functions $f : V \to \mathbb{C}$ to be

$$(T_v f)(x) \;:=\; f(x - i \log q \cdot v) \quad (x, v \in V).$$

In particular, for $\lambda \in P$ we have

$$T_v e^\lambda \;=\; q^{(v,\lambda)} e^\lambda,$$

so that we have $T_v : A \to A$.

Recall that by Proposition 3.7 we have $\alpha_* = \alpha/u_\alpha \in S$ for any $\alpha \in R$.

Proposition 3.12. For an admissible pair (R, S) with S not being of type E_8, F_4 nor G_2, there is a $\sigma \in V$ such that

$$\{(\sigma, \alpha_*) \mid \alpha \in R^+\} \subset \{0, 1\}.$$

In the other cases, there is a $\sigma \in V$ such that

$$\{(\sigma, \alpha_*) \mid \alpha \in R^+\} \subset \{0, 1, 2\}.$$

Remark. In the case $S \neq \mathrm{E}_8, \mathrm{F}_4, \mathrm{G}_2$, σ is a minuscule fundamental weight for $S^\vee$.

Exercise 3.3 ($*$)**.** Assume $R = S = \mathrm{A}_n$ as in Exercise 3.2 (2). As for σ in Proposition 3.12, show that we can put $\sigma = \omega_r$ for each $r = 1, \ldots, n$.

Definition. We set

$$\Delta^+ \;:=\; \prod_{\alpha\in R^+} \frac{(t_{2\alpha}^{1/2} e^\alpha; q_\alpha)_\infty}{(t_\alpha t_{2\alpha}^{1/2} e^\alpha; q_\alpha)_\infty}.$$

Remark. Since $R = R^+ \cup (-R^+)$, we have $\Delta = \Delta^+ \overline{\Delta^+}$.

Definition 3.13. Using σ in the previous Proposition 3.12, we set

$$\Phi_\sigma \;:=\; \frac{T_\sigma \Delta^+}{\Delta^+}, \quad D_\sigma f \;:=\; |W_\sigma|^{-1} \sum_{w \in W} w(\Phi_\sigma (T_\sigma f - f)).$$

The operator $D_\sigma : A^W \to A^W$ is called the **Macdonald difference operator**.

Exercise 3.4 ($*$)**.** Assume $R = S = \mathrm{A}_n$ as in Exercise 3.3, and take $\sigma = \omega_r$. Using the notation $x_i = e^{\varepsilon_i}$, check the following formula.

$$\frac{T_{\omega_r}\Delta^+}{\Delta^+} \;=\; \prod_{j=r+1}^{n+1} \frac{1 - t x_r/x_j}{1 - x_r/x_j}.$$

Theorem 3.14. Assume R is reduced. The operator D_σ preserves A^W, and the Macdonald polynomial P_λ is an eigenfunction of D_σ with eigenvalue

$$q^{(\sigma,\rho_k)}(\widetilde{m}_\sigma(\lambda+\rho_k)-\widetilde{m}_\sigma(\rho_k)).$$

where ρ_k is given in (3.2) and

$$\widetilde{m}_\sigma(\lambda) := |W_\sigma|^{-1}\sum_{w\in W} q^{(w\sigma,\lambda)}.$$

Remark. (1) The case R is non-reduced will be treated in Lecture 4, where the corresponding Theorems 3.9, 3.10, 3.11 and 3.14 will be explained.

(2) The generalization of the theory of Macdonald symmetric polynomial to general root systems was started by Macdonald in the years around 1988. Theorems 3.9, 3.10 and 3.14 are established in [M88]. As mentioned before, the evaluation conjecture (Theorem 3.11) was remained unsolved until the work of Cherednik [C95].

3.5 GL_n case

In this subsection we check that the Macdonald polynomial in the case $R=S=\mathrm{A}_{n-1}$ is essentially the same as the Macdonald symmetric polynomial explained in §2. Precisely speaking, we replace $A=\mathbb{C}[P]$ by $\widetilde{A}=\mathbb{C}[e^{\pm\varepsilon_1},\dots,e^{\pm\varepsilon_n}]$. This replacement can be regarded as switching SL_n picture to GL_n picture. The references of this subsection are [Mi04, §4.11, §4.12] and [N97].

Recall the setting for $R=S=\mathrm{A}_{n-1}$:

$$\begin{aligned}
&\mathbb{R}^n=\textstyle\sum_{i=1}^n\mathbb{R}\varepsilon_i \supset V=\{\sum_{i=1}^n c_i\varepsilon_i \mid \sum_{i=1}^n c_i=0\},\\
&R=\{\varepsilon_i-\varepsilon_j \mid 1\le i\ne j\le n\}\supset R^+=\{\varepsilon_i-\varepsilon_j \mid 1\le i<j\le n\},\quad \alpha_i=\varepsilon_i-\varepsilon_{i+1}\quad(1\le i\le n-1),\\
&Q=\mathbb{Z}[R]=\textstyle\sum_{i=1}^{n-1}\mathbb{Z}\alpha_i\supset Q^+=\mathbb{N}[R^+]=\sum_{i=1}^{n-1}\mathbb{N}\alpha_i,\\
&W=S_n.
\end{aligned}$$

Definition. We set

$$\widetilde{P}:=\textstyle\sum_{i=1}^n\mathbb{Z}\varepsilon_i,\quad \widetilde{\omega}_i:=\varepsilon_1+\cdots+\varepsilon_i\ (1\le i\le n),\quad \widetilde{A}:=\mathbb{C}[\widetilde{P}]=\mathbb{C}[e^{\pm\varepsilon_1},\dots,e^{\pm\varepsilon_n}].$$

Note that $(\lambda,\alpha)=0$ for any $\lambda\in P$ and $\alpha\in R$, and $(\widetilde{\omega}_i,\alpha_j)=\delta_{i,j}$.

Recall Proposition 3.12 and Exercise 3.3 on the element σ. We have the following alternative choices.

Proposition. For $r=1,\dots,n$, the following holds.

$$\{(\widetilde{\omega}_r,\alpha)\mid \alpha\in R^+\}\subset\{0,1\}.$$

Now we can recover the operator $D^{(r)}$ in §2.2 from the operator $D_{\widetilde{\omega}_r}$ in Definition 3.13. We slightly change the definition of $D_{\widetilde{\omega}_r}$ as follows.

Proposition 3.15. For $v\in V$, consider the operator

$$E_v f := |W_v|^{-1}\sum_{w\in W} w(\Phi_v T_v f).$$

Then for $r = 1, \ldots, n$, we have

$$E_{\widetilde{\omega}_r} = \sum_{I \subset \{1,\ldots,n\},\, |I|=r} \widetilde{A}_I(x;t) T_{q,x}^I, \quad \widetilde{A}_I := \prod_{i \in I,\, j \notin I} \frac{tx_i - x_j}{x_i - x_j}.$$

Here we used the notation $x_i = e^{\varepsilon_i}$. Thus we have $t^{\binom{n}{r}} E_{\widetilde{\omega}_r} = D^{(r)}$.

Exercise 3.5 $(*)$**.** Show Proposition 3.15.

Since $E_{\widetilde{\omega}_r}$ is essentially the same as the Macdonald difference operators in §2.2, the Macdonald symmetric function

$$P_\lambda(x;q,t) \in \mathbb{C}[x]^W \subset \mathbb{C}[x^{\pm 1}]^W = \widetilde{A}^{S_n}$$

is the simultaneous eigenfunction of $E_{\widetilde{\omega}_r}$'s. As for the eigenvalue, we have

Proposition 3.16. Under the notation in Theorem 3.14, the eigenvalue of $P_\lambda(x;q,t)$ with respect to $E_{\widetilde{\omega}_r}$ is given by

$$q^{(\widetilde{\omega}_r, \rho_k)} \widetilde{m}_{\widetilde{\omega}_r}(\lambda + \rho_k).$$

Exercise 3.6 $(*)$**.** Check that Proposition 3.16 is consistent with Theorem 2.6, which says that the eigenvalue of $P_\lambda(x;q,t)$ with respect to $D^{(r)}$ is $e_r(q^\lambda t^\delta)$.

References

[B68] N. Bourbaki, *Groupes et algèbres Lie*, Chapitres 4, 5 et 6, Hermann, 1968.

[C95] I. Cherednik, *Macdonald's evaluation conjectures and difference Fourier transform*, Inv. Math. **122** (1995), 119–145.

[K92] T. Koornwinder, *Askey-Wilson polynomials for root systems of type BC*, Contemporary Mathematics **138** (1992), 189–204.

[M88] I. G. Macdonald, *Orthogonal polynomials associated with root systems*, preprint, 1988; available on the web.

[M95] I. G. Macdonald, *Symmetric functions and orthogonal polynomials*, University Lecture Series, **12**, American Mathematical Society, 1998.

[M03] I. G. Macdonald, *Affine Hecke algebras and orthogonal polynomials*, Cambridge tracts in mathematics, **157**, Cambridge University Press, 2003.

[Mi04] 三町勝久, マクドナルド多項式入門, 代数学百科 I 群論の進化 第 4 章, 朝倉書店, 2004.

[N97] 野海正俊述, 長谷川浩司記, **アフィン Hecke 環と多変数直交多項式**— Macdonald-Cherednik 理論—, 東北大学集中講義講義録, 1997;
available at `https://www.math.nagoya-u.ac.jp/~yanagida/others-j.html`