Summary and Problems of Lecture 2 [*1]

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The assignments are Exercises 2.1–2.8. The deadline of the report is **October 29th (Monday)**.

2 Macdonald symmetric polynomials

$\mathbb{K} = \mathbb{Q}(q,t)$, $\mathbb{K}[x^{\pm 1}] = \mathbb{K}[x_1^{\pm}, \dots, x_n^{\pm}]$, $W = S_n$, $P^+ = \mathcal{P}_n = \{\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{N}^n \mid \lambda_1 \geq \cdots \geq \lambda_n\}$.

2.1 The Macdonald difference operator and polynomials

Definition. The (first order) **Macdonald difference operator** $D_x^{(1)} = D_x$ is defined to be

$$D_x := \textstyle\sum_{1 \leq k \leq n} \big(\prod_{j \neq k} (t x_k - x_j)/(x_k - x_j)\big) T_{q,x_k},$$

where $T_{q,x_k} : \mathbb{K}[x^{\pm 1}] \to \mathbb{K}[x^{\pm 1}]$, $(T_{q,x_k} f)(x_1, \dots, x_n) := f(x_1, \dots, q x_k, \dots, x_n)$ is the **q-shift operator**.

Proposition 2.1. D_x satisfies the following properties.

(1) D_x is W-invariant, i.e., $w D_x = D_x w$ for any $w \in W$.

(2) Let $\Delta(x) := \prod_{1 \leq i < j \leq n} (x_i - x_j)$. Then for any $f(x) \in K[x]$ we have

$$(D_x f)(x) = (\Delta(x))^{-1} \textstyle\sum_{k=1}^n \Delta(x_1, \dots, t x_k, \dots, x_n) f(x_1, \dots, q x_k, \dots, x_n).$$

(3) D_x preserves $\mathbb{K}[x]^W$, i.e., $D_x(\mathbb{K}[x]^W) \subset \mathbb{K}[x]^W$.

Exercise 2.1 ($*$)**.** Give a proof of Proposition 2.1.

Theorem 2.2. For each $\lambda \in P^+$ there exists $P_\lambda = P_\lambda(x;q,t) \in \mathbb{K}[x]^W$ such that

(1) $P_\lambda \in m_\lambda + \sum_{\mu < \lambda} \mathbb{K} m_\mu$, where $\lambda \geq \mu$ is the dominance ordering on P^+.

(2) $D_x P_\lambda(x;q,t) = P_\lambda(x;q,t) \cdot e_1(q^\lambda t^\delta)$, $\quad e_1(q^\lambda t^\delta) := \sum_{k=1}^n q^{\lambda_k} t^{n-k}$.

Moreover such P_λ is uniquely determined. It is called **the Macdonald symmetric polynomials**.

Corollary 2.3. $\{P_\lambda \mid \lambda \in P^+\}$ is a $\mathbb{K}$-basis of $\mathbb{K}[x]^W$.

Exercise 2.2 ($*$)**.** Give a proof of Corollary 2.3 using Theorem 2.2.

Exercise 2.3 ($**$)**.** Using Theorem 2.2, show that P_λ satisfies the following properties.

(1) Let $(x;q)_l := (1-x)(1-qx)\cdots(1-q^{l-1}x)$ for $l \in \mathbb{N}$. For $\lambda = (k)$ we have

$$\frac{(t;q)_k}{(q;q)_k} P_{(k)}(x;q,t) = \sum_m \frac{(t;q)_{m_1} \cdots (t;q)_{m_n}}{(q;q)_{m_1} \cdots (q;q)_{m_n}} x_1^{m_1} \cdots x_n^{m_n},$$

where $m = (m_1, \dots, m_n)$ runs over $\mathbb{N}^n$ with $m_1 + \cdots + m_n = k$.

(2) $P_\lambda(x;q,1) = m_\lambda(x)$. (3) $P_\lambda(x;q,q) = s_\lambda(x)$.

2.2 Family of difference operators

Definition. We set $D_x^{(0)} := 1$, and for $r = 1, \dots, n$

$$D_x^{(r)} := \textstyle\sum_{I \subset \{1,\dots,n\},\, |I| = r} A_I(x;t) T_{q,x}^I, \quad A_I(x;t) := t^{\binom{r}{2}} \prod_{i \in I,\, j \notin I} (t x_i - x_j)/(x_i - x_j).$$

[*1] 2018/10/16, ver. 0.4.

$D^{(r)}$ is called the r**-th order Macdonald difference operator**. We also set $D_x(u) := \sum_{r=0}^{n}(-u)^r D_x^{(r)}$.

Exercise 2.4 ($*$)**.** Show that $A_I(x;t) = (T_{t,x}^{I}\Delta(x))/\Delta(x)$.

Proposition 2.4. Let $\delta = (\delta_1, \delta_2, \ldots, \delta_n) := (n-1, n-2, \ldots, 0)$. Then we have

$$D_x(u)m_\lambda(x) \in (1-uq^{\lambda_1}t^{\delta_1})\cdots(1-uq^{\lambda_n}t^{\delta_n})m_\lambda(x) + \textstyle\sum_{\mu<\lambda}\mathbb{K}m_\mu(x).$$

Corollary 2.5. $D_x^{(r)}m_\lambda(x) = e_r(q^\lambda t^\delta)m_\lambda(x) + $ (lower order terms), and $D_x^{(1)} = D_x$ satisfies

$$\forall\lambda \in P^+ \quad \Phi\, m_\lambda \;=\; c_{\lambda\lambda}m_\lambda + \textstyle\sum_{\mu<\lambda}c_{\lambda\mu}m_\mu, \quad c_{\lambda\lambda} \neq 0; \qquad \forall\,\mu<\lambda \quad c_{\mu\mu} \neq c_{\lambda\lambda}.$$

Exercise 2.5 ($*$)**.** Check Corollary 2.5 using Proposition 2.4.

2.3 Commutativity of Macdonald difference operators

Theorem 2.6. (1) The difference operators $D_x^{(r)}$ are commutative:

$$[D_x^{(r)}, D_x^{(s)}] \;=\; 0 \quad (r,s = 0,\ldots,n).$$

(2) P_λ is a join eigenfunctionof the operators $D_x^{(r)}$ $(r = 0,\ldots,n)$. More precisely, we have

$$D_x^{(r)}P_\lambda(x) \;=\; P_\lambda(x)e_r(q^\lambda t^\delta).$$

Assume either of the following two conditions is satisfied.

- $q,t \in \mathbb{C}$ and $0 < |q|, |t| < 1$.
- $t = q^k$, $k \in \mathbb{N}$.

Define the pairing $\langle\cdot,\cdot\rangle_{q,t}$ on $\mathbb{K}[x^{\pm1}]^W$ by

$$\langle f, g\rangle_{q,t} \;:=\; |W|^{-1}\,\mathrm{C.T.}[f(x^{-1})g(x)w(x;q,t)],$$

where $\mathrm{C.T.}[\varphi(x)]$ denotes the constant term of the Laurent expansion of $\varphi(x)$ in terms of x, and

$$w(x;q,t) \;:=\; \textstyle\prod_{i\neq j}(x_i/x_j;q)_\infty/(tx_i/x_j;q)_\infty, \quad (a;q)_\infty \;:=\; \displaystyle\lim_{n\to\infty}(a;q)_n \;=\; \textstyle\prod_{i=0}^{\infty}(1-aq^i).$$

Proposition 2.7. (1) $\langle D^{(r)}f, g\rangle_{q,t} = \langle f, D^{(r)}g\rangle_{q,t}$. for any $f,g \in \mathbb{K}[x^{\pm1}]^W$.

(2) For $\lambda,\mu \in P^+$ with $\lambda \neq \mu$ we have $\langle P_\lambda, P_\mu\rangle_{q,t} = 0$.

Exercise 2.6 ($**$)**.** Prove Proposition 2.7.

2.4 Macdonald-Cauchy kernel function

Theorem 2.8. Set $\Pi(x,y) = \Pi(x,y;q,t) := \prod_{i,j=1}^{n}(tx_iy_j;q)_\infty/(x_iy_j;q)_\infty$. Then

$$\Pi(x,y) \;=\; \textstyle\sum_{\lambda\in P^+}b_\lambda P_\lambda(x)P_\lambda(y), \quad b_\lambda \;:=\; \prod_{s\in\lambda}(1-q^{a(s)}t^{l(s)+1})/(1-q^{a(s)+1}t^{l(s)}).$$

The function $\Pi(x,y)$ is called **the Macdonald-Cauchy kernel function**.

Exercise 2.7 ($*$)**.** Check that Theorem 2.8 in the case $t = q$ reduces to the Schur case.

Exercise 2.8 ($***$)**.** Give a proof of Theorem 2.8 by taking the following steps.

(1) Using Proposition 2.7 check that $\Pi(x,y) = \sum_{\lambda\in P^+}c_\lambda P_\lambda(x)P_\lambda(y)$ with some $c_\lambda \in \mathbb{K}$ $\iff$ $D_x(u)\Pi(x,y) = D_y(u)\Pi(x,y)$.

(2) Check that the equality $D_x(u)\Pi(x,y;q,t) = D_y(u)\Pi(x,y;q,t)$ can be reduced to a q-independent identity, which is nothing but the Schur case. (Since the Schur case is already known, the proof is completed.)