Lecture 2: Macdonald symmetric polynomials [*1]

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2 Macdonald symmetric polynomials

Today we set $\mathbb{K} := \mathbb{Q}(q,t)$. We denote $\mathbb{K}[x^{\pm 1}] := \mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, and switch some notation:

$$W := S_n \quad \text{(symmetric group)}, \quad P^+ := \mathcal{P}_n = \{\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{N}^n \mid \lambda_1 \geq \dots \geq \lambda_n\}.$$

2.1 The Macdonald difference operator and polynomials

For $k = 1, \dots, n$, we denote by T_{q,x_k} the **q-shift operator** or the **q-difference operator**[*2] with respect to x_k. It is a $\mathbb{K}$-algebra isomorphism $\mathbb{K}[x^{\pm 1}] \to \mathbb{K}[x^{\pm 1}]$ given by

$$T_{q,x_k} : \mathbb{K}[x^{\pm 1}] \longrightarrow \mathbb{K}[x^{\pm 1}], \quad (T_{q,x_k} f)(x_1, \dots, x_n) := f(x_1, \dots, q x_k, \dots, x_n).$$

Definition. The (first order) **Macdonald difference operator** $D_x^{(1)} = D_x$ is defined to be

$$D_x := \sum_{k=1}^n \Big(\prod_{j \neq k} \frac{t x_k - x_j}{x_k - x_j} \Big) T_{q,x_k}.$$

Proposition 2.1. D_x satisfies the following properties.

(1) D_x is W-invariant, i.e., $w D_x = D_x w$ for any $w \in W$.

(2) Let $\Delta(x) := \prod_{1 \leq i < j \leq n} (x_i - x_j)$. Then for any $f(x) \in K[x]$ we have

$$(D_x f)(x) = \frac{1}{\Delta(x)} \sum_{k=1}^n \Delta(x_1, \dots, t x_k, \dots, x_n) f(x_1, \dots, q x_k, \dots, x_n).$$

(3) D_x preserves $\mathbb{K}[x]^W$, i.e., $D_x(\mathbb{K}[x]^W) \subset \mathbb{K}[x]^W$.

Exercise 2.1 ($*$)**.** Give a proof of Proposition 2.1.

The Macdonald symmetric polynomial is defined as an eigenfunction of the operator D_x.

Theorem 2.2 ([M88])**.** For each $\lambda \in P^+$ there exists $P_\lambda = P_\lambda(x; q, t) \in \mathbb{K}[x]^W$ such that

(1) $P_\lambda \in m_\lambda + \sum_{\mu < \lambda} \mathbb{K} m_\mu$, where $\lambda \geq \mu$ is the dominance ordering on P^+.

(2) P_λ is an eigenfunction of D_x. More precisely, we have

$$D_x P_\lambda(x; q, t) = P_\lambda(x; q, t) \cdot e_1(q^\lambda t^\delta), \quad e_1(q^\lambda t^\delta) := \textstyle\sum_{k=1}^n q^{\lambda_k} t^{n-k}.$$

Moreover such P_λ is uniquely determined. It is called **the Macdonald symmetric polynomial**.

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[*2] q 差分作用素

Corollary 2.3. $\{P_\lambda \mid \lambda \in P^+\}$ is a $\mathbb{K}$-basis of $\mathbb{K}[x]^W$.

Exercise 2.2 $(*)$**.** Give a proof of Corollary 2.3 using Theorem 2.2.

The proof of Theorem 2.2 is based on

Theorem. Assume that a $\mathbb{K}$-linear operator $\Phi : \mathbb{K}[x]^W \to \mathbb{K}[x]^W$ satisfies

$$\begin{aligned} &\forall \lambda \in P^+ \quad \Phi\, m_\lambda \;=\; c_{\lambda\lambda} m_\lambda + \textstyle\sum_{\mu<\lambda} c_{\lambda\mu} m_\mu, \quad c_{\lambda\lambda} \neq 0, \\ &\forall\, \mu < \lambda \quad c_{\mu\mu} \neq c_{\lambda\lambda}. \end{aligned} \tag{2.1}$$

Then there exists uniquely $P_\lambda \in \mathbb{K}[x]^W$ for any $\lambda \in P^+$ such that

- $P_\lambda(x) \in m_\lambda + \sum_{\mu<\lambda} \mathbb{K} m_\mu$
- $\Phi P_\lambda(x) = P_\lambda(x) c_{\lambda\lambda}$.

Exercise 2.3 $(**)$**.** Using Theorem 2.2, show that P_λ satisfies the following properties.

(1) Let $(x;q)_l := (1-x)(1-qx)\cdots(1-q^{l-1}x)$ for $l \in \mathbb{N}$. For $\lambda = (k)$ we have

$$\frac{(t;q)_k}{(q;q)_k} P_{(k)}(x;q,t) \;=\; \sum_m \frac{(t;q)_{m_1}\cdots(t;q)_{m_n}}{(q;q)_{m_1}\cdots(q;q)_{m_n}} x_1^{m_1}\cdots x_n^{m_n},$$

where $m = (m_1, \ldots, m_n)$ runs over $\mathbb{N}^n$ with $m_1 + \cdots + m_n = k$.

(2) $P_\lambda(x;q,1) = m_\lambda(x)$ (monomial symmetric polynomial).

(3) $P_\lambda(x;q,q) = s_\lambda(x)$ (Schur symmetric polynomial).

2.2 Family of difference operators

The purpose of this subsection is to check the condition (2.1). We will introduce higher order analogue of D_x using the following symbol for q-difference operators.

$$T^I_{q,x} \;:=\; T_{q,x_{i_1}} T_{q,x_{i_2}} \cdots T_{q,x_{i_r}}, \quad I \;=\; \{i_1, \ldots, i_r\}.$$

Definition. We set $D_x^{(0)} := 1$, and for $r = 1, \ldots, n$

$$D_x^{(r)} \;:=\; \sum_{I \subset \{1,\ldots,n\},\, |I|=r} A_I(x;t) T^I_{q,x}, \quad A_I(x;t) \;:=\; t^{\binom{r}{2}} \prod_{i\in I,\ j\notin I} \frac{tx_i - x_j}{x_i - x_j}.$$

$D^{(r)}$ is called the r**-th order Macdonald difference operator**. We also set

$$D_x(u) \;:=\; \sum_{r=0}^{n} (-u)^r D_x^{(r)} \;=\; \sum_{I\subset\{1,\ldots,n\}} (-u)^{|I|} A_I(x;t) T^I_{q,x}.$$

Exercise 2.4 $(*)$**.** Recalling $\Delta(x) = \prod_{1\le i<j\le n}(x_i - x_j)$, show the following formula:

$$A_I(x;t) \;=\; \frac{T^I_{t,x}\Delta(x)}{\Delta(x)}.$$

Proposition 2.4 (Triangularity of Macdonald operators)**.** Let $\delta = (\delta_1, \delta_2, \ldots, \delta_n) := (n-1, n-2, \ldots, 0)$. Then we have

$$D_x(u) m_\lambda(x) \in (1 - uq^{\lambda_1} t^{\delta_1}) \cdots (1 - uq^{\lambda_n} t^{\delta_n}) m_\lambda(x) + \sum_{\mu<\lambda} \mathbb{K} m_\mu(x).$$

Outline of Proof. Use the formula $D_x(u) = \frac{1}{\Delta(x)} \det \left[x_j^{\delta_i}(1-ut^{\delta_i}T_{q,x_j})\right]_{i,j=1}^n$. □

Corollary 2.5. $D_x^{(r)} m_\lambda(x) = e_r(q^\lambda t^\delta) m_\lambda(x) +$(lower order terms), and $D_x^{(1)} = D_x$ satisfies the condition (2.1).

Exercise 2.5 $(*)$**.** Check Corollary 2.5 using Proposition 2.4.

2.3 Commutativity of Macdonald difference operators

Theorem 2.6. (1) The difference operators $D_x^{(r)}$ are commutative:

$$[D_x^{(r)}, D_x^{(s)}] = 0 \quad (r,s = 0,\ldots,n).$$

(2) P_λ is a join eigenfunction[*3]of the operators $D_x^{(r)}$ $(r = 0,\ldots,n)$. More precisely, we have

$$D_x^{(r)} P_\lambda(x) = P_\lambda(x) \cdot e_r(q^\lambda t^\delta).$$

Outline of Proof. It is enough to show $[D_x(u), D_x(v)] = 0$. For that, let us introduce an inner product $\langle \cdot, \cdot \rangle_{q,t}$ on $\mathbb{K}[x]^W$. Assume either of the following two conditions is satisfied.

- $q, t \in \mathbb{C}$ and $0 < |q|, |t| < 1$
- $t = q^k$, $k = 0, 1, \ldots$.

Define the pairing $\langle \cdot, \cdot \rangle_{q,t}$ on $\mathbb{K}[x^{\pm 1}]^W$ by

$$\langle f, g \rangle_{q,t} := \frac{1}{|W|} \mathrm{C.T.}[f(x^{-1})g(x)w(x;q,t)],$$

where $\mathrm{C.T.}[\varphi(x)]$ denotes the constant term of the Laurent expansion of $\varphi(x)$ in terms of x, and

$$w(x;q,t) := \prod_{i \neq j} \frac{(x_i/x_j;q)_\infty}{(tx_i/x_j;q)_\infty}, \quad (a;q)_\infty := \lim_{n\to\infty}(a;q)_n = \prod_{i=0}^\infty (1-aq^i).$$

Proposition 2.7. (1) $D^{(r)} = D_x^{(r)}$ is self-adjoint with respect to $\langle \cdot, \cdot \rangle_{q,t}$. In other words, for any $f, g \in \mathbb{K}[x^{\pm 1}]^W$ we have

$$\langle D^{(r)} f, g \rangle_{q,t} = \langle f, D^{(r)} g \rangle_{q,t}.$$

(2) For $\lambda, \mu \in P^+$ with $\lambda \neq \mu$ we have $\langle P_\lambda, P_\mu \rangle_{q,t} = 0$.

Exercise 2.6 $(**)$**.** Prove Proposition 2.7.

Proposition. There exists a unique subset $\{Q_\lambda\}_{\lambda \in P^+}$ of $\mathbb{K}[X]^W$ such that

(1) $Q_\lambda = m_\lambda +$ (lower order terms), (2) $\langle Q_\lambda, Q_\mu \rangle_{q,t} = 0$ for $\lambda, \mu \in P^+$ with $\lambda \neq \mu$.

Corollary. We have

$$[D_x(u), D_x(v)]\Big|_{\mathbb{K}[x]^W} = 0.$$

Proposition. For any $\Phi \in \mathbb{K}(x)[T_{q,x}]$, if $\Phi|_{\mathbb{K}[x]^W} = 0$ then $\Phi = 0$.

Thus the proof of Theorem 2.6 is finished. □

[*3] 同時固有関数

2.4 Macdonald-Cauchy kernel function

Recall the Cauchy kernel for Schur polynomials:

$$\prod_{i,j=1}^{n} \frac{1}{1-x_i y_j} = \sum_{\lambda \in P^+} s_\lambda(x) s_\lambda(y). \tag{2.2}$$

Macdonald introduced a q,t-analogue of this Cauchy kernel. To explain it, let us introduce the **arm** and **leg** functions. Regard $\lambda \in P^+ \subset \mathcal{P}$ as a Young diagram and set the coordinate (i,j) of boxes in λ such that $1 \le i \le \ell(\lambda)$, $1 \le j \le \lambda_i$. Then for a box $s = (i,j)$ we define

$$a(s) = a_\lambda(s) := \lambda_i - j, \quad l(s) = a_{{}^t\lambda}(j,i),$$

where ${}^t\lambda$ is the transpose of λ.

Theorem 2.8. Set

$$\Pi(x,y) = \Pi(x,y;q,t) := \prod_{i,j=1}^{n} \frac{(t x_i y_j ; q)_\infty}{(x_i y_j ; q)_\infty}.$$

Then we have

$$\Pi(x,y) = \sum_{\lambda \in P^+} b_\lambda P_\lambda(x) P_\lambda(y), \quad b_\lambda := \prod_{s \in \lambda} \frac{1 - q^{a(s)} t^{l(s)+1}}{1 - q^{a(s)+1} t^{l(s)}}.$$

Here s runs over the boxes of the Young diagram associated to λ. The function $\Pi(x,y)$ is called **the Macdonald-Cauchy kernel function**.

Exercise 2.7 $(*)$**.** Check that Theorem 2.8 in the case $t = q$ reduces to the Schur case (2.2).

Exercise 2.8 $(***)$**.** Give a proof of Theorem 2.8 by taking the following steps.

(1) Using Proposition 2.7 check that $\Pi(x,y) = \sum_{\lambda \in P^+} c_\lambda P_\lambda(x) P_\lambda(y)$ with some $c_\lambda \in \mathbb{K}$ $\iff$ $D_x(u)\Pi(x,y) = D_y(u)\Pi(x,y)$.

(2) Check that the equality $D_x(u)\Pi(x,y;q,t) = D_y(u)\Pi(x,y;q,t)$ can be reduced to a q-independent identity, which is nothing but the Schur case. (Since the Schur case is already known, the proof is completed.)

References

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