

Summary and Problems of Lecture 1 ^{*1}

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The assignments of Lecture 1 are Exercises 1.1–1.8 below. The deadline of the report is **October 29th (Monday)**.

1 Symmetric polynomials

$\mathbb{N} := \mathbb{Z}_{\geq 0} = \{0, 1, 2, \dots\}$, \mathbb{K} : a field of characteristic 0.

1.1 Symmetric groups and symmetric polynomials

Definition. • S_n : the n -th **symmetric group**.

- $\mathbb{K}[x]^{S_n} = \mathbb{K}[x_1, \dots, x_n]^{S_n} := \{\text{symmetric polynomials}\} = \{f \in \mathbb{K}[x] \mid \sigma.f = f \quad \forall \sigma \in S_n\}$:
the ring of **symmetric polynomials**.
- The r -th **elementary symmetric polynomial**

$$e_r(x) := \sum_{1 \leq j_1 < \dots < j_r \leq n} x_{j_1} \cdots x_{j_r} \in \mathbb{Z}[x]^{S_n}$$

Theorem 1.3. $\mathbb{K}[x]^{S_n} = \mathbb{K}[e_1(x), \dots, e_n(x)]$.

1.2 Partitions

Definition. • $\mathcal{P}^d := \{\text{partitions of } d\}$: the set of **partitions** of d (d の分割の集合),

- $\mathcal{P} := \bigsqcup_{d \in \mathbb{N}} \mathcal{P}^d$: the set of partitions (分割の集合).
- $p(d) := |\mathcal{P}^d|$: the **partition number** of d (d の分割数).
- For $\lambda = (\lambda_1, \lambda_2, \dots) \in \mathcal{P}_n$, $|\lambda| := \sum_i \lambda_i$, $\ell(\lambda) := \text{length of } \lambda$, ${}^t \lambda$: the **transpose** of λ .
- Young diagram for a partition λ :

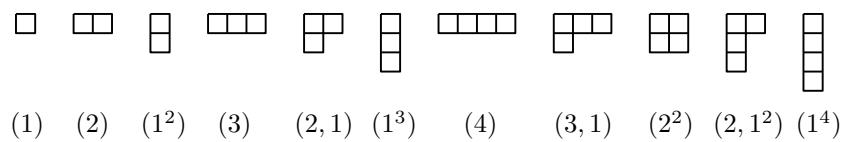


Figure 1 Young diagrams corresponding to partitions λ with $|\lambda| \leq 4$

Exercise 1.1 (*). Explain that the generating function $G(z) := \sum_{d \geq 0} p(d)z^d$ of the partition numbers

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is equal to the following infinite product.

$$G(z) = \prod_{m \in \mathbb{N}} \frac{1}{1 - z^m} = \frac{1}{1 - z} \frac{1}{1 - z^2} \frac{1}{1 - z^3} \cdots$$

1.3 Classical symmetric polynomials

Definition. • The action of S_n on \mathbb{N}^n : $w.\alpha = w.(\alpha_1, \dots, \alpha_n) := (\alpha_{w(1)}, \dots, \alpha_{w(n)})$.

- $x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$.
- $w.x^\alpha = x^{w^{-1}.\alpha}$ for $w \in S_n$.
- $\mathcal{P}_n := \{\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{N}^n \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\}$.
- $\mathcal{P}_n^d := \{\lambda \in \mathcal{P}_n \mid |\lambda| = d\} = \{\lambda \in \mathcal{P}^d \mid \ell(\lambda) \leq n\}$.

Proposition 1.5. The orbit decomposition of the action of S_n on \mathbb{N}^n is given by

$$\mathbb{N}^n = \sqcup_{\lambda \in \mathcal{P}_n} S_n.\lambda.$$

For the subset $\{\alpha \in \mathbb{N}^n \mid |\alpha| = d\}$, we have the orbit decomposition $\{\alpha \in \mathbb{N}^n \mid |\alpha| = d\} = \sqcup_{\lambda \in \mathcal{P}_n^d} S_n.\lambda$.

Exercise 1.2 (*). Show Proposition 1.5.

Definition. • $\Lambda_n = \Lambda_n(x) := \mathbb{Z}[x]^{S_n}$, $\Lambda_n^d = \Lambda_n^d(x) := \{f(x) \in \Lambda_n \mid \deg f(x) = d\}$.

- The **monomial symmetric polynomial** for $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathcal{P}_n$:

$$m_\lambda(x) := \sum_{\alpha \in S_n.\lambda} x^\alpha = \sum_{\substack{\alpha=(\alpha_1, \dots, \alpha_n): \\ \text{different permutations of } \lambda}} x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}.$$

Proposition 1.7. Λ_n is a free \mathbb{Z} -module, and $\{m_\lambda \mid \lambda \in \mathcal{P}_n\}$ is a basis of Λ_n .

Definition. • For $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathcal{P}_n^d$, $e_\lambda := e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_n} \in \Lambda_n^d$.

- The **dominance ordering** $\mu \leq \lambda$ of partitions:

$$\mu \leq \lambda \iff |\mu| = |\lambda| \text{ and } \mu_1 + \cdots + \mu_k \leq \lambda_1 + \cdots + \lambda_k \quad \forall k = 1, \dots, n. \quad (1.1)$$

Example. The dominance ordering is a total order on \mathcal{P}_n^n with $n \leq 5$:

$$\begin{aligned} (2) &> (1, 1), \\ (3) &> (2, 1) > (1, 1, 1), \\ (4) &> (3, 1) > (2, 2) > (2, 1, 1) > (1, 1, 1, 1), \\ (5) &> (4, 1) > (3, 2) > (3, 1^2) > (2^2, 1) > (2, 1^3) > (1^5). \end{aligned}$$

However, on \mathcal{P}_n^n with $n \geq 6$ the dominance ordering is a partial order (半順序).

$$(6) > (5, 1) > (4, 2) > (4, 1^2) > (3, 2, 1) > (3, 1^3) > (2^2, 1^2) > (2, 1^4) > (1^6).$$

Theorem 1.9. For any $\lambda \in \mathcal{P}_n^d$, one can expand e_λ in terms of $\{m_\mu \mid \mu \in \mathcal{P}_n^d\}$ as

$$e_\lambda = m_\lambda + \sum_{\mu < \lambda} a_{\lambda, \mu} m_\mu, \quad a_{\lambda, \mu} \in \mathbb{Z}.$$

Exercise 1.3 ().** Give a proof of Theorem 1.9 (see [O06, 定理 9.2] for example).

Exercise 1.4 ().** The r -th **completely homogeneous symmetric polynomial** $h_r \in \Lambda_n^r$ is defined to be

$$h_r(x) := \sum_{1 \leq i_1 \leq i_2 \leq \dots \leq i_r \leq n} x_{i_1} x_{i_2} \cdots x_{i_r}.$$

For $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathcal{P}_n^d$, we set $h_\lambda := h_{\lambda_1} \cdots h_{\lambda_n} \in \Lambda_n^d$.

- (1) Check the equality $h_d(x) = \sum_{\lambda \in \mathcal{P}^d} m_\lambda(x)$.
- (2) Show that $\{h_\lambda \mid \lambda \in \mathcal{P}_n^d\}$ is a basis of Λ_n^d .

1.4 Schur polynomials

Definition 1.11. (1) For $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n$, we define an alternate function $a_\mu(x)$ by

$$a_\mu(x) := \begin{vmatrix} x_1^{\mu_1} & x_1^{\mu_2} & \cdots & x_1^{\mu_n} \\ x_2^{\mu_1} & x_2^{\mu_2} & \cdots & x_2^{\mu_n} \\ \vdots & & \ddots & \vdots \\ x_n^{\mu_1} & x_n^{\mu_2} & \cdots & x_n^{\mu_n} \end{vmatrix}.$$

(2) For $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathcal{P}_n$, we define the **Schur symmetric polynomial** to be

$$s_\lambda(x) := \frac{a_{\delta+\lambda}(x)}{a_\delta(x)}, \quad \delta + \lambda := (\lambda_1 + n - 1, \lambda_2 + n - 2, \dots, \lambda_n), \delta := (n-1, n-2, \dots, 0).$$

Example 1.12. For $n = |\lambda| \leq 3$, Schur symmetric polynomials look as follows.

$$\begin{aligned} s_{(1)} &= x_1, \\ s_{(2)} &= \begin{vmatrix} x_1^3 & 1 \\ x_2^3 & 1 \end{vmatrix} / \begin{vmatrix} x_1 & 1 \\ x_2 & 1 \end{vmatrix} = x_1^2 + x_1 x_2 + x_2^2 = h_2, \quad s_{(1^2)} = \begin{vmatrix} x_1^2 & x_1^1 \\ x_2^2 & x_1^1 \end{vmatrix} / \begin{vmatrix} x_1 & 1 \\ x_2 & 1 \end{vmatrix} = x_1 x_2 = e_2, \\ s_{(3)} &= \begin{vmatrix} x_1^5 & x_1 & 1 \\ x_2^5 & x_2 & 1 \\ x_3^5 & x_3 & 1 \end{vmatrix} / \begin{vmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{vmatrix} = x_1^3 + x_2^3 + x_3^3 + x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 = h_3, \\ s_{(2,1)} &= \begin{vmatrix} x_1^4 & x_1^2 & 1 \\ x_2^4 & x_2^2 & 1 \\ x_3^4 & x_3^2 & 1 \end{vmatrix} / \begin{vmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{vmatrix} = (x_1 + x_2)(x_2 + x_3)(x_3 + x_1), \\ s_{(1^3)} &= \begin{vmatrix} x_1^3 & x_1^2 & x_1 \\ x_2^3 & x_2^2 & x_1 \\ x_3^3 & x_3^2 & x_1 \end{vmatrix} / \begin{vmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{vmatrix} = x_1 x_2 x_3 = e_3. \end{aligned}$$

Exercise 1.5 (*). Check that $s_{(1^n)} = e_n$ and $s_{(n)} = h_n$ in $\mathbb{K}[x]^{S_n}$.

Proposition 1.13. The Schur symmetric polynomial s_λ given in Definition 1.11 is a symmetric polynomial for each $\lambda \in \mathcal{P}_n$.

Exercise 1.6 ().** Give a proof of Proposition 1.13. More precisely, show that for any $\lambda \in \mathcal{P}_n$

- (1) $s_\lambda(x) \in \mathbb{Z}[x]$,
- (2) $s_\lambda(x) \in \Lambda_n = \mathbb{Z}[x]^{S_n}$.

Theorem 1.14. For any $\lambda \in \mathcal{P}_n$ we have

$$s_\lambda = m_\lambda + \sum_{\mu < \lambda} K_{\lambda,\mu} m_\mu, \quad K_{\lambda,\mu} \in \mathbb{N}.$$

Here $\mu < \lambda$ means the dominance ordering (1.1). The number $K_{\lambda,\mu}$ is called the **Kostka number**.

Example. For $|\lambda| \leq 4$ we have

$$\begin{aligned} s_{(1)} &= m_{(1)}, \\ s_{(2)} &= m_{(2)} + m_{(1^2)}, \quad s_{(1^2)} = m_{(1^2)}, \\ s_{(3)} &= m_{(3)} + m_{(2,1)} + m_{(1^3)}, \quad s_{(2,1)} = m_{(2,1)} + 2m_{(1^3)}, \quad s_{(1^3)} = m_{(1^3)}, \\ s_{(4)} &= m_{(4)} + m_{(3,1)} + m_{(2,2)} + m_{(2,1^2)} + m_{(1^4)}, \quad s_{(3,1)} = m_{(3,1)} + m_{(2^2)} + 2m_{(2,1^2)} + 3m_{(1^4)}, \\ s_{(2^2)} &= m_{(2^2)} + m_{(2,1^2)} + 2m_{(1^4)}, \quad s_{(2,1^2)} = m_{(2,1^2)} + 3m_{(1^4)}, \quad s_{(1^4)} = m_{(1^4)}. \end{aligned}$$

Corollary 1.15. $\{s_\lambda \mid \lambda \in \mathcal{P}_n\}$ is a basis of the \mathbb{Z} -module Λ_n .

Exercise 1.7 (*). Give a proof of Corollary 1.15 using Theorem 1.14.

Theorem 1.17 (Cauchy formula). For $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ we have

$$\sum_{\lambda \in \mathcal{P}_n} s_\lambda(x) s_\lambda(y) = \prod_{i,j=1}^n \frac{1}{1 - x_i y_j}.$$

The right hand side term $\prod_{i,j=1}^n \frac{1}{1 - x_i y_j}$ is called the **Cauchy kernel** function.

Exercise 1.8 ().** Give a proof of Theorem 1.17.

References

- [M95] I. G. Macdonald, *Symmetric functions and Hall polynomials*, 2nd ed., Oxford University Press, 1995.
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