Lecture 1: classical theory of symmetric polynomials [*1]

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1 Symmetric polynomials

General notation:
$\mathbb{N} := \mathbb{Z}_{\geq 0} = \{0, 1, 2, \ldots\}$.
n will denote a positive integer unless otherwise stated[*2].
$\mathbb{K}$ denotes a field of characteristic 0[*3].

1.1 Symmetric groups and symmetric polynomials

Let us denote by S_n the n-th **symmetric group**[*4]. It consists of permutations[*5] of the set $\{1, 2, \ldots, n\}$. One can express an element $\sigma \in S_n$ as

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & i & \cdots & n-1 & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(i) & \cdots & \sigma(n-1) & \sigma(n) \end{pmatrix}.$$

The multiplication of the group S_n is defined to be the composition[*6] of permutation. In other words, we have

$$\sigma\tau := \sigma \circ \tau, \quad (\sigma\tau)(i) = \sigma(\tau(i)).$$

Then the associativity condition $(\sigma\tau)\mu = \sigma(\tau\mu)$ holds for any $\sigma, \tau, \mu \in S_n$. The unit[*7] of the group S_n is the identity permutation[*8]

$$e = \mathrm{id} = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ 1 & 2 & \cdots & n-1 & n \end{pmatrix}.$$

Recall that S_n is generated by the transposition[*9]. For $i = 1, 2, \ldots, n-1$, set

$$s_i := (i, i+1) = \begin{pmatrix} 1 & \cdots & i & i+1 & \cdots & n \\ 1 & \cdots & i+1 & i & \cdots & n \end{pmatrix}.$$

The element s_i is called a **simple reflection**[*10]. Then S_n is generated by the simple reflections $s_1, s_2, \ldots, s_{n-1}$. Simple reflections enjoy the following relations.

$$s_i^2 = 1, \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \quad s_i s_j = s_j s_i. \tag{1.1}$$

[*1] 2018/10/02, ver. 0.3.
[*2] 断らない限り
[*3] 標数 0 の体
[*4] n 次対称群
[*5] 置換
[*6] 合成
[*7] (群の) 単位元
[*8] 恒等置換
[*9] 互換
[*10] 単純鏡映

Let us denote the polynomial ring[*11]of n variables $x = (x_1, \dots, x_n)$ over $\mathbb{K}$ by the symbol

$$\mathbb{K}[x] \;=\; \mathbb{K}[x_1, \dots, x_n].$$

The n-th symmetric group S_n acts[*12]on $k[x]$ by permuting variables x[*13]. In other words, let an element $\sigma \in S_n$ act on polynomials of $x_1, \dots, x_n$ by the rule

$$\sigma.x_i \;=\; x_{\sigma(i)}. \tag{1.2}$$

Then this action extends naturally to that on a polynomial $f \in \mathbb{K}[x]$, and we have

$$e.f \;=\; f, \quad \sigma.(\tau.f) = (\sigma\tau).f.$$

Definition 1.1. A **symmetric polynomial**[*14] of n variables is an element $f \in \mathbb{K}[x]$ such that $\sigma.f = f$ for any $\sigma \in S_n$. Then the $\mathbb{K}$-linear space

$$\mathbb{K}[x]^{S_n} \;:=\; \{\text{symmetric polynomials}\} \;=\; \{f \in \mathbb{K}[x] \mid \sigma.f = f \quad \forall\, \sigma \in S_n\}$$

is a commutative ring, which is called **the ring of symmetric polynomials**[*15].

The same construction works if we replace $\mathbb{K}$ by a commutative ring R. In particular, it works for $\mathbb{Z}$, the ring of integers. We denote by

$$R[x]^{S_n}, \quad \mathbb{Z}[x]^{S_n}$$

the ring of symmetric polynomials over R or $\mathbb{Z}$.

Definition 1.2. For $r = 0, 1, \dots, n$, the r-th **elementary symmetric polynomial** [*16] e_r is given by[*17]

$$e_r(x) \;:=\; \sum_{1 \le j_1 < \cdots < j_r \le n} x_{j_1} \cdots x_{j_r} \in \mathbb{Z}[x]^{S_n}.$$

The generating function[*18] of e_r's is given by

$$\sum_{r=0}^{n} z^r e_r(x) \;=\; (1 + z x_1)(1 + z x_2) \cdots (1 + z x_n).$$

Recall the following well-known statement.

Theorem 1.3. $\mathbb{K}[x]^{S_n} = \mathbb{K}[e_1(x), \dots, e_n(x)]$.

A proof of this theorem will be sketched in §1.3. As a preliminary let us introduce notations on partitions.

[*11] 多項式環
[*12] (群が) 作用する
[*13] 変数 $x = (x_1, \dots, x_n)$ を置換する (ことで)
[*14] 対称多項式
[*15] 対称多項式環
[*16] 基本対称多項式
[*17] modified in ver. 0.3.
[*18] 母函数

1.2 Partitions

A **partition**[*19] means a finite non-increasing sequence of positive integers. In other words, a partition λ is a sequence expressed as

$$\lambda = (\lambda_1, \ldots, \lambda_k), \quad \lambda_i \in \mathbb{Z}, \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0.$$

For a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$, we set

$$|\lambda| := \sum_i \lambda_i, \quad \ell(\lambda) := (\text{length of } \lambda) = k.$$

We identify a partition with the sequence padded with 0's. Thus

$$\lambda = (\lambda_1, \ldots, \lambda_k) = (\lambda_1, \ldots, \lambda_k, 0) = (\lambda_1, \ldots, \lambda_k, 0, 0, \ldots).$$

We also regard $\emptyset := () = (0)$ as a partition.

If a partition λ satisfies $|\lambda| = d$, then we say λ is a **partition of** d. We set

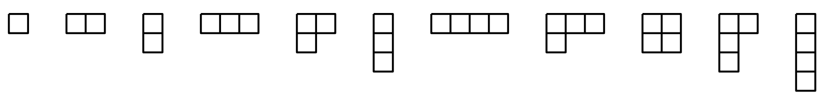
$$\mathcal{P}^d := \{\text{partitions of } d\}, \quad \mathcal{P} := \bigsqcup_{d \in \mathbb{N}} \mathcal{P}^d$$

The integer $p(d) := |\mathcal{P}^d|$ is called the **partition number**[*20] of d.

Here are the partitions of $d \leq 6$. We use the abbreviations like $(1^2) = (1,1)$, $(2^3) = (2,2,2)$.

d	$p(d)$	$\mathcal{P}^d$
0		$()$
1	1	(1)
2	2	(2), (1^2)
3	3	(3), $(2,1)$, (1^3)
4	5	(4), $(3,1)$, (2^2), $(2,1^2)$, (1^4)
5	7	(5), $(4,1)$, $(3,2)$, $(3,1^2)$, $(2^2,1)$, $(2,1^3)$, (1^5)
6	11	(6), $(5,1)$, $(4,2)$, $(4,1^2)$, (3^2), $(3,2,1)$, $(3,1^3)$, (2^3), $(2^2,1^2)$, $(2,1^4)$, (1^6)

Dealing with partitions, it is sometimes very convenient to use **Young diagrams**[*21]. We will use the English style[*22] of Young diagrams as in Figure 1.



(1) (2) (1^2) (3) $(2,1)$ (1^3) (4) $(3,1)$ (2^2) $(2,1^2)$ (1^4)

Figure 1 Young diagrams corresponding to partitions λ with $|\lambda| \leq 4$

[*19] 分割

[*20] 分割数

[*21] Young 図形

[*22] There is another way of drawing Young diagram called French style.

Exercise 1.1 ($*$)**.** [*23] Explain that the generating function $G(z) := \sum_{d \ge 0} p(d) z^d$ of the partition numbers is equal to the following infinite product.

$$G(z) \;=\; \prod_{m \in \mathbb{N}} \frac{1}{1-z^m} \;=\; \frac{1}{1-z}\frac{1}{1-z^2}\frac{1}{1-z^3}\cdots.$$

$G(z)$ is sometimes called **the partition function**.

Definition 1.4. For a partition λ, its **transpose**[*24] ${}^t\lambda$ means the partition whose Young diagram is obtained by the transposition of the Young diagram of λ.

For example, we have

$${}^t(n) \;=\; (1^n), \quad {}^t(2,1) \;=\; (2,1), \quad {}^t(3,1) \;=\; (2,1^2), \quad {}^t(2,2) \;=\; (2,2).$$

We also have ${}^t({}^t\lambda) = \lambda$.

1.3 Classical symmetric polynomials

We continue to use the notation $x = (x_1, \ldots, x_n)$. Hereafter we denote the ring of symmetric polynomials over $\mathbb{Z}$ by

$$\Lambda_n \;=\; \Lambda_n(x) \;:=\; \mathbb{Z}[x]^{S_n}.$$

Its degree d part is denoted by

$$\Lambda_n^d \;=\; \Lambda_n^d(x) \;:=\; \{f(x) \in \Lambda_n \mid \deg f(x) = d\}.$$

In this subsection we introduce several well-known bases of Λ_n and explain a proof of Theorem 1.3.

It is convenient to introduce the following symbol. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, we set

$$x^\alpha \;:=\; x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}.$$

We also set $|\alpha| := \alpha_1 + \cdots + \alpha_n$. So we have $\deg x^\alpha = |\alpha|$. The action of $w \in S_n$ on x^α is given by

$$w.x^\alpha \;=\; w.(x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}) \;=\; x_{w(1)}^{\alpha_1} x_{w(2)}^{\alpha_2} \cdots x_{w(n)}^{\alpha_n} \;=\; x_1^{\alpha_{w^{-1}(1)}} x_2^{\alpha_{w^{-1}(2)}} \cdots x_n^{\alpha_{w^{-1}(n)}}.$$

Therefore if we define the action of S_n on $\mathbb{N}^n$ by

$$w.\alpha \;=\; w.(\alpha_1, \ldots, \alpha_n) \;:=\; (\alpha_{w(1)}, \ldots, \alpha_{w(n)}), \quad w \in S_n,\ \alpha \in \mathbb{N}^n, \tag{1.3}$$

then we have

$$w.x^\alpha = x^{w^{-1}.\alpha}. \tag{1.4}$$

Let us also introduce

$$\begin{aligned} \mathcal{P}_n &:= \{\text{non-increasing sequences of non-negative integers of length } n\} \\ &= \{\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{N}^n \mid \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ge 0\}. \end{aligned}$$

[*23] The number of $*$'s denotes the difficulty of the exercise.
[*24] 転置

Similarly as in the case of partitions, we set $|\lambda| := \sum_i \lambda_i$ for $\lambda \in \mathcal{P}_n$. We also set $\ell(\lambda)$ to be the maximal number k such that $\lambda_k \neq 0$. Finally we set

$$\mathcal{P}_n^d := \{\lambda \in \mathcal{P}_n \mid |\lambda| = d\}.$$

So $\mathcal{P}_n = \sqcup_{d \geq 0} \mathcal{P}_n^d$. We have an obvious identification

$$\mathcal{P}_n^d = \{\lambda \in \mathcal{P}^d \mid \ell(\lambda) \leq n\}.$$

Now we have

Proposition 1.5. Under the action (1.3) of S_n on $\mathbb{N}^n$, the orbit decomposition[*25] is given by

$$\mathbb{N}^n = \sqcup_{\lambda \in \mathcal{P}_n} S_n.\lambda.$$

For the subset $\{\alpha \in \mathbb{N}^n \mid |\alpha| = d\}$, we have the orbit decomposition $\{\alpha \in \mathbb{N}^n \mid |\alpha| = d\} = \sqcup_{\lambda \in \mathcal{P}_n^d} S_n.\lambda$.

Exercise 1.2 ($*$)**.** Show Proposition 1.5.

Definition 1.6. For $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathcal{P}_n$, the **monomial symmetric polynomial** [*26]$m_\lambda \in \Lambda_n$ is defined to be

$$m_\lambda(x) := \sum_{\alpha \in S_n.\lambda} x^\alpha = \sum_{\substack{\alpha = (\alpha_1, \ldots, \alpha_n): \\ \text{different permutations of } \lambda}} x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}.$$

In the first expression we denoted by $S_n.\lambda$ the orbit of $\lambda \in \mathbb{Z}^n$ under the action of S_n.

Example. In the case $n = 3$, we have

$$\begin{aligned}
m_{(3)}(x) &= m_{(3)}(x_1, x_2, x_3) = \sum_{\alpha \in S_3.(3,0,0)} x^\alpha = x^{(3,0,0)} + x^{(0,3,0)} + x^{(0,0,3)} = x_1^3 + x_2^3 + x_3^3, \\
m_{(2,1)}(x) &= \sum_{\alpha \in S_3.(2,1,0)} x^\alpha = x^{(2,1,0)} + x^{(2,0,1)} + x^{(1,2,0)} + x^{(1,0,2)} + x^{(0,2,1)} + x^{(0,1,2)} \\
&= x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 - 2x_3 + x_2 x_3^2, \\
m_{(1^3)}(x) &= \sum_{\alpha \in S_3.(1,1,1)} x^\alpha = x^{(1,1,1)} = x_1 x_2 x_3.
\end{aligned}$$

Note also that $e_r = m_{(1^r)}$ for any $r \in \mathbb{N}$.

Proposition 1.7. Λ_n is a free $\mathbb{Z}$-module, and $\{m_\lambda \mid \lambda \in \mathcal{P}_n\}$ is a basis of Λ_n. In other words

$$\Lambda_n = \bigoplus_{\lambda \in \mathcal{P}_n} \mathbb{Z}\, m_\lambda$$

Proof. It is enough to show $\Lambda_n^d = \oplus_{\lambda \in \mathcal{P}_n^d} \mathbb{Z}\, m_\lambda$ for each $d \in \mathbb{N}$. Any $f \in \Lambda_n^d$ can be expressed as $f(x) = \sum_{\alpha \in \mathbb{N}^n, |\alpha| = d} c_\alpha x^\alpha$. Since f is a symmetric polynomial, we have $w.f = f$ for any $w \in S_n$. Recalling (1.4), we see that $w.f = \sum_\alpha c_\alpha x^{w^{-1}.\alpha} = \sum_\alpha c_{w.\alpha} x^\alpha$. Thus $w.f = f$ implies $c_{w.\alpha} = c_\alpha$. Then using Proposition 1.5 we have

$$f = \sum_{\alpha \in \mathbb{N}^n, |\alpha| = d} c_\alpha x^\alpha = \sum_{\lambda \in \mathcal{P}_n^d} \sum_{\alpha \in S_n.\lambda} c_\alpha x^\alpha = \sum_{\lambda \in \mathcal{P}_n^d} c_\lambda \sum_{\alpha \in S_n.\lambda} c_\alpha x^\alpha = \sum_{\lambda \in \mathcal{P}_n^d} c_\lambda m_\lambda(x).$$

[*25] 軌道分解

[*26] 単項対称多項式, またはモノミアル対称多項式

So any $f \in \Lambda_n^d$ can be expressed as a summation of m_λ with integer coefficients, and such an expression is unique. Therefore $\Lambda_n^d = \oplus_{\lambda \in \mathcal{P}_n^d} \mathbb{Z}\, m_\lambda$. $\square$

Now we introduce

Definition 1.8. For $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathcal{P}_n^d$, we define

$$e_\lambda \;:=\; e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_n} \;\in\; \Lambda_n^d.$$

By Proposition 1.7, we find that each $f \in \Lambda_n^d$ can be expressed as a linear combination of $\{m_\mu \mid \mu \in \mathcal{P}_n^d\}$. In the case $f = e_{{}^t\lambda}$, we have the following statement.

Theorem 1.9. [*27] For any $\lambda \in \mathcal{P}^d$ one can expand $e_{{}^t\lambda}$ in terms of $\{m_\mu \mid \mu \in \mathcal{P}^d\}$ as

$$e_{{}^t\lambda} \;=\; m_\lambda + \sum_{\mu < \lambda} a_{\lambda,\mu} m_\mu, \quad a_{\lambda,\mu} \in \mathbb{Z}.$$

Here we used the **dominance ordering** [*28] $\mu \le \lambda$, which is defined by

$$\mu \le \lambda \iff |\mu| \;=\; |\lambda| \;\text{ and }\; \mu_1 + \cdots + \mu_k \le \lambda_1 + \cdots + \lambda_k \;\; \forall\, k = 1, 2, \ldots. \tag{1.5}$$

Actually the dominance ordering is a total order[*29] on $\mathcal{P}^d$ with $d \le 5$. We have

$$\begin{aligned}
&(2) > (1,1), \\
&(3) > (2,1) > (1,1,1), \\
&(4) > (3,1) > (2,2) > (2,1,1) > (1,1,1,1), \\
&(5) > (4,1) > (3,2) > (3,1^2) > (2^2,1) > (2,1^3) > (1^5).
\end{aligned}$$

However, on $\mathcal{P}^d$ with $n \ge 6$ the dominance ordering is a partial order[*30].

$$(6) > (5,1) > (4,2) \begin{matrix} > (4,1^2) > \\ > (3^2) > \end{matrix} (3,2,1) \begin{matrix} > (3,1^3) > \\ > (2^3) > \end{matrix} (2^2,1^2) > (2,1^4) > (1^6).$$

For $d = 7, 8$, it looks as in Figure 2.

Exercise 1.3 (∗∗)**.** Give a proof of Theorem 1.9 (see [O06, 定理 9.2] for example).

As a corollary of Theorem 1.9, we have

Corollary 1.10. Then $\{e_\lambda \mid \lambda \in \mathcal{P}_n^d\}$ is a basis of Λ_n^d. Thus

$$\Lambda_n^d \;=\; \bigoplus\nolimits_{\lambda \in \mathcal{P}_n^d} \mathbb{Z}\, e_\lambda.$$

In particular, Theorem 1.3 holds.

[*27] modified in ver. 0.3.

[*28] 支配順序またはドミナンス順序

[*29] 全順序

[*30] 半順序

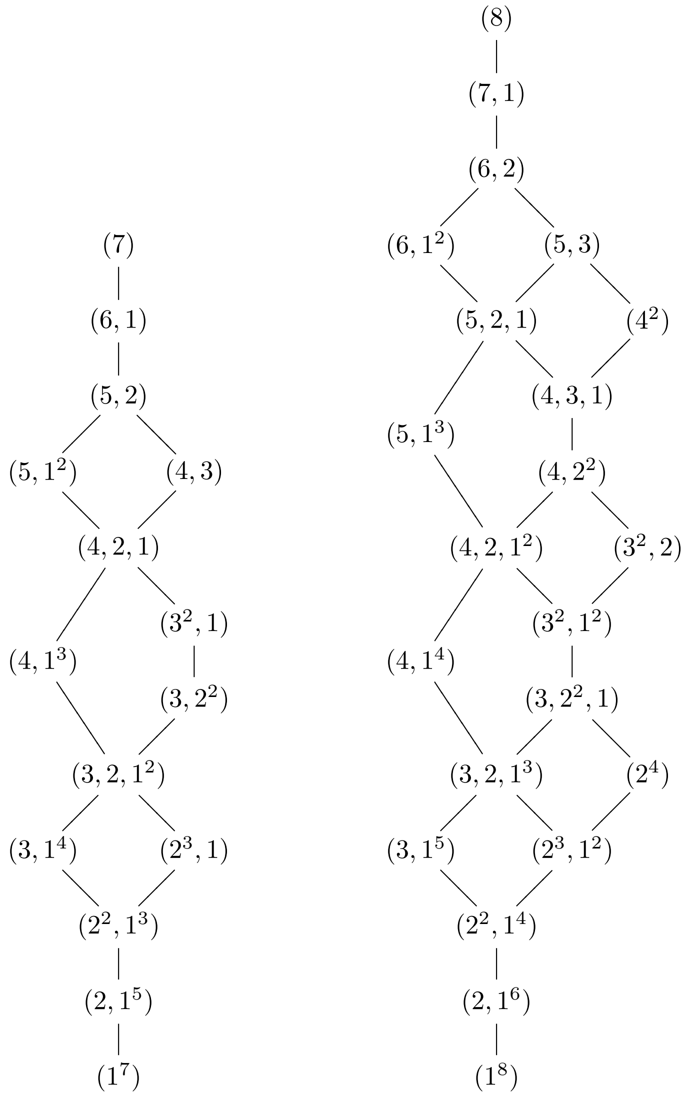


Figure 2 The Hasse diagram of dominance ordering on partitions of $d = 7, 8$

Proof. By Theorem 1.9, if we express $e_{{}^t\lambda} = \sum_\mu a_{\lambda,\mu} m_\mu$, then we have $a_{\lambda,\mu} = 0$ if $\mu \not\leq \lambda$, and $a_{\lambda,\lambda} = 0$.

Now consider the matrix $A = (a_{\lambda,\mu})_{\lambda,\mu\in\mathcal{P}_n^d}$, where columns and rows are ordered by the inverse lexicographic ordering[*31]. Since this ordering is a total ordering and respects the dominance ordering, A is an upper triangular matrix[*32]with integer coefficients and 1's on the diagonal. In particular A^{-1} exists and is also an upper triangular matrix with integer coefficients and 1's on the diagonal.

Then the vectors $e := (e_{{}^t\lambda})_{\lambda\in\mathcal{P}_n^d}$ and $m := (m_\lambda)_{\lambda\in\mathcal{P}_n^d}$ are related by $e = Am$. So $m = A^{-1}e$ and

$$m_\lambda \;=\; e_{{}^t\lambda} + \sum_{\mu<\lambda} b_{\lambda,\mu} m_{{}^t\mu}, \quad b_{\lambda,\mu} \in \mathbb{Z}.$$

Since $\{m_\lambda\}$ is a basis of Λ_n^d, we find that $\{e_\lambda\}$ is also a basis of Λ_n^d. $\square$

[*31] 逆辞書式順序

[*32] 上三角行列

Exercise 1.4 $(**)$**.** The r-th **completely homogeneous symmetric polynomial** [*33] $h_r \in \Lambda_n^r$ is defined to be

$$h_r(x) := \sum_{1 \le i_1 \le i_2 \le \cdots \le i_r \le n} x_{i_1} x_{i_2} \cdots x_{i_r}.$$

For $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathcal{P}_n^d$, we set

$$h_\lambda := h_{\lambda_1} \cdots h_{\lambda_n} \in \Lambda_n^d.$$

(1) Check the equality $h_d(x) = \sum_{\lambda \in \mathcal{P}^d} m_\lambda(x)$.

(2) Show that $\{h_\lambda \mid \lambda \in \mathcal{P}_n^d\}$ is a basis of Λ_n^d.

1.4 Schur polynomials

Definition 1.11. (1) For $\mu = (\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{Z}^n$, we define an alternate function[*34] $a_\mu(x)$ by

$$a_\mu(x) := \begin{vmatrix} x_1^{\mu_1} & x_1^{\mu_2} & \cdots & x_1^{\mu_n} \\ x_2^{\mu_1} & x_2^{\mu_2} & \cdots & x_2^{\mu_n} \\ \vdots & & \ddots & \vdots \\ x_n^{\mu_1} & x_n^{\mu_2} & \cdots & x_n^{\mu_n} \end{vmatrix}.$$

(2) For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathcal{P}_n$, we define the **Schur symmetric polynomial** to be

$$s_\lambda(x) := \frac{a_{\delta+\lambda}(x)}{a_\delta(x)}, \quad \delta + \lambda := (\lambda_1 + n - 1, \lambda_2 + n - 2, \ldots, \lambda_n), \delta := (n-1, n-2, \ldots, 0).$$

Example 1.12. For $n = |\lambda| \le 3$, Schur symmetric polynomials look as follows.

$$s_{(1)} = x_1,$$

$$s_{(2)} = \begin{vmatrix} x_1^3 & 1 \\ x_2^3 & 1 \end{vmatrix} \Big/ \begin{vmatrix} x_1 & 1 \\ x_2 & 1 \end{vmatrix} = x_1^2 + x_1 x_2 + x_2^2 = h_2, \qquad s_{(1^2)} = \begin{vmatrix} x_1^2 & x_1^1 \\ x_2^2 & x_1^1 \end{vmatrix} \Big/ \begin{vmatrix} x_1 & 1 \\ x_2 & 1 \end{vmatrix} = x_1 x_2 = e_2,$$

$$s_{(3)} = \begin{vmatrix} x_1^5 & x_1 & 1 \\ x_2^5 & x_2 & 1 \\ x_3^5 & x_3 & 1 \end{vmatrix} \Big/ \begin{vmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{vmatrix} = x_1^3 + x_2^3 + x_3^3 + x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 = h_3,$$

$$s_{(2,1)} = \begin{vmatrix} x_1^4 & x_1^2 & 1 \\ x_2^4 & x_2^2 & 1 \\ x_3^4 & x_3^2 & 1 \end{vmatrix} \Big/ \begin{vmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{vmatrix} = (x_1 + x_2)(x_2 + x_3)(x_3 + x_1),$$

$$s_{(1^3)} = \begin{vmatrix} x_1^3 & x_1^2 & x_1 \\ x_2^3 & x_2^2 & x_1 \\ x_3^3 & x_3^2 & x_3 \end{vmatrix} \Big/ \begin{vmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{vmatrix} = x_1 x_2 x_3 = e_3.$$

In the above calculation we used e_r (see Definition 1.2) and h_r (see Exercise 1.4).

Exercise 1.5 $(*)$**.** Check that $s_{(1^n)} = e_n$ and $s_{(n)} = h_n$ in $\mathbb{K}[x]^{S_n}$.

Proposition 1.13. The Schur symmetric polynomial s_λ given in Definition 1.11 is a symmetric polynomial for each $\lambda \in \mathcal{P}_n$.

[*33] 完全対称多項式
[*34] 交代式

Exercise 1.6 (∗∗)**.** Give a proof of Proposition 1.13. More precisely, show that for any $\lambda \in \mathcal{P}_n$

(1) $s_\lambda(x) \in \mathbb{Z}[x]$, (2) $s_\lambda(x) \in \Lambda_n = \mathbb{Z}[x]^{S_n}$.

By Proposition 1.7, one can expand s_λ in terms of m_μ.

Theorem 1.14. For any $\lambda \in \mathcal{P}_n$ we have

$$s_\lambda = m_\lambda + \sum_{\mu<\lambda} K_{\lambda,\mu} m_\mu, \quad K_{\lambda,\mu} \in \mathbb{N}.$$

Here $\mu < \lambda$ means the dominance ordering (1.5). The number $K_{\lambda,\mu}$ is called the **Kostka number**.

Example. For $|\lambda| \le 4$ we have

$$\begin{aligned}
&s_{(1)} = m_{(1)}, \\
&s_{(2)} = m_{(2)} + m_{(1^2)}, \quad s_{(1^2)} = m_{(1^2)}, \\
&s_{(3)} = m_{(3)} + m_{(2,1)} + m_{(1^3)}, \quad s_{(2,1)} = m_{(2,1)} + 2m_{(1^3)}, \quad s_{(1^3)} = m_{(1^3)}, \\
&s_{(4)} = m_{(4)} + m_{(3,1)} + m_{(2,2)} + m_{(2,1^2)} + m_{(1^4)}, \quad s_{(3,1)} = m_{(3,1)} + m_{(2^2)} + 2m_{(2,1^2)} + 3m_{(1^4)}, \\
&s_{(2^2)} = m_{(2^2)} + m_{(2,1^2)} + 2m_{(1^4)}, \quad s_{(2,1^2)} = m_{(2,1^2)} + 3m_{(1^4)}, \quad s_{(1^4)} = m_{(1^4)}.
\end{aligned}$$

We will not give a proof of Theorem 1.14. See [M95, p. 73, Chap. I §6 (6.5)] or [O06, p. 160, §9.6 系 9.35, 問 9.13] for example. The proofs of these references use the tableau formula (Theorem 1.16).

Corollary 1.15. $\{s_\lambda \mid \lambda \in \mathcal{P}_n\}$ is a basis of the $\mathbb{Z}$-module Λ_n.

Exercise 1.7 (∗)**.** Give a proof of Corollary 1.15 using Theorem 1.14.

There is an explicit formula of Schur polynomials.

Theorem 1.16. For any $\lambda \in \mathcal{P}_n$ we have

$$s_\lambda(x) := \sum_{T \in \mathrm{SSTab}(\lambda;n)} x^T.$$

This theorem is called the **tableau formula** for Schur polynomial.

Some explanations are in order. $\mathrm{SSTab}(\lambda;n)$ denotes the set of **semi-standard tableaux**[*35] of shape λ. A semi-standard tableau $T \in \mathrm{SSTab}(\lambda;n)$ is a Young diagram of λ whose boxes are numbered by $1, 2, \ldots, n$ such that in each column numbers appear increasingly, and in each row numbers appear non-decreasingly. For example, $\mathrm{SSTab}(\lambda;3)$ looks as in Figure 3. For each $T \in \mathrm{SSTab}(\lambda;n)$, we set

$$x^T := x_1^{m_1(T)} x_2^{m_2(T)} \cdots x_n^{m_n(T)},$$

where $m_i(T)$ denotes the times of the number i appearing in the tableau T.

Example. Let us check Theorem 1.16 in the case $|\lambda| = 3$. We can use Figure 3 and Example 1.12 The result is

$$\sum_{T \in \mathrm{SSTab}((3),3)} x^T = x_1^3 + x_1^2 x_2 + x_1 x_2 x_3 + x_2^3 + x_2^2 x_3 + x_2 x_3^2 + x_3^2 = s_{(3)},$$

[*35] 半標準盤

$$\sum_{T\in \mathrm{SSTab}((2,1),3)} x^T \;=\; x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + 2x_1x_2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2 \;=\; s_{(2,1)},$$
$$\sum_{T\in \mathrm{SSTab}((1^3),3)} x^T \;=\; x_1x_2x_3 \;=\; s_{(1^3)}.$$

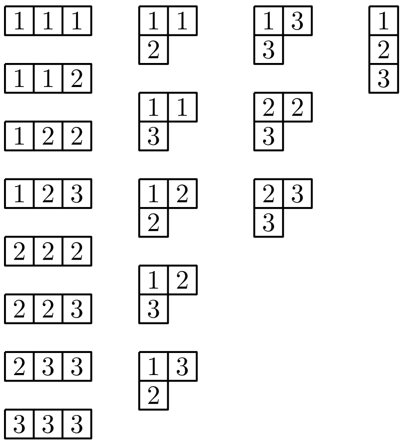


Figure 3 Semi-standard tableaux of shape λ with $|\lambda| = 3$.

We will not give a proof of Theorem 1.16. See [M95, p. 73, Chap. I §5 (5.12)] or [O06, p. 159, §9.6 定理 9.33] for example. But let us mention the following key identity.

Theorem 1.17 (The Cauchy formula)**.** For $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ we have

$$\sum_{\lambda\in\mathcal{P}_n} s_\lambda(x)s_\lambda(y) \;=\; \prod_{i,j=1}^{n} \frac{1}{1 - x_iy_j}.$$

The right hand side term $\prod_{i,j=1}^{n} \frac{1}{1-x_iy_j}$ is called **the Cauchy kernel** (function)[*36].

Exercise 1.8 ($**$)**.** Give a proof of Theorem 1.17.

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*36 Cauchy 核 (関数)