Spectral analysis of analytic functionals on certain polynomial hypergroups

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$G = \langle s_1, \cdots, s_l \rangle$: the free group of finite rank $l \geq 1$.

$G_n = \{g \in G; |g| = n\}$: the set of words of length n.

$G_1 = \{s_1^{\pm 1}, \cdots, s_l^{\pm 1}\}$ and $|G_n| = 2l(2l-1)^{n-1}$ for $n \geq 1$.

$\mathbb{C}G = \sum_{g \in G} \mathbb{C}g$: the algebraic group algebra of G ($g^* = g^{-1}$).

$C^*(G)$: the full group C*-algebra of G.

τ: the standard trace, $\tau(g) = 0$ ($g \neq e$) and $\tau(e) = 1$.

$\tau^{1/2}$: the GNS vector of τ.

The regular representation of G on $\ell^2(G)$ is identified with the GNS representation of $\mathbb{C}G$ or $C^*(G)$ with respect to τ.

$$g\tau^{1/2} = \tau^{1/2}g, \quad (g\tau^{1/2}|g'\tau^{1/2}) = \tau(g^{-1}g') = \delta_{g,g'}.$$

$C^*_{\text{red}}(G) \subset \mathcal{B}(\ell^2(G))$: the reduced group C*-algebra.

$(\mathbb{C}G)'' = C^*_{\text{red}}(G)''$: the left von Nemann algebra of G.

A function $f : G \to \mathbb{C}$ can be viewed in several ways:

- $\sum_{g \in G} f(g)g \in \mathbb{C}G$ if f is finitely supported.
- $\sum_{g \in G} f(g)g \in C^*(G)$ if $f \in \ell^1(G)$.
- A linear functional $\mathbb{C}G \to \mathbb{C}$ by

$$\sum_g x(g)g \mapsto \sum_{g \in G} f(g)x(g).$$

A function of $g \in G$ is **radial** if it is a function of the length $|g|$.

Elementary radial functions $h_n = h_n^* \in \mathbb{C}G$ are defined to be

$$h_n = \frac{1}{|G_n|} \sum_{g \in G_n} g \quad \text{for } n \geq 0,$$

which satisfy

$$h_1 h_n = \frac{1}{2l} h_{n-1} + \frac{2l-1}{2l} h_{n+1} = h_n h_1$$

and give a linear basis of a commutative *-subalgebra $\mathcal{A}$ of $\mathbb{C}G$ with $h_0 = e$ the unit element in $\mathbb{C}G$.

Here are some of known results:

- $\|h_1\| = 1$ in $C^*(G)$.
- $\|h_1\|_{\text{red}} = \sqrt{2l-1}/l \leq 1$ (J.M. Cohen).
- The spectral measure μ of h_1 relative to τ (the Plancherel measure of G) was worked out by P. Cartier (1972):

$$\mu(dt) = \frac{1}{\pi}\frac{\sqrt{2l-1-l^2t^2}}{1-t^2}\,dt,$$

which is supported by the interval

$$[-\sqrt{2l-1}/l, \sqrt{2l-1}/l] \subset [-1,1].$$

The measure appeared in a 1959 paper by H. Kesten and is referred to as the Kesten measure in the following.

A conditional expectation (unit-preserving $\mathcal{A}$-bilinear projection) $E : \mathbb{C}G \to \mathcal{A}$ is defined so that $\tau(E(x)) = \tau(x)$ $(x \in \mathbb{C}G)$:

$$E(g) = h_{|g|} \quad (g \in G).$$

If e is a projection to the radial subspace $\overline{\mathcal{A}\tau^{1/2}} \subset \ell^2(G)$,

$$exe = E(x)e \quad (x \in \mathbb{C}G)$$

and the same formula gives rise to

$$E : (\mathbb{C}G)'' \to \mathcal{A}'', \quad E(C^*_{\text{red}}(G)) = A_{\text{red}}.$$

Here $\mathcal{A}'' \subset (\mathbb{C}G)''$ is the von Neumann subalgebra generated by $\mathcal{A}$ and A_{red} is the C*-closure of $\mathcal{A}$ in $C^*_{\text{red}}(G)$.

The conditional expectation $E : \mathbb{C}G \to \mathcal{A}$ is norm-continuously extended to $C^*(G) \to A$ (U. Haagerup).
Here A is the closure of $\mathcal{A}$ in $C^*(G)$.

We have a commutative diagram of conditional expectations:

$$\begin{array}{ccccccc} \mathbb{C}G & \longrightarrow & C^*(G) & \longrightarrow & C^*_{\text{red}}(G) & \longrightarrow & (\mathbb{C}G)'' \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{A} & \longrightarrow & A & \longrightarrow & A_{\text{red}} & \longrightarrow & \mathcal{A}'' \end{array}.$$

Positive definite functions of Haagerup (1979): For $\lambda > 1$,

$$\varphi_\lambda(g) = \lambda^{-|g|} \quad (g \in G)$$

is positive definite, i.e., it gives a positive functional on $C^*(G)$.

The standard trace is the limit

$$\tau = \lim_{\lambda \to \infty} \varphi_\lambda$$

as positive linear functionals on $C^*(G)$.

Notice that $g \mapsto (-1)^{|g|} g$ induces a *-automorphism of $\mathbb{C}G$, whence φ_λ is positive definite even for $\lambda < -1$.

Problem: Describe the spectral measure of h_1 relative to φ_λ; solve the moment problem

$$\varphi_\lambda(h_1^n) = \int_{[-1,1]} t^n \, \mu(dt) \quad (n \geq 0).$$

Polynomial Hypergroups

Given a real $1 \neq r \in \mathbb{R}$, let $\mathcal{A} = \sum_{n \geq 0} \mathbb{C} h_n$ be a *-algebra with h_0 the unit element and $h_n = h_n^*$ satisfying and

$$h_1 h_n = r h_{n-1} + (1-r) h_{n+1} \quad (n \geq 1),$$

which is *-isomorphic to the polynomial algebra $\mathbb{C}[t]$ by the correspondence $h_n \leftrightarrow P_n(t)$.

Here $P_n(t)$ is a polynomial of degree n specified by the recurrence relation

$$tP_n(t) = rP_{n-1}(t) + (1-r)P_{n+1}(t)$$

with the initial condition $P_0(t) = 1$ and $P_1(t) = t$.

Generally a *-algebra with a distinguished basis $\{h_n = h_n^*\}_{n\geq 0}$ with h_0 the unit element is called a (hermitian) **hypergroup** if, in the expansion $h_j h_k = \sum_{n\geq 0} c_{j,k,n} h_n$, coefficients satisfy

$$c_{j,k,n} \geq 0, \quad \sum_n c_{j,k,n} = 1 \quad \text{and} \quad c_{j,k,0} > 0 \iff j = k.$$

For the *-algebra $\mathcal{A}$, coefficients are easily computed to see that

$$\mathcal{A} = \sum_{n\geq 0} \mathbb{C} h_n \text{ is a hypergroup} \iff 0 < r \leq \frac{1}{2}.$$

Note that $0 < r = 1/2l \leq 1/2$ for radial functions.

The hypergroup $\mathcal{A}$ is completed to a Banach *-algebra with respect to the ℓ^1-norm $\|\sum_{n\geq 0}\alpha_n h_n\|_1 = \sum_{n\geq 0}|\alpha_n|$.

Let A be the accompanied C*-algebra.
In the case of radial functions, we see that $A \subset C^*(G)$.

By the correspondence between a linear functional $\phi : \mathcal{A} \to \mathbb{C}$ and a sequence $(\phi_n = \phi(h_n))$, we can easily check the following.

1. A linear functional ϕ is ℓ^1-bounded $\iff$ (ϕ_n) is bounded.
2. A multiplicative functional $\phi \iff c = \phi(h_1) \in \mathbb{C}$.
3. A *-homomorphism $\phi \iff c = \phi(h_1) \in \mathbb{R}$.
4. A bounded *-homomorphism $\phi \iff c = \phi(h_1) \in [-1, 1]$.

The spectrum $\sigma(h_1)$ of h_1 in A is therefore $[-1, 1]$ and

$$A = C^*(h_0, h_1) \cong C([-1, 1]).$$

Analytic Functionals

Definition

A linear functional $\phi : \mathcal{A} \to \mathbb{C}$ is **analytic** if the generating function $\phi(z) = \sum_{n\geq 0} \phi_n z^n$ $(z \in \mathbb{C})$ is convergent at some $z \neq 0$.

A C*-bounded functional ϕ is analytic: $\|a\| \leq \|a\|_1$ $(a \in \mathcal{A})$, whence (ϕ_n) is bounded.
Moreover, by C*-boundedness, ϕ is represented by a Radon measure μ in $[-1, 1]$ in such a way that

$$\phi(h_n) = \int_{[-1,1]} P_n(t)\, \mu(dt).$$

Thus the problem is to find μ for a given sequence $(\phi(h_n))$.

Thanks to the recurrence relation, the generating function

$$P(z,t) = \sum_{n\geq 0} z^n P_n(t)$$

is expressed by

$$P(z,t) = \frac{1-r-rzt}{1-r-zt+rz^2}$$

and we see that

$$\begin{aligned}\phi(z) = \sum_{n\geq 0} \phi(h_n) z^n &= \int_{[-1,1]} \frac{1-r-rzt}{1-r-zt+rz^2}\,\mu(dt) \\ &= r\phi_0 + \frac{r^2z^2-(1-r)^2}{z}\int_{[-1,1]} \frac{1}{t-rz-(1-r)/z}\,\mu(dt).\end{aligned}$$

Introduce a new variable by $w = rz + (1-r)/z \iff$

$$z = \frac{w - \sqrt{w^2 - 4r(1-r)}}{2r} \sim \frac{1-r}{w} \ (w \to \infty).$$

so that

$$\int_{[-1,1]} \frac{1}{t-w}\,\mu(dt) = \frac{z}{r^2z^2 - (1-r)^2}(\phi(z) - r\phi_0).$$

Thus, finding μ is reduced to computing the inversion formula of Stieltjes transform.

Notice here that, given a finite Radon measure μ on $[-1,1]$, the left side is analytic in $1/w$ and vanishes at $w=\infty$, which in turn determines an analytic function $\phi(z)$ of z around $z=0$ by equating it with the right hand side.

Recall that the Cauchy-Stieltjes transform

$$C(w) = \int_{[-1,1]} \frac{1}{t-w}\,\mu(dt)$$

of μ is holomorphic on $\overline{\mathbb{C}} \setminus [-1,1]$ and vanishes at $w = \infty$.
The Stieltjes inversion formula: In the weak* topology, we have

$$2\pi i \mu(dt) = \lim_{\epsilon \to +0} \Big(C(t+i\epsilon) - C(t-i\epsilon) \Big)\, dt.$$

Example

When $0 < r \le 1/2$ and $\phi(z) \equiv 1$, μ is the Kesten measure:

$$C(w) = \frac{(2r-1)w + \sqrt{w^2 - 4r(1-r)}}{2r(1-w^2)}$$

$$\iff \mu(dt) = \frac{1}{2\pi r} \frac{\sqrt{4r(1-r) - t^2}}{1-t^2}\, dt.$$

Analytic functionals of geometric series

For $0 \neq \lambda \in \mathbb{C}$, let

$$\phi(z) = \frac{\lambda}{\lambda - z} = \sum_{n=0}^{\infty} \frac{z^n}{\lambda^n} \Longleftrightarrow \phi_n = \phi(h_n) = \lambda^{-n} \ (n \geq 0).$$

Example

Note that, if $\mathcal{A} \subset \mathbb{C}G$ consists of radial functions ($r = 1/2l$) and $\lambda > 1$ or $\lambda < -1$,
ϕ is the restriction of the Haagerup functional φ_λ to $\mathcal{A} \subset \mathbb{C}G$.

Let $c_r(\lambda) = r\lambda + (1-r)\frac{1}{\lambda}$ be a Joukowsky transform of λ.

Theorem

Analytic functional ϕ of geometric series is C-bounded if and only if λ falls into one of the following cases.*

1. $|\lambda| > \sqrt{(1-r)/r}$ *or* $\lambda = \pm\sqrt{(1-r)/r}$,

$$\mu_{red}(dt) = \frac{\lambda - \lambda^{-1}}{2\pi} \frac{\sqrt{4r(1-r) - t^2}}{(1-t^2)(c_r(\lambda) - t)}\, dt.$$

2. $\lambda \in \mathbb{R}$ *and* $1 \le |\lambda| < \sqrt{(1-r)/r}$
($c_r(\lambda) \in \mathbb{R}$ *and* $2\sqrt{r(1-r)} < |c_r(\lambda)| \le 1$ *then),*

$$\mu(dt) = \mu_{red}(dt) + \frac{1 - c_r(\lambda^2)}{1 - c_r(\lambda)^2}\, \delta(t - c_r(\lambda)).$$

When ϕ is C-bounded, it is positive if and only if $\pm\lambda \ge 1$.*

Comments:

(i) The Cauchy-Stieltjes transform of geometric series is

$$C(w) = \frac{\lambda - \lambda^{-1}}{2} \frac{N(w)}{(1-w^2)(r\lambda + (1-r)\lambda^{-1} - w)},$$

$$N(w) = 2(\lambda - \lambda^{-1})^{-1}(1-w^2) + (2r-1)w + \sqrt{w^2 - 4r(1-r)}.$$

(ii) H. Yoshida observed that

$$C(w) = \cfrac{1}{\lambda^{-1} - w + \cfrac{-r(1-\lambda^{-2})}{-r\lambda^{-1} - w + \cfrac{-r(1-r)}{-w + \cfrac{-r(1-r)}{-w + \ddots}}}}.$$

Applications to free group C*-algebras

Our spectral formula is compatible with known results on representations of free groups.
Recall again that $r = 1/2l$ for radial functions.

1. The spectral measure μ of h_1 relative to τ $(\phi \equiv 1)$ is supported by $[-2\sqrt{r(1-r)}, 2\sqrt{r(1-r)}]$, whence we have $\sigma_{\mathrm{red}}(h_1) = [-2\sqrt{r(1-r)}, 2\sqrt{r(1-r)}]$.
2. $A \subset C^*(G)$ and the conditional expectation $E : \mathbb{C}G \to \mathcal{A}$ is extended to a contraction $C^*(G) \to A$ because the measure support of φ_λ for various $\lambda \in \mathbb{R} \setminus (-1, 1)$ covers $\sigma_A(h_1) = [-1, 1]$.
3. φ_λ $(\pm\lambda > 1)$ splits through $C^*_{\mathrm{red}}(G)$ if and only if $|\lambda| \geq \sqrt{(1-r)/r}$.

References

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S. Yamagami and H. Yoshida, On shifting semicircular roots, arXiv:2209.04585.