G-Hilb for trihedral groups

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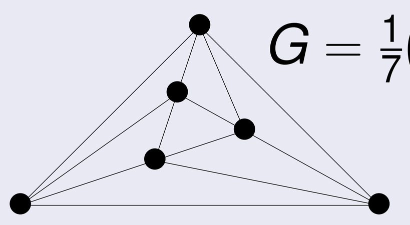
Setting

The McKay correspondence originally classified the finite subgroups $G \subset \mathrm{SL}_2(\mathbb{C})$ in terms of simply-laced Dynkin diagrams, where the Dynkin type can be extracted from geometric or representation theoretic data associated to G. To elaborate on the geometry, take the quotient singularity $\mathbb{A}^2/G$ and resolve it via blowups. By virtue of G lying in $\mathrm{SL}_2(\mathbb{C})$ this produces a 'crepant' resolution $\varphi : Y \to \mathbb{A}^2/G$; that is, $\varphi^* K_{\mathbb{A}^2/G} = K_Y$. For surfaces, crepant resolutions are unique, and this Y turns out to have a natural interpretation as the moduli space of 'G-clusters': scheme-theoretic group orbits of G acting on $\mathbb{A}^2$, which is called G-Hilb $\mathbb{A}^2$. In particular, a G-cluster Z must have $H^0(\mathcal{O}_Z) \cong \mathbb{C}[G]$ as G-modules. When G is abelian, G-Hilb $\mathbb{A}^2$ is just the Hirzebruch-Jung toric resolution. Asides from being crepant, G-Hilb $\mathbb{A}^2$ comes with compatibilities that make it preferred: there's a bijection between irreducible representations of G and a natural basis for $H^*(G\text{-Hilb}\,\mathbb{A}^2)$.
In $\mathrm{SL}_3(\mathbb{C})$ there's no concise classification of finite subgroups. However, the McKay correspondence is categorified in the following result...
Theorem (Bridgeland-King-Reid): Suppose $n \leq 3$ and $G \subset \mathrm{SL}_n(\mathbb{C})$. Then the Hilbert-Chow morphism $G\text{-Hilb}\,\mathbb{A}^n \to \mathbb{A}^n/G$ is a resolution, and there is an equivalence between the derived categories $D^b G\text{-Hilb}\,\mathbb{A}^n \simeq D^b_G \mathbb{A}^n$.
The equivalence is given by the Fourier-Mukai transform. This recovers the classical McKay correspondence by taking K-theory on both sides: $H^*(G\text{-Hilb}\,\mathbb{A}^n) \cong K_0 D^b(G\text{-Hilb}\,\mathbb{A}^n) \cong K_0 D^b_G(\mathbb{A}^n) \cong \mathrm{Rep}(G)$. The first isomorphism is the Chern character, the last follows from the Quillen-Suslin theorem. **One remaining problem is to actually compute and study G-Hilb $\mathbb{A}^3$ for many $G \subset \mathrm{SL}_3(\mathbb{C})$.**

The abelian case



$G = \frac{1}{7}(1,2,4)$

When G is abelian, its action can be diagonalised and so it commutes with the torus action on $\mathbb{A}^3$. Consequently, $\mathbb{A}^3/G$ is toric. Nakamura and Craw-Reid produced an algorithm to run the appropriate sequence of toric blowups to compute G-Hilb $\mathbb{A}^3$ as illustrated for $G = \frac{1}{7}(1,2,4)$.

G-graphs

Toric geometry enables a global computation of G-Hilb $\mathbb{A}^3$ in the abelian case. In general, only a local computation is possible: that is, one can compute an open affine cover of G-Hilb $\mathbb{A}^3$ with transition functions. There is a 'standard' affine cover in the abelian situation as follows. If Z is a G-cluster, $H^0(\mathcal{O}_Z)$ admits a monomial basis of eigenfunctions, or a G-graph. Fixing such a basis Γ gives an affine piece U_Γ of G-Hilb $\mathbb{A}^3$ by considering all G-clusters for which $H^0(\mathcal{O}_Z) = \mathbb{C}[x,y,z]/I_Z$ is based by Γ. It's easy to see from an example why this is affine...

$$\begin{array}{cccc} 1 & x & x^2 & x^3 \\ z & xz & x^2z & \end{array}$$

...if $G = \frac{1}{7}(1,2,4)$, then a G-graph Γ is shown above. Ideals I for which $\mathbb{C}[x,y,z]/I$ is based by Γ are those generated by relations of the form $x^4 = \alpha z, y = \beta x^2, z^2 = \gamma x, x^3 z = \alpha\gamma, xyz = \alpha\beta\gamma$ from matching characters (as I is G-invariant). The choices of α, β, γ produce an affine piece of G-Hilb $\mathbb{A}^3$ isomorphic to $\mathbb{A}^3$, which is smooth as expected. In fact, all standard affine pieces will be isomorphic to $\mathbb{A}^3$ as they arise from smooth simplicial cones in the Craw-Reid triangulation. Another way of viewing a G-graph is as the set of nonzero monomials in the toric zero-stratum or origin of the relevant affine piece. This paraphrasal breaks down in the nonabelian case.

Trihedral groups

While the abelian subgroups of $\mathrm{SL}_2(\mathbb{C})$ correspond to Dynkin types A_n the subgroups for D_n are binary dihedral groups for which a similar treatment to the abelian case was performed by Nolla de Celis. The analog in three dimensions is 'trihedral groups': $G = A \rtimes T$ where $A \subset \mathrm{SL}_3(\mathbb{C})$ is abelian and $T = \langle \tau \rangle$ with $\tau : x \mapsto y \mapsto z \mapsto x$ in coordinates. When A is cyclic, it must be of the form $\frac{1}{r}(1,s,s^2)$ with $1 + s + s^2 \equiv 0 \bmod r$. By Clifford theory, the irreducible representations of G are either three dimensional induced representations $\rho_a := \mathrm{Ind}_A^G \psi_a$ from one dimensional representations of A, or one dimensional representations L_i lifted from $T = G/A$ or coming from fixed points of the twisting T-action on $\mathrm{Irr}\,A$ if they exist (which is precisely when $3 \mid r$). This action of T on $\mathrm{Irr}(A) = A^\vee \cong \mathbb{Z}/r$ when $A = \frac{1}{r}(1,s,s^2) = \langle \sigma \rangle$ is cyclic is given by $\tau : a \mapsto sa$. Thus, as A-modules $\rho_a \downarrow_A = \psi_a \oplus \psi_{sa} \oplus \psi_{s^2 a}$ and so $\rho_a = \rho_{sa} = \rho_{s^2 a}$. One has $\mathbb{C}[G] = \bigoplus 3\rho_a \oplus \bigoplus L_i$ for suitable indices a, i. Throughout, $G = A \rtimes T$ will be a trihedral group. Finding an analog of G-graphs by which to study G-Hilb $\mathbb{A}^3$ requires some structural results.

References

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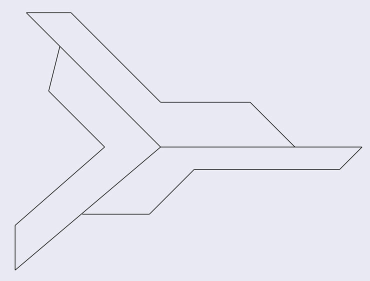
Trihedral boats

Lemma: Every G-cluster Z has a 'quasimonomial' basis for its global sections $H^0(\mathcal{O}_Z)$.
Quasimonomial means mostly monomial, except inside certain one dimensional representations. It's easier to see in a representative example for $G = \frac{1}{13}(1,3,9) \rtimes T$...

$$\begin{array}{ccccccccccc} 1 & & x & & x^2 & & x^3 & & x^4 & & x^5 \\ & xz & & x^2z & & x^3z & & (x^4z) & & [x^5z] & \\ xz^2 & & x^2z^2 & & x^3z^2 & & x^4z^2 & & & & \end{array}$$



By the symmetry of the situation, the Newton polygon of a basis has trihedral symmetry. I have illustrated only a third of it; the rest is obtained by rotating the figure by $2\pi/3$ and $4\pi/3$. There are three behaviours exhibited. For any monomial m_1, say of character a, except $1, x^4z, x^4z^2, x^5z$ one obtains three basic monomials $m_1, \tau m_1, \tau^2 m_1$ that together span a copy of ρ_a in $H^0(\mathcal{O}_Z)$. But one can check that every A-character is represented by some monomial in the figure. Hence, it also contains monomials m_2 and m_3 with characters sa and s^2a, and so one obtains three copies of ρ_a as required in $\mathbb{C}[G]$. The one dimensional representations are $L_i : \sigma \mapsto 1, \tau \mapsto \omega^i$ for $i = 0,1,2$ where ω is a primitive cube root of unity. These come from the invariant monomials 1, which spans L_0, and $x^4z \rightsquigarrow x^4z + \omega y^4x + \omega^2 z^4 y$ and $x^4z + \omega^2 y^4 x + \omega z^4 y$ that span L_1 and L_2 respectively. *Quasimonomial* means allowing these trinomials in place of A-invariant monomials. Lastly, there is 'twinning': either of x^4z^2 or x^5z can be chosen to be basic, but not both due to the relation coming from x^4z. Choosing x^4z^2 has the benefit that every nontrivial A-character is represented exactly once. With these conventions, such a fundamental domain for the T-action on the Newton polygon of $H^0(\mathcal{O}_Z)$ in which every monomial except 1 is divisible by x is called a *trihedral boat* for Z.
Theorem/Conjecture: Every G-cluster has a trihedral boat.
This is a theorem when $r = 1 + s + s^2$. The power of the Nakamura/Craw-Reid approach is that it describes *all* standard affine pieces of G-Hilb $\mathbb{A}^3$ for *all* abelian G in a uniform way. There is no known classification result for trihedral boats, though Reid and I have progress and conjectures in this direction. In particular, it is straightforward to algorithmically compute all boats for a given G, as well as any anomalous clusters that don't admit a boat. No exceptions have been found so far. The partial classification strongly suggests a description of the locus where the iterated Hilbert scheme T-Hilb A-Hilb $\mathbb{A}^3$ is isomorphic to G-Hilb $\mathbb{A}^3$.
A feature of the nonabelian case is that many manipulations with monomials 'don't converge' in that the exponents appearing after applying relations increase rather than decrease to reduce to basic monomials. Representations of quivers turn out to be a better way to package the computations and allow one to explicitly compute the standard affine pieces, which will *not* always be $\mathbb{A}^3$.

Acknowledgments

This research was partly funded by an URSS bursary provided by the University of Warwick. I am grateful for the generous supervision of Prof. Miles Reid, as well as for much input from many others.