NONCOMMUTATIVE KNÖRRER PERIODICITY
FOR CYCLIC QUOTIENT SURFACE SINGULARITIES

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ABSTRACT

Knörrer periodicity produces an equivalence between the singularity categories of Gorenstein cyclic quotient surface singularities and commutative finite dimensional algebras $\mathbb{C}[z]/(z^n)$.
We produce an equivalence between the singularity categories of arbitrary cyclic quotient surface singularities and certain (not necessarily commutative) finite dimensional algebras.

KNÖRRER PERIODICITY

Buchweitz introduced the singularity category in [2].

Definition. Suppose R is a Noetherian $\mathbb{C}$-algebra.

- The *singularity category* is the triangulated category defined by the Verdier quotient
$$D_{sg}(R) := \frac{D^b(R)}{\mathrm{Perf}(R)}.$$

He showed the singularity category generalises:

- The category of matrix factorisations for hypersurface rings.
- The stable category of maximal Cohen-Macaulay modules (MCMs) for Gorenstein rings.

Buchweitz also restated Knörrer periodicity.

Theorem ([3], [2], Knörrer Periodicity). *Suppose*

- $K = \mathbb{C}[z_1, \dots, z_n]/(f)$
- $R = \mathbb{C}[x, y, z_1, \dots, z_n]/(x^2 + y^2 + f)$.

Then there is an equivalence
$$D_{sg}(R) \cong D_{sg}(K)$$
of singularity categories.

Related equivalences for NC resolutions have been studied using relative singularity categories.

Definition. Suppose Λ is a Noetherian $\mathbb{C}$-algebra and P is a projective Λ-module.

- The *relative singularity category* is the triangulated category defined by (the idempotent completion of) the Verdier quotient
$$\Delta_P(\Lambda) := \frac{D^b(\Lambda)}{\langle P \rangle}.$$

Theorem ([4]). *Suppose that R, K are MCM-representation finite, complete, Gorenstein, $\mathbb{C}$-algebras. TFAE*

- *There is an equivalence $D_{sg}(R) \cong D_{sg}(K)$ of singularity categories.*
- *There is an equivalence $\Delta_R(A) \cong \Delta_K(\Lambda)$ of relative singularity categories.*

where A and Λ are the Auslander algebra NC resolutions of R and K respectively.

As (relative) singularity categories are defined for non-Gorenstein rings it is natural to ask whether similar equivalences of triangulated categories exist for non-Gorenstein examples.
Cyclic quotient surface singularities include non-Gorenstein examples, and we will consider equivalences analogous to Knörrer periodicity between general cyclic quotient surface singularities and finite dimensional algebras.

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NONCOMMUTATIVE KNÖRRER PERIODICITY

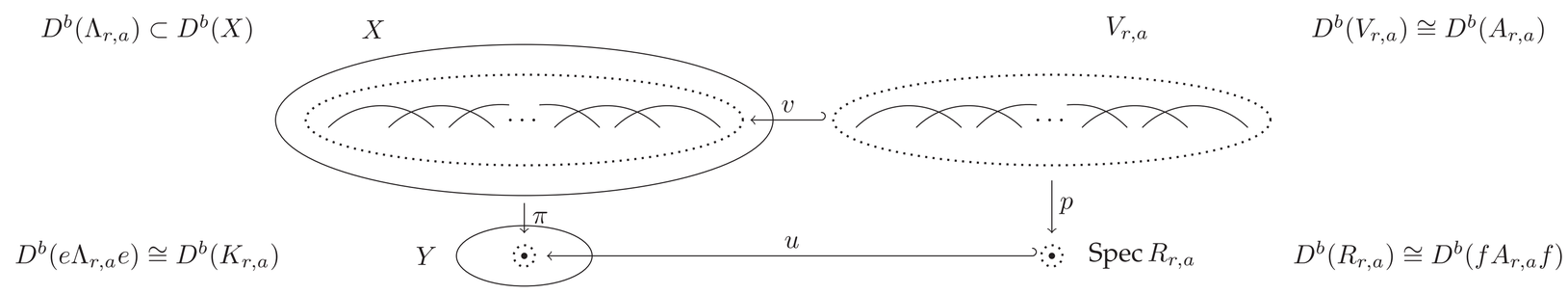
Suppose that $0 < a < r$ are coprime integers and $p : V_{r,a} \to \operatorname{Spec} R_{r,a}$ is the minimal resolution of a type $\frac{1}{r}(1, a)$ cyclic quotient surface singularity.

Theorem ([1]). *There is a finite dimensional algebra $\Lambda_{r,a}$ and idempotent $e \in \Lambda_{r,a}$ such that there is an equivalence*
$$\Delta_{R_{r,a}}(V_{r,a}) := \frac{D^b(V_{r,a})}{\langle \mathcal{O}_{V_{r,a}} \rangle} \cong \frac{D^b(\Lambda_{r,a})}{\langle e\Lambda_{r,a} \rangle} =: \Delta_{e\Lambda_{r,a}}(\Lambda_{r,a})$$
of relative singularity categories. The algebra $\Lambda_{r,a}$ is quasi-hereditary and global dimension 2.

Theorem ([1]). *For the finite dimensional algebra $K_{r,a} := e\Lambda_{r,a}e$ there is an equivalence*
$$D_{sg}(R_{r,a}) \cong D_{sg}(K_{r,a}).$$
of singularity categories. We call the algebra $K_{r,a}$ the Knörrer invariant algebra.

OUTLINE OF PROOF

The strategy of the proof is to construct an equivalence of relative singularity categories between (NC) resolutions that induces an equivalence of singularity categories. We sketch an outline the main steps below.



- Consider $\pi : X \to Y$, the contraction of a type A_n collection of curves C_i in a smooth, projective variety X such that $C_i \cong \mathbb{P}^1$ and $C_i \cdot C_i = -\alpha_i \leq -2$.
- Also consider $p : V \to \operatorname{Spec} R$, the complete local, affine neighbourhood of the contraction.
- $R \cong R_{r,a}$ is a cyclic quotient surface singularity such that $\frac{r}{a} = [\alpha_1, \dots, \alpha_n]$ and $V \cong V_{r,a}$ is its minimal resolution.
- The minimal resolution $V_{r,a}$ is derived equivalent to the reconstruction algebra $A_{r,a}$ of Wemyss [5].
- There is an idempotent $f \in A_{r,a}$ such that $fA_{r,a}f \cong R_{r,a}$ and $\Delta_{fA_{r,a}}(A_{r,a}) \cong \Delta_{R_{r,a}}(V_{r,a})$.
- The full thick subcategory generated by line bundles $\langle \mathcal{O}, \mathcal{O}(-C_1), \mathcal{O}(-C_1 - C_2), \dots, \mathcal{O}(-C_1 \cdots - C_n) \rangle \subset D^b(X)$ was studied by Hille and Ploog and is derived equivalent to a finite dimensional algebra $\Lambda_{r,a}$.
- There is a line bundle $L \in \operatorname{Coh} V_{r,a}$ and idempotent $e \in \Lambda_{r,a}$ such that the functor
$$F : D^b(X) \to D^b(V_{r,a});\ \mathcal{E} \mapsto \mathbf{L}v^*\mathcal{E} \otimes L$$
descends to a functor $F : \Delta_{e\Lambda_{R,a}}(\Lambda_{r,a}) \to \Delta_{fA_{r,a}}(A_{r,a})$ between relative singularity categories.
- This functor can be checked to be an equivalence of relative singularity categories $\Delta_{e\Lambda_{r,a}}(\Lambda_{r,a}) \cong \Delta_{fA_{r,a}}(A_{r,a})$.
- The equivalence of relative singularity categories descends to an equivalence of singularity categories
$$D_{sg}(e\Lambda_{r,a}e) \cong D_{sg}(fA_{r,a}f)$$
via methods of Kalck and Yang [4].
- We define $K_{r,a} := e\Lambda_{r,a}e$, and hence there is an equivalence of singularity categories $D_{sg}(R) \cong D_{sg}(K)$.

CYCLIC QUOTIENT SURFACE SINGULARITIES

For $0 < a < r$ coprime integers define

- The finite group $\frac{1}{r}(1, a) := \left\langle \begin{pmatrix} \varepsilon_r & 0 \\ 0 & \varepsilon_r^a \end{pmatrix} \right\rangle < \mathrm{GL}_2(\mathbb{C})$ where ε_r is a primitive r^{th} root of unity.
- The invariant algebra $R_{r,a} := \mathbb{C}[x, y]^{\frac{1}{r}(1,a)}$.
- The Hirzebruch-Jung (HJ) continued fraction expansion
$$\frac{r}{a} = \alpha_1 - \cfrac{1}{\alpha_2 - \cfrac{1}{\dots - \cfrac{1}{\alpha_n}}} = [\alpha_1, \dots, \alpha_n].$$

$\operatorname{Spec} R_{r,a}$ is a cyclic quotient surface singularity:

- It has a minimal resolution $p : V_{r,a} \to \operatorname{Spec} R_{r,a}$.
- The exceptional divisor $E = \bigcup_{i=1}^n C_i$ is a type A_n configuration of curves $C_i \cong \mathbb{P}^1$.
- $C_i \cdot C_i := -\alpha_i$ for $\frac{r}{a} = [\alpha_1, \dots, \alpha_n]$.
- The invariant algebra is Gorenstein if and only if $a = r - 1$ and hence $\alpha_i = 2$ for all i.
- For HJ fraction $\frac{r}{r-a} = [\beta_1, \dots, \beta_l]$, $R_{r,a} = \dfrac{\mathbb{C}[z_0, \dots, z_{l+1}]}{\langle z_{i-1}z_{k+1} - z_i \left(\prod_{i \leq j \leq k} z_j^{(\beta_j - 2)} \right) z_k : 1 \leq i \leq k \leq l \rangle}$.

KNÖRRER INVARIANT ALGEBRAS

For $0 < a < r$ coprime integers with Hirzebruch-Jung continued fractions $\frac{r}{r-a} = [\beta_1, \dots, \beta_l]$ and $\frac{r}{a} = [\alpha_1, \dots, \alpha_n]$ consider the algebra $K_{r,a}$.

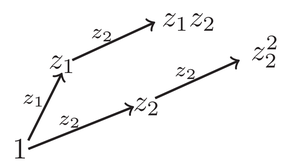
Proposition. *The algebra $K_{r,a}$ can be presented as the monomial algebra*
$$K_{r,a} := \frac{\mathbb{C}\langle z_1, \dots, z_l \rangle}{\langle z_i \left(\prod_{i \leq j \leq k} z_j^{(\beta_j - 2)} \right) z_k \mid \text{for } 1 \leq i, k \leq l \rangle}.$$

Further:

- *The dimension of $K_{r,a}$ is r.*
- *As right $K_{r,a}$-modules there are $n+1$ isomorphism classes of indecomposable ideals $I_i \lhd K_{r,a}$.*
- *The largest of these indecomposable ideals has dimension a.*
- *There is an isomorphism $\Lambda_{r,a} \cong \operatorname{End}_{K_{r,a}}(\bigoplus_{i=0}^n I_i)$.*

Examples.

- $K_{n+1,n} = \frac{\mathbb{C}[z]}{(z^{n+1})}$, which occurs in Knörrer periodicity.
- $K_{n+1,1} = \frac{\mathbb{C}\langle z_1, \dots z_n \rangle}{(z_1, \dots, z_n)^2}$ is the radical square zero algebra.
- $K_{5,2} = \frac{\mathbb{C}\langle z_1, z_2 \rangle}{(z_1^2, z_2^3, z_1 z_2^2, z_2 z_1)}$ is 5 dimensional and right monomial multiplication can be presented by the diagram



- $K_{17,7}$ is 17 dimensional and right monomial multiplication can be presented by the diagram