Hörmander's L^2-estimate and the Ohsawa-Takegoshi extension theorem

Bo-Yong Chen

Young mathematician workshop on SCV
July 24-July 27, 2013, Nagoya

Hörmander's L^2-estimate

We start with the following fundamental

Hörmander's Theorem.

Let Ω *be a pseudoconvex domain in* $\mathbb{C}^n$ *and* φ *a smooth strictly psh function on* Ω. *Then for any* $\bar{\partial}-$*closed* $(0,1)$ *form* v *with* $\int_\Omega |v|^2_{i\partial\bar{\partial}\varphi} e^{-\varphi} < \infty$, $\exists$ *solution* u *to* $\bar{\partial} u = v$ *s.t.*

$$\int_\Omega |u|^2 e^{-\varphi} \leq \int_\Omega |v|^2_{i\partial\bar{\partial}\varphi} e^{-\varphi}. \tag{0.1}$$

Hörmander's L^2-estimate

Notations:

$$|v|^2_{i\partial\bar\partial\varphi} = \sum \varphi^{j,k} v_j \bar{v}_k$$

where $v = \sum v_j d\bar{z}_j$ and $(\varphi^{j,k})$ is the matrix inverse to $(\partial^2\varphi/\partial z_j \partial\bar{z}_k)$.

Equivalently,

$$|v|^2_{i\partial\bar\partial\varphi} = \sup\{|\langle v, X\rangle|^2 : X \in T^{0,1}(\Omega), |X|_{i\partial\bar\partial\varphi} \leq 1\}$$

where $|X|^2_{i\partial\bar\partial\varphi} = \sum \frac{\partial^2\varphi}{\partial z_j \partial\bar{z}_k} X_j \bar{X}_k$. In particular,

$$i\partial\bar\partial\varphi \geq i\partial\varphi \wedge \bar\partial\varphi \Rightarrow |\bar\partial\varphi|_{i\partial\bar\partial\varphi} \leq 1.$$

Hörmander's L^2-estimate

The proof of Hörmander's theorem is based on

Morrey-Kohn-Hörmander (MKH) Formula.

Let $\Omega = \{\rho < 0\}$, $\rho : C^2$ *defining function,* $u \in C^1_{(0,1)}(\overline{\Omega})$ *s.t.* $\partial\rho \cdot u = 0$ *on* $\partial\Omega$, $\varphi \in C^2(\overline{\Omega}, \mathbb{R})$ $\Rightarrow$

$$\begin{aligned} \|\bar\partial u\|^2_\varphi + \|\bar\partial^*_\varphi u\|^2_\varphi &= \int_{\partial\Omega} \sum_{j,k=1}^n \frac{\partial^2 \rho}{\partial z_j \partial \bar z_k} u_j \bar u_k e^{-\varphi} \frac{dS}{|\nabla \rho|} \\ &+ \sum_{j=1}^n \int_\Omega \left| \frac{\partial u_j}{\partial \bar z_j} \right|^2 e^{-\varphi} \\ &+ \int_\Omega \sum_{j,k=1}^n \frac{\partial^2 \varphi}{\partial z_j \partial \bar z_k} u_j \bar u_k e^{-\varphi}. \end{aligned}$$

Donnelly-Fefferman type estimate

Ohsawa and Berndtsson generalized the original L^2 estimate of Donnelly-Fefferman as follows

Donnelly-Fefferman Type Theorem. *Suppose furthermore* $\exists\ \psi : C^2$ *psh on* Ω*, and* $0 < \alpha < 1$ *s.t.* $i\alpha\partial\bar{\partial}\psi \geq i\partial\psi \wedge \bar{\partial}\psi$*, then the* $L^2(\Omega, \varphi)-$*minimal solution of* $\bar{\partial}u = v$ *satisfies*

$$\int_\Omega |u|^2 e^{\psi-\varphi} \leq C_\alpha \int_\Omega |v|^2_{i\partial\bar{\partial}(\varphi+\psi)} e^{\psi-\varphi} \qquad (0.2)$$

provided the right-hand side is finite.

Donnelly-Fefferman type estimate

There are two important candidates for the Donnelly-Fefferman condition:

a) Ω is hyperconvex, i.e., $\exists \rho \in PSH^{-}(\Omega)$ s.t. $\{\rho < c\}$ is relatively compact in Ω for each $c < 0$,

$$\psi := -\alpha \log(-\rho);$$

b) $f_j \in \mathscr{O}(\Omega)$, $\sum_j |f_j|^2 < 1$,

$$\psi := -\alpha \log \left(-\log \sum |f_j|^2 \right).$$

Berndtsson-Charpentier '00 observed that *Hörmander* $\Rightarrow$ *Donnelly-Fefferman*.

Ohsawa-Takegoshi L^2 extension theorem

One of the deepest result in complex analysis is the following

Ohsawa-Takegoshi Extension Theorem. $\Omega \subset \mathbb{C}^n$ *pseudoconvex with* $\sup_\Omega |z_n|^2 < 1$.
$\Rightarrow \exists$ *constant* $C_n > 0$ *s.t.* $\forall \varphi \in PSH(\Omega)$, $\forall f$ *holomorphic on* $\Omega' := \Omega \cap \{z_n = 0\}$ *with*

$$\int_{\Omega'} |f|^2 e^{-\varphi} < \infty,$$

$\exists$ *holomorphic extension* F *of* f *to* Ω *s.t.*

$$\int_\Omega |F|^2 e^{-\varphi} \leq C_n \int_{\Omega'} |f|^2 e^{-\varphi}.$$

Ohsawa-Takegoshi L^2 extension theorem

The original proof relies on

Twisted MKH Formula.

Let ρ, u, φ *be as in MKH formula, and* $\eta \in C^2(\overline{\Omega}, \mathbb{R}_+)$. *Then*

$$\begin{aligned}
&\|\sqrt{\eta}\bar\partial u\|^2_\varphi + \|\sqrt{\eta}\bar\partial^*_\varphi u\|^2_\varphi \\
= \ & \sum_{j,k=1}^n \int_{\partial\Omega} \eta \frac{\partial^2 \rho}{\partial z_j \partial \bar z_k} u_j \bar u_k e^{-\varphi} \frac{d\sigma}{|\nabla \rho|} \\
+ \ & \sum_{j,k=1}^n \int_{\Omega} \left(\eta \frac{\partial^2 \varphi}{\partial z_j \partial \bar z_k} - \frac{\partial^2 \eta}{\partial z_j \partial \bar z_k} \right) u_j \bar u_k e^{-\varphi} \\
+ \ & \sum_{j=1}^n \int_{\Omega} \eta \left| \frac{\partial u_j}{\partial \bar z_j} \right|^2 e^{-\varphi} + 2\mathrm{Re} \int_{\Omega} (\partial \eta \cdot u) \overline{\bar\partial^*_\varphi u} e^{-\varphi}.
\end{aligned}$$

Hörmander ⇒ Ohsawa-Takegoshi

The original extension theorem was generalized by many people, in particular, Blocki obtained the optimal constant in the L^2 extension theorem.

Here I would like to present a proof based only on Hörmander's theorem.

1. A standard reduction: It suffices to show that there is a holomorphic extension F of f to Ω s.t.

$$\int_\Omega |F|^2 e^{-\varphi} \leq C_n \int_V |f|^2 e^{-\varphi}$$

under the stronger hypothesis that Ω is bounded with $\partial\Omega \in C^\infty$, φ is C^∞ strictly psh on $\overline{\Omega}$, f is holomorphic in an open neighborhood V of $\overline{\Omega'}$ in $\{z_n = 0\}$.

Hörmander $\Rightarrow$ Ohsawa-Takegoshi

To see this, choose an increasing sequence of bounded smooth pseudoconvex domains Ω_j with $\cup\Omega_j = \Omega$, and take a sequence of smooth strictly psh functions φ_j on Ω with $\varphi_j \downarrow \varphi$ (note that Ω is pseudoconvex).

For each j, there is a holomorphic function F_j on Ω_j s.t. $F_j = f$ on $\Omega'_j := \Omega_j \cap \{z_n = 0\}$ and

$$\int_{\Omega_j} |F_j|^2 e^{-\varphi_j} \leq C_n \int_{\Omega'_{j+1}} |f|^2 e^{-\varphi_j} \leq C_n \int_{\Omega'} |f|^2 e^{-\varphi}.$$

Thus $\{F_j\}$ is a normal family and we only need to take a limit of $\{F_j\}$.

Hörmander $\Rightarrow$ Ohsawa-Takegoshi

2. Put

$$\phi = \varphi + \log |z_n|^2,\ v_\varepsilon = f\bar{\partial}\chi(|z_n|^2/\epsilon^2)$$

where $\varepsilon \ll 1$ and χ is a cut-off function s.t. $\chi|_{(-\infty,1/2)} = 1$ and $\chi|_{(1,\infty)} = 0$. Note that v_ε is a C^∞ $(0,1)$ form in a neighborhood of $\overline{\Omega}$ provided ε small enough.

By Hörmander's theorem, there is a solution of $\bar{\partial}u = v_\varepsilon$ on $\hat{\Omega} := \Omega \backslash \{z_n = 0\}$ verifying

$$\int_{\hat{\Omega}} |u|^2 e^{-\phi} \leq \int_{\hat{\Omega}} |v_\varepsilon|^2_{i\partial\bar{\partial}\phi} e^{-\phi}.$$

Unfortunately, the RHS depends on ε!

3. To overcome this difficulty, we use a trick goes back to Berndtsson and Charpentier '00 as follows: without loss of generality, we may assume that u is the $L^2(\hat{\Omega}, \phi)-$minimal solution of $\bar{\partial} u = v_\varepsilon$, i.e.,

$$u \perp \mathrm{Ker}\, \bar{\partial} \quad \text{in} \quad L^2(\hat{\Omega}, \phi).$$

Thus for any *bounded* smooth psh function ψ on $\hat{\Omega}$,

$$u e^{\psi} \perp \mathrm{Ker}\, \bar{\partial} \quad \text{in} \quad L^2(\hat{\Omega}, \phi + \psi) = L^2(\hat{\Omega}, \phi),$$

i.e., ue^{ψ} is $L^2(\hat{\Omega}, \phi + \psi)-$minimal solution of

$$\bar{\partial} \tilde{u} = \tilde{v} := \bar{\partial}(u e^{\psi}).$$

Hörmander $\Rightarrow$ Ohsawa-Takegoshi

By Hörmander's theorem again, we have

$$\int_{\hat{\Omega}} |\tilde{u}|^2 e^{-\phi-\psi} \leq \int_{\hat{\Omega}} |\tilde{v}|^2_{i\partial\bar{\partial}(\phi+\psi)} e^{-\phi-\psi},$$

that is

$$\begin{aligned} \int_{\hat{\Omega}} |u|^2 e^{\psi-\phi} &\leq \int_{\hat{\Omega}} |v_\varepsilon + \bar{\partial}\psi \wedge u|^2_{i\partial\bar{\partial}(\phi+\psi)} e^{\psi-\phi} \\ &\leq (1+1/r) \int_{\hat{\Omega}} |v_\varepsilon|^2_{i\partial\bar{\partial}(\phi+\psi)} e^{\psi-\phi} \\ &\quad + \int_{\hat{\Omega}} |\bar{\partial}\psi|^2_{i\partial\bar{\partial}(\phi+\psi)} |u|^2 e^{\psi-\phi} \\ &\quad + r \int_{\operatorname{supp} v_\varepsilon} |\bar{\partial}\psi|^2_{i\partial\bar{\partial}(\phi+\psi)} |u|^2 e^{\psi-\phi}. \end{aligned}$$

in view of Schwarz's inequality.

Hörmander $\Rightarrow$ Ohsawa-Takegoshi

How to choose ψ?

Donnelly-Fefferman type estimate suggests to look at

$$\psi = -\alpha \log(-\log(|z_n|^2 + \varepsilon^2))$$

with $\alpha \leq 1$. In order that the first term in RHS of the previous inequality is bounded independent of ε, one has to choose $\alpha = 1$. But then the role of the LHS of previous inequality would be canceled by the second term of RHS!

Thus it is reasonable to add some psh function to ψ with slower growth.

Hörmander $\Rightarrow$ Ohsawa-Takegoshi

4. Put

$$\rho = \log(|z_n|^2 + \varepsilon^2)$$

$$\eta = -\rho + \log(-\rho)$$

$$\psi = -\log \eta.$$

A direct calculation shows

$$i\partial\bar{\partial}\psi \geq \left(\frac{1}{\eta^2} + \frac{1}{\eta(-\rho+1)^2}\right) i\partial\eta \wedge \bar{\partial}\eta + \frac{i\partial\bar{\partial}\rho}{\eta}.$$

Since

$$\bar{\partial}\psi = -\frac{\bar{\partial}\eta}{\eta} = \frac{1}{\eta}\left(1 - \frac{1}{\rho}\right)\bar{\partial}\rho,$$

it follows that

Hörmander $\Rightarrow$ Ohsawa-Takegoshi

$$|\bar\partial\psi|^2_{i\partial\bar\partial(\phi+\psi)} \leq \frac{1}{1+\frac{\eta}{(-\rho+1)^2}}$$

$$i\partial\bar\partial\psi \geq \frac{i\partial\bar\partial\rho}{\eta} = \frac{\epsilon^2 i dz_n d\bar z_n}{\eta(|z_n|^2+\epsilon^2)^2}$$

and on $\operatorname{supp} v_\varepsilon \subset \{\varepsilon^2/2 \leq |z_n|^2 \leq \varepsilon^2\}$,

$$|\bar\partial\psi|^2_{i\partial\bar\partial(\phi+\psi)} \leq \frac{1}{\eta^2}\left(1-\frac{1}{\rho}\right)^2 |\bar\partial\rho|^2_{i\partial\bar\partial\psi}$$

$$\leq \frac{1}{\eta^2}\left(1-\frac{1}{\rho}\right)^2 \frac{\eta(|z_n|^2+\varepsilon^2)^2}{\varepsilon^2}\frac{|z_n|^2}{(|z_n|^2+\varepsilon^2)^2}$$

$$\leq \frac{4}{\eta} \qquad (\text{provided } \varepsilon \ll 1).$$

Thus

$$\int_{\hat{\Omega}} |\bar{\partial}\psi|^2_{i\partial\bar{\partial}(\phi+\psi)}|u|^2 e^{\psi-\phi} \leq \int_{\hat{\Omega}} \frac{|u|^2}{1+\frac{\eta}{(-\rho+1)^2}} e^{\psi-\phi}$$

$$\int_{\operatorname{supp} v} |\bar{\partial}\psi|^2_{i\partial\bar{\partial}(\phi+\psi)}|u|^2 e^{\psi-\phi} \leq \int_{\hat{\Omega}} \frac{4}{\eta}|u|^2 e^{\psi-\phi}.$$

Fubini's theorem implies

$$\begin{aligned} &\int_{\hat{\Omega}} |v_\varepsilon|^2_{i\partial\bar{\partial}(\phi+\psi)} e^{\psi-\phi} \\ \leq\ &\int_{\hat{\Omega}} |\chi'|^2|f|^2 \frac{|z_n|^2}{\varepsilon^4}\frac{\eta(|z_n|^2+\varepsilon^2)^2}{\varepsilon^2}\frac{1}{|z_n|^2}\frac{1}{\eta}e^{-\varphi} \\ \leq\ &2\left(\int_{\{\frac{\epsilon^2}{2}<|z_n|^2<\epsilon^2\}} |\chi'|^2\frac{(|z_n|^2+\epsilon^2)^2}{\epsilon^6}\right)\int_V |f|^2 e^{-\varphi}. \end{aligned}$$

It follows that

$$\int_{\hat{\Omega}} |v_\varepsilon|^2_{i\partial\bar{\partial}(\phi+\psi)} e^{\psi-\phi} \leq C_n \int_V |f|^2 e^{-\varphi}.$$

Substituting these inequalities to the inequality in Step 3, we get

$$\int_{\hat{\Omega}} \left(\frac{\frac{\eta}{(-\rho+1)^2}}{1+\frac{\eta}{(-\rho+1)^2}} - \frac{4r}{\eta} \right) |u|^2 e^{\psi-\phi}$$
$$\leq \; (1+r^{-1}) C_n \int_V |f|^2 e^{-\varphi}.$$

Hörmander $\Rightarrow$ Ohsawa-Takegoshi

Since $\eta \asymp -\rho$ provided $\varepsilon \ll 1$, so that

$$\frac{\frac{\eta}{(-\rho+1)^2}}{1+\frac{\eta}{(-\rho+1)^2}} \asymp \frac{1}{\eta},$$

we may choose $r = r_n \ll 1$ s.t. the LHS is bounded below by

$$c_n \int_{\hat{\Omega}} \frac{|u|^2}{|z_n|^2 \eta^2} e^{-\varphi},$$

i.e.,

$$\int_{\hat{\Omega}} \frac{|u|^2}{|z_n|^2 |\log(|z_n|^2+\epsilon^2)|^2} e^{-\varphi} \le C_n \int_{\Omega'} |f|^2 e^{-\varphi}. \tag{0.3}$$

5. By virtue of (0.3) and removable singularities of L^2 holomorphic functions, we conclude that

$$F_\epsilon = \chi(|z_n|^2/\epsilon^2)f - u$$

is a holomorphic extension of f to Ω s.t.

$$\int_\Omega \frac{|F_\epsilon|^2}{|z_n|^2(-\log|z_n|^2)^2} e^{-\varphi} \le C_n' \int_V |f|^2 e^{-\varphi}.$$

By taking a limit of F_ϵ as $\epsilon \to 0$, we get the desired extension.

It is interesting to see that the gain in previous estimate is the Poincaré metric of the punctured disk, which was first noticed by Demailly.

A Hörmander type estimate

If we take

$$\psi = -\frac{\alpha}{\alpha_0}\log(-\rho),\ 0 < \alpha < \alpha_0$$

in Donnelly-Fefferman type estimate, where $\rho < 0$, C^2 psh, $|\rho| \asymp \delta_\Omega^{\alpha_0}$, then

$$\int_\Omega |u|^2 e^{-\varphi}\delta_\Omega^{-\alpha} \leq C_{\alpha,\Omega}\int_\Omega |v|^2_{i\partial\bar\partial\varphi} e^{-\varphi}\delta_\Omega^{-\alpha}. \qquad (0.4)$$

This estimate is useful in the study of regularity of the Bergman projection (see e.g., Berndtsson and Charpentier '00, and Michel-Shaw '01).

A Hörmander type estimate

Diederich-Fornaess Theorem.
Ω : *bounded, pseudoconvex,* $\partial\Omega \in C^2$, *then*
(1) $\exists\ 0 < \alpha < 1$, $\rho < 0$, C^2 *psh, s.t.* $|\rho| \asymp \delta_\Omega^\alpha$.
(2) $\forall\, 0 < \alpha < 1$, $\forall\, p \in \partial\Omega$, $\exists$ *neighborhood* $U \ni p$ *and* $\rho < 0$, C^2 *psh on* $\Omega \cap U$, *s.t.* $|\rho| \asymp \delta_\Omega^\alpha$.

The Diederich-Fornaess index is defined by

$$\alpha_0 := \sup\{\alpha > 0 : \exists \rho \in PSH^-(\Omega), s.t. |\rho| \asymp \delta_\Omega^\alpha\}$$

Diederich-Fornaess also constructed so-called worm domains with α_0 arbitrarily small.

A Hörmander type estimate

On the other side, we still have

Theorem 1 (Chen '12)
Let $\Omega \subset\subset \mathbb{C}^n$ *be pseudoconvex with* $\partial\Omega \in C^2$ *and* φ *a* C^2 *psh function on* Ω*. Then* $\forall\, \alpha < 1$, $\forall$ $(0,1)$ *form* v *with* $\bar{\partial} v = 0$ *and*

$$\int_\Omega |v|^2_{i\partial\bar{\partial}\varphi} e^{-\varphi} \delta_\Omega^{-\alpha} < \infty,$$

$\exists$ *solution* u *of* $\bar{\partial} u = v$ *satisfying* (0.4), *i.e.,*

$$\int_\Omega |u|^2 e^{-\varphi} \delta_\Omega^{-\alpha} \le C_{\alpha,\Omega} \int_\Omega |v|^2_{i\partial\bar{\partial}\varphi} e^{-\varphi} \delta_\Omega^{-\alpha}.$$

A Hörmander type estimate

Now we sketch the idea of the proof as follows

1. Diederich-Fornaess theorem $\Rightarrow \forall\, 0 < \alpha < 1$, $\exists$ cover $\{U_j\}_{0 \le j \le m_\alpha}$ of $\overline{\Omega}$, C^2 psh functions $\rho_j < 0$ on $\Omega \cap U_j$ s.t.

$$C_\alpha^{-1} \delta_\Omega(z)^{\frac{\alpha+1}{2}} \le -\rho_j(z) \le C_\alpha \delta_\Omega(z)^{\frac{\alpha+1}{2}}.$$

Standard procedure: solving first $\bar\partial u_\varepsilon = v$ on

$$\Omega_\epsilon := \{z \in \Omega : \delta_\Omega(z) > \epsilon\},$$

then take a weak limit of u_ε as $\varepsilon \to 0$.

A Hörmander type estimate

2. Let $\varphi_\tau(z) = \varphi(z) + \tau|z|^2$ with $\tau \gg 1$ to be determined later. Twisted Kohn-Morrey-Hörmander formula $\Rightarrow$

$$\int_{\Omega_\epsilon} (\eta + c(\eta)^{-1})|\bar\partial^*_{\varphi_\tau} w|^2 e^{-\varphi_\tau} + \int_{\Omega_\epsilon} \eta|\bar\partial w|^2 e^{-\varphi_\tau}$$

$$\geq \sum_{k,l} \int_{\Omega_\epsilon} \left(\eta \frac{\partial^2 \varphi_\tau}{\partial z_k \partial \bar z_l} - \frac{\partial^2 \eta}{\partial z_k \partial \bar z_l}\right) w_k \bar w_l \; e^{-\varphi_\tau}$$

$$- \int_{\Omega_\epsilon} c(\eta) \left|\sum_k \frac{\partial \eta}{\partial z_k} w_k\right|^2 e^{-\varphi_\tau} \qquad (0.5)$$

where $w \in \mathrm{Dom}\, \bar\partial^*_{\varphi_\tau} \cap C^1(\overline{\Omega}_\epsilon)$, $\eta \geq 0$, $\eta \in C^2(\Omega)$, $c \in C(\mathbb{R}, \mathbb{R}_+)$.

A Hörmander type estimate

3. Let $\{\chi_j\}$ be partition of unity subordinate to $\{U_j\}$. Clearly,

$$w^j := \chi_j w \in \mathrm{Dom}\,\bar{\partial}^*_{\varphi_\tau}.$$

Choose $\tilde{\chi}_j \in C_0^\infty(U_j)$ s.t. $\tilde{\chi}_j = 1$ on $\mathrm{supp}\,\chi_j$. Put

$$\phi_j = -\frac{2\alpha}{\alpha+1}\log(-\rho_j),\ \eta = e^{-\tilde{\chi}_j\phi_j},$$

$$c(\eta) = \frac{1-\alpha}{2\alpha}e^{\tilde{\chi}_j\phi_j}.$$

A Hörmander type estimate

4. Applying (0.5) to w^j,

$$\sum_{k,l} \int_{\Omega_\epsilon \cap U_j} \frac{\partial^2 \varphi_\tau}{\partial z_k \partial \bar{z}_l} |\chi_j|^2 w_k \bar{w}_l \; e^{-\varphi_\tau - \phi_j}$$

$$\leq \; C_\alpha \int_{\Omega_\epsilon \cap U_j} (|\bar{\partial}(\chi_j w)|^2 + |\bar{\partial}^*_{\varphi_\tau}(\chi_j w)|^2) e^{-\varphi_\tau - \phi_j}.$$

Since $e^{-\phi_j} \asymp \delta_\Omega^\alpha$ on $\Omega \cap U_j$,

$$\sum_{k,l} \int_{\Omega_\epsilon \cap U_j} \frac{\partial^2 \varphi_\tau}{\partial z_k \partial \bar{z}_l} |\chi_j|^2 w_k \bar{w}_l \; e^{-\varphi_\tau} \delta_\Omega^\alpha$$

$$\leq \; C_\alpha \int_{\Omega_\epsilon \cap U_j} (|\bar{\partial}(\chi_j w)|^2 + |\bar{\partial}^*_{\varphi_\tau}(\chi_j w)|^2) e^{-\varphi_\tau} \delta_\Omega^\alpha.$$

A Hörmander type estimate

5. Thus

$$\sum_{k,l} \int_{\Omega_\epsilon} \frac{\partial^2 \varphi_\tau}{\partial z_k \partial \bar{z}_l} w_k \bar{w}_l \; e^{-\varphi_\tau} \delta_\Omega^\alpha$$

$$= \sum_{k,l} \int_{\Omega_\epsilon} \frac{\partial^2 \varphi_\tau}{\partial z_k \partial \bar{z}_l} \left(\sum_{j=0}^{m_\alpha} \chi_j w_k \right) \left(\sum_{j=0}^{m_\alpha} \chi_j \bar{w}_l \right) \; e^{-\varphi_\tau} \delta_\Omega^\alpha$$

$$\leq (m_\alpha + 1) \sum_{j=0}^{m_\alpha} \sum_{k,l} \int_{\Omega_\epsilon \cap U_j} \frac{\partial^2 \varphi_\tau}{\partial z_k \partial \bar{z}_l} |\chi_j|^2 w_k \bar{w}_l \; e^{-\varphi_\tau} \delta_\Omega^\alpha$$

$$\leq C_\alpha \sum_{j=0}^{m_\alpha} \int_{\Omega_\epsilon \cap U_j} (|\bar{\partial}(\chi_j w)|^2 + |\bar{\partial}^*_{\varphi_\tau}(\chi_j w)|^2) e^{-\varphi_\tau} \delta_\Omega^\alpha.$$

A Hörmander type estimate

6. Since

$$\begin{aligned}\bar\partial(\chi_j w) &= \chi_j \bar\partial w + \bar\partial \chi_j \wedge w \\ \bar\partial^*_{\varphi_\tau}(\chi_j w) &= \chi_j \bar\partial^*_{\varphi_\tau} w - \bar\partial \chi_j \lrcorner w,\end{aligned}$$

thus by Schwarz's inequality,

$$\begin{aligned}&\sum_{k,l} \int_{\Omega_\epsilon} \frac{\partial^2 \varphi_\tau}{\partial z_k \partial \bar z_l} w_k \bar w_l \; e^{-\varphi_\tau} \delta_\Omega^\alpha \\ \leq \;& C_\alpha \int_{\Omega_\epsilon} (|\bar\partial w|^2 + |\bar\partial^*_{\varphi_\tau} w|^2) e^{-\varphi_\tau} \delta_\Omega^\alpha \\ &+ C_\alpha \int_{\Omega_\epsilon} |w|^2 \sum_j |\bar\partial \chi_j|^2 e^{-\varphi_\tau} \delta_\Omega^\alpha.\end{aligned}$$

A Hörmander type estimate

7. Since $\partial\bar{\partial}\varphi_\tau = \partial\bar{\partial}\varphi + \tau\partial\bar{\partial}|z|^2$, thus when $\tau = \tau(\alpha, \Omega) \gg 1$, **the second term in the right-hand side may be absorbed by the left-hand side** and we get the following

Fundamental Inequality.

$$\sum_{k,l} \int_{\Omega_\epsilon} \frac{\partial^2 \varphi}{\partial z_k \partial \bar{z}_l} w_k \bar{w}_l \; e^{-\varphi_\tau} \delta_\Omega^\alpha$$

$$\leq \; C_{\alpha,\Omega} \int_{\Omega_\epsilon} (|\bar{\partial} w|^2 + |\bar{\partial}^*_{\varphi_\tau} w|^2) e^{-\varphi_\tau} \delta_\Omega^\alpha.$$

The remaining argument is standard.

Berndtsson's boundary value estimate

In an attempt to generalize Carleson's Corona Theorem to the unit ball in $\mathbb{C}^n$, Berndtsson '87 proved the following

Theorem. *Let* Ω *be a pseudoconvex domain with* C^2 *boundary, and* $\rho, \varphi \in C^2(\Omega)$, $\rho < 0$, *such that*

$$\Theta := (-\rho)\partial\bar{\partial}\varphi + \partial\bar{\partial}\rho > 0.$$

If u *is the* $L^2(\Omega, \varphi)-$*minimal solution of* $\bar{\partial}u = v$, *then* $\forall\, 0 < \alpha < 1$,

$$(1-\alpha)\int_\Omega |u|^2(-\rho)^{-\alpha}e^{-\varphi} \leq \frac{1}{\alpha}\int_\Omega |v|^2_{i\Theta}(-\rho)^{1-\alpha}e^{-\varphi}. \tag{0.6}$$

Berndtsson's boundary value estimate

Let $\alpha \to 1-$ in (0.6), one immediately gets the following boundary value estimate:

$$\int_{\partial\Omega} \frac{|u|^2}{|\nabla \rho|} e^{-\varphi} \leq \int_{\Omega} |v|^2_{i\Theta} e^{-\varphi},$$

which plays a crucial role in the study of the Szegö kernel (see e.g. Chen-Fu '11).

Thus it seems to be interesting to know the precise estimate of the constant $C_{\alpha,\Omega}$ in Theorem 1.

There are numerous applications of L^2 estimates for $\bar{\partial}$ equation, but I mention only a few, depending on my personal taste.

One of the most remarkable application of Hörmander's theorem to the Bergman kernel is

Theorem (Berndtsson '05)
Let Ω be a pseudoconvex domain in $\mathbb{C}^n \times \mathbb{C}^k$ and $\varphi \in PSH(\Omega)$. Let K_{φ_t} be the Bergman kernel of the slice Ω_t with weight φ_t. Then $\log K_{\varphi_t}(z)$ is psh on Ω or identically equal to $-\infty$.

Regularity dependence on the parameter lies in the intersection of complex and real analysis, a little bit likes the Hardy space.

Example 1. Let $\{\Omega_t\}$ be an increasing (resp. decreasing) sequence of bounded domains over an interval I. Let K_t be the Bergman kernel of Ω_t. Then $\log K_t$ is decreasing (resp. increasing) in t, so that $\frac{\partial}{\partial t}\log K_t$ exists a.e. and belongs to $L^1_{\text{loc}}(I)$, in view of Lebesgue's theorem.

Example 2. If $K_{\varphi_t} > 0$ in Berndtsson's theorem, then $\log K_{\varphi_t} \in W^{1,2}_{\text{loc}}(\Omega)$.

Very strong stability properties of the Bergman kernels under small C^∞ perturbations of strongly pseudoconvex domains were obtained by Greene and Krantz '82.

Put $I = [0, 1]$. One has

Theorem (Diederich-Ohsawa '91)

(1) *Let* $\{\Omega_t\}_{t \in I}$ *be a* C^∞ *family of hyperconvex domains. Then* $K_t(z)$ *is continuous in* (t, z).

(2) *Let* Ω *be a bounded pseudoconvex domain and* $\{\varphi_t\}_{t \in I}$ *a* C^∞ *family of psh functions on* Ω *s.t.* $i\partial\bar\partial\varphi_t \geq Ci\partial\bar\partial|z|^2$ *and* $\sup_\Omega |\frac{\partial^k}{\partial t^k}\bar\partial\varphi_t| \leq C_k$ *for all* $k \geq 1$. *Then* $K_{\varphi_t}(z)$ *is* C^∞ *in* (t, z).

Donnelly-Fefferman type theorem implies

Proposition 1. *Let* $\{\Omega_t\}_{t\in I}$ *be a* C^2 *family of bounded pseudoconvex domains with* $\partial\Omega_t \in C^2$*. Then locally uniformly,*

$$|\log K_t - \log K_{t_0}| \leq \frac{C}{|\log|t-t_0||}.$$

Proposition 2.

Let $\{\Omega_t\}_{t\in I}$ *be a* nondecreasing C^2-*family of bounded pseudoconvex domains with* $\partial\Omega_t \in C^2$*. Suppose for each* $w \in \Omega_0$*,*

$$\sup_{z\in\Omega_0} |K_0(z,w)| < \infty.$$

Then locally uniformly on Ω_0,

$$|\log K_t - \log K_0| \le C t^{1/2}.$$

Proposition 3. *Let* Ω *be pseudoconvex with* $\partial\Omega \in C^2$ *and* $\{\varphi_t : t \in I\}$ *a family of psh functions on* Ω. *Put* $\delta(z) := d(z, \partial\Omega)$. *Then locally uniformly,*

$$\begin{aligned} &|\log K_t - \log K_{t_0}| \\ \le\ & C \inf_{0<\varepsilon<1} \left(\frac{1}{|\log \varepsilon|} + \sup_{\delta \ge \varepsilon} |\varphi_t - \varphi_{t_0}| \right). \end{aligned}$$

Corollary.

(i) *If* $\sup_\Omega |\varphi_t - \varphi_{t'}|/|\log\delta| \leq C|t-t'|$ *for any* $t, t' \in I$, *then* $\log K^t$ *is Hölder continuous of order* $1/2$ *in* t.

(ii) *If* $\sup_\Omega |\varphi_t - \varphi_{t'}|/\log|\log\delta| \leq C|t-t'|$ *for any* $t, t' \in I$, *then* $\log K^t$ *is log-Lipschitz continuous in* t, *i.e.,*

$$|\log K_t - \log K_{t_0}| \leq C|t-t_0||\log|t-t_0||$$

holds locally uniformly.

Essay on reproducing kernels

Let Ω be a bounded domain with $\partial\Omega \in C^2$. For $\forall\, \alpha < 1$, the Bergman space $A^2_\alpha(\Omega)$ is the set of $f \in \mathscr{O}(\Omega)$ s.t.

$$\|f\|^2_\alpha := \int_\Omega |f|^2 \delta^{-\alpha} < \infty.$$

The Hardy space $H^2(\Omega)$ is the set of $f \in \mathscr{O}(\Omega)$ s.t.

$$\|f\|^2_{L^2(\partial\Omega)} = \lim_{\alpha \to 1-} (1-\alpha) \int_\Omega |f|^2 \delta^{-\alpha} < \infty.$$

Recall also that

$$A^2_\alpha(\Omega) = W^{\alpha/2}(\Omega) \cap \mathscr{O}(\Omega)$$

$$H^2(\Omega) = W^{1/2}(\Omega) \cap \mathscr{O}(\Omega)$$

where $W^s(\Omega)$ is the Sobolev space.

These Hilbert spaces induce reproducing kernels: the Bergman kernel K_α and the Szegö kernel S. The standard Bergman kernel is $K = K_0$.

By Theorem 1 and Corollary, we conclude that $\log K_\alpha$ is $1/2-$Hölder continuous in α.

Proposition 4. *Let* Ω *be a bounded domain with* $\partial\Omega \in C^2$. *Then*

$$(1-\alpha)^{-1} K_\alpha(z,w) \to S(z,w)$$

locally uniformly in z, w *as* $\alpha \to 1-$.

Corollary.

$$\left.\frac{\partial K_\alpha(z,w)}{\partial \alpha}\right|_{\alpha=1-} = -S(z,w).$$

Question 1.
Is there an expansion

$$K_\alpha(z,w) = (1-\alpha)S(z,w) + \sum_{j=2}^{\infty} a_j(z,w)(1-\alpha)^j$$

as $\alpha \to 1-$*?*

A related question is

Question 2.
Does $K_\alpha(z)$ *depend real analytically on* α*?*

Note that the answer to Question 2 is positive when Ω is strongly pseudoconvex (Englis '10).

In his 1972 book, E. M. Stein asked the following

Stein's Question. *What are the relations between K and S?*

Theorem (Chen-Fu '11)
Let Ω be a bounded pseudoconvex domain with C^2 boundary in $\mathbb{C}^n$. Then
(1) $\forall\, 0 < \alpha < 1$, $\exists\, C_\alpha > 0$ *s.t.*

$$S/K \leq C_\alpha \delta_\Omega |\log \delta_\Omega|^{\frac{n}{\alpha}}$$

(2) *Suppose furthermore Ω is $\delta-$regular, then* $\exists\, 0 < \alpha \leq 1$, $C > 0$ *s.t.*

$$S/K \geq C \delta_\Omega |\log \delta_\Omega|^{-1/\alpha}.$$

Definition. A domain Ω is called $\delta-$regular if there are $\varphi \in PSH^{-}(\Omega)$ and a defining function ρ s.t.

$$i\partial\bar{\partial}\varphi \geq \rho^{-1} i\partial\bar{\partial}\rho \ \text{ near } \partial\Omega.$$

Examples:

(1) Pseudoconvex domains of finite D'Angelo type.

(2) Domains admitting a psh defining function.

The proof of relies on Berndtsson's estimate (0.6).