Tate classes and L-functions of twisted quaternionic Shimura surfaces

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1 Introduction

Let X be a smooth projective variety defined over a number field F and let

$$\bar{X} = X \times_F \bar{\mathbb{Q}}.$$

For a prime number l, let $H^i_{et}(X, \bar{\mathbb{Q}}_l)$ be the l-adic cohomology of X. If K is a number field, we denote $\Gamma_K := \mathrm{Gal}(\bar{\mathbb{Q}}/K)$. The Galois group Γ_F acts on $H^i_{et}(X, \bar{\mathbb{Q}}_l)$ by a representation $\rho_{i,l}$. For any $j \in \mathbb{Z}$, let $H^i_{et}(X, \bar{\mathbb{Q}}_l)(j)$ denote the representation of Γ_F on $H^i_{et}(X, \bar{\mathbb{Q}}_l)$ defined by $\rho_{i,l} \otimes \xi_l^j$, where ξ_l is the l-adic cyclotomic character.

Let $U^i(X)$ denote the $\mathbb{Q}$-linear space of the algebraic subvarieties of X of codimension i. We have the l-adic cycle map

$$d_{i,l} : U^i(X) \otimes \bar{\mathbb{Q}}_l \to H^{2i}_{et}(X, \bar{\mathbb{Q}}_l)(i).$$

The cohomology classes in the image of this map are said to be *algebraic*.

The action of Γ_F on $\bar{X}$ gives a continuous l-adic representation

$$\rho_{i,l} : \Gamma_F \to \mathrm{Aut}_{\bar{\mathbb{Q}}_l}(H^i_{et}(X, \bar{\mathbb{Q}}_l)).$$

For each finite extension E of F, we denote by $U^i(X, E)$ the subspace of $d_{i,l}(U^i(X) \otimes \bar{\mathbb{Q}}_l)$ left fixed by Γ_E. The first part of the Tate's conjecture states that

$$U^i(X, E) = (H^{2i}_{et}(X, \bar{\mathbb{Q}}_l)(i))^{\Gamma_E}.$$

The elements of $V^i(X, E) := (H^{2i}_{et}(X, \bar{\mathbb{Q}}_l)(i))^{\Gamma_E}$ are called *Tate classes*. We denote by $\rho^{ss}_{i,l}$ the semisimplification of $\rho_{i,l}$ and define the space of *semisimple Tate classes* $V^i(X, E)'$ replacing the action of $\rho_{i,l}$ in the definition of $V^i(X, E)$ by $\rho^{ss}_{i,l}$.

The L-function $L^i(s, X_{/F})$ attached to the representation $\rho_{i,l}$ converges for $\mathrm{Re}(s) > 1 + i/2$. The second part of the Tate's conjecture states that the L-function $L^{2i}(s, X_{/E})$ has a pole at $s = i + 1$ of order equal to

$$\dim_{\bar{\mathbb{Q}}_l} U^i(X, E).$$

In their work [HLR], Harder, Langlands and Rapoport had proved the first part of the Tate's conjecture for Hilbert modular surfaces for non-CM sub-motives. In [K] and [MR] it was proved the first part of the Tate's conjecture for Hilbert modular surfaces for CM sub-motives and thus using the two results, one gets the full first part of the Tate's conjecture asserting the algebraicity of all the Tate classes of Hilbert modular surfaces over an arbitrary number field.

The second part of the Tate's conjecture for Hilbert modular surfaces was proved in [HLR], [K] and [MR] only for solvable number fields.

In this paper we consider a totally real field F and a quaternion algebra D over F, which is unramified at exactly 2 infinite places of F. Let G be the algebraic group over F defined by the multiplicative group $D^\times$ of D and let $\bar{G} = \mathrm{Res}_{F/\mathbb{Q}}(G)$. We consider a prime ideal $\wp$ of O_F, such that $G(F_\wp)$ is isomorphic to $GL_2(F_\wp)$. Let $S_{\bar{G},\hat{\Gamma}(\wp)} = S_{\hat{\Gamma}(\wp)}$ be the canonical model of the quaternionic Shimura surface associated to the adelic principal congruence subgroup $\hat{\Gamma}(\wp)$ of $\bar{G}(\mathbb{A}_f)$ of level $\wp$, where $\mathbb{A}_f$ is the finite adeles ring of $\mathbb{Q}$. Then $S_{\hat{\Gamma}(\wp)}$ is a quasi-projective surface defined over a finite extension $E/\mathbb{Q}$ called the canonical field of definition.

The surface $S_{\hat{\Gamma}(\wp)}$ has a natural action of $GL_2(O/\wp)$. Consider a continuous Galois representation $\varphi : \Gamma_E \to GL_2(O/\wp)$ and let $S'_{\hat{\Gamma}(\wp)}$ be the surface defined over E obtained from $S_{\hat{\Gamma}(\wp)}$ via twisting by φ composed with the natural action of $GL_2(O/\wp)$ on $S_{\hat{\Gamma}(\wp)}$ (see §2 for details).

The Shimura surface $S_{\hat{\Gamma}(\wp)}$ is not smooth and in this article we use the *é*tale cohomology of the smooth toroidal compactification of $S_{\hat{\Gamma}(\wp)}$ and to simplify the notations, we denote this compactification also by $S_{\hat{\Gamma}(\wp)}$.

The surfaces $S_{\hat{\Gamma}(\wp)}$ and $S'_{\hat{\Gamma}(\wp)}$ become isomorphic over $\bar{\mathbb{Q}}$ and by descend we deduce that the first part of the Tate's conjecture for the surface $S_{\hat{\Gamma}(\wp)}$ over a given field K is true if and only if the first part of the Tate's conjecture for the surface $S'_{\hat{\Gamma}(\wp)}$ over the field K is true. As we mentioned above the first part of the Tate's conjecture is known for Hilbert modular surfaces [HLR], [K], [MR]. Also the first part of the Tate's conjecture is known in the non-CM case for the Shimura surfaces treated in [L], corresponding to a quadratic real field F and to a quaternion algebra $D = B \otimes_{\mathbb{Q}} F$, where B is a quaternion algebra over $\mathbb{Q}$, such that D splits at the real places and F splits over the places where B ramifies.

In this article we want to generalize the results in [HLR], [K] and [MR] and prove that if the field $L := \bar{\mathbb{Q}}^{\ker(\varphi)}$ is a solvable extension of a totally real field, then the L-function $L^2(s, S'_{\hat{\Gamma}(\wp)_{/k}})$ has a pole at $s = 2$ of order equal to

$$\dim_{\bar{\mathbb{Q}}_l} V^1(S'_{\hat{\Gamma}(\wp)}, k)',$$

if k is a solvable extension of a totally real field which contains the field of definition E.

Since in the case of quaterninic Shimura varieties, the representations $\rho_{i,l}$ defined above are conjectured to be semisimple, this result should imply the equality between the order of the pole of $L^2(s, S'_{\hat{\Gamma}(\wp)/k})$ at $s = 2$ and

$$\dim_{\bar{\mathbb{Q}}_l} V^1(S'_{\hat{\Gamma}(\wp)}, k),$$

if k is a solvable extension of a totally real field which contains E (see section 6.4 for details).

2 Twisted quaternionic Shimura surfaces

Let F be a totally real field of degree d over $\mathbb{Q}$ and $O := O_F$ be its the ring of integers. Let $\mathbb{A}_{\mathbb{Q}} = \mathbb{R} \times \mathbb{A}_f$ be the adeles ring of $\mathbb{Q}$ and $\mathbb{A}_F$ the adeles ring of F. We denote by $I_{\mathbb{Q}}$ and I_F the ideles ring of $\mathbb{Q}$ and F, respectively.

We consider a quaternion algebra D over F which is unramified at exactly 2 infinite places of F. We denote by S_∞ the set of infinite places of F and we identify S_∞ as a $\Gamma_{\mathbb{Q}}$-set with $\Gamma_F \setminus \Gamma_{\mathbb{Q}}$. Let S'_∞ be the subset of S_∞ at which D is ramified. Thus the cardinal of $S_\infty - S'_\infty$ is equal to 2.

Let G be the algebraic group over F defined by the multiplicative group $D^\times$. By restricting the scalars, we obtain the algebraic group $\bar{G} = \mathrm{Res}_{F/\mathbb{Q}}(G)$ over $\mathbb{Q}$ defined by the propriety: $\bar{G}(A) = G(A \otimes_{\mathbb{Q}} F)$ for all $\mathbb{Q}$-algebras A. It is easy to see that $\bar{G}(\mathbb{R})$ is isomorphic to $GL_2(\mathbb{R})^2 \times \mathbf{H}^{*(d-2)}$, where $\mathbf{H}$ is the algebra of quaternions over $\mathbb{R}$.

For $v \in S_\infty - S'_\infty$, we fix an isomorphism of $G(F_v)$ with $GL_2(\mathbb{R})$. We have $\bar{G}(\mathbb{R}) = \prod_{v \in S_\infty} G(F_v)$. Let $J = (J_v) \in \bar{G}(\mathbb{R})$, where

$$J_v = \begin{cases} 1 & \text{for } v \in S'_\infty; \\ 1/\sqrt{2}\left(\begin{smallmatrix} 1 & 1 \\ -1 & 1 \end{smallmatrix}\right) & \text{for } v \in S_\infty - S'_\infty. \end{cases}$$

Let K_∞ be the centralizer of J in $\bar{G}(\mathbb{R})$. Set

$$X = \bar{G}(\mathbb{R})/K_\infty.$$

It is well known that X is complex analitically isomorphic to $(h_\pm)^2$ where $h_\pm = \mathbb{C} - \mathbb{R}$. For each open compact subgroup $K \subseteq \bar{G}(\mathbb{A}_f)$ set

$$S_K(\mathbb{C}) = \bar{G}(\mathbb{Q}) \setminus X \times \bar{G}(\mathbb{A}_f)/K.$$

For K sufficiently small, $S_K(\mathbb{C})$ is a complex manifold which is the set of complex points of a quasi projective variety. The canonical field of definition of $S_K(\mathbb{C})$ is by definition the subfield E of $\bar{\mathbb{Q}}$ such that Γ_E is the stabilizer of $S'_\infty \subseteq \Gamma_F \setminus \Gamma_{\mathbb{Q}}$. It is known that $S_K(\mathbb{C})$ has a canonical model over E which is denoted by S_K.

Actually this model can be defined over $O_E[1/N]$ for some integer N, where O_E is the ring of integers of E. The dimension of S_K is equal to 2.

Let $\wp$ be a prime ideal of O_F such that $G(F_\wp)$ is isomorphic to $GL_2(F_\wp)$. Consider $\hat{\Gamma}(\wp) =: 1 + \wp O_D \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$, where O_D is the ring of integers of D. The group $GL_2(O/\wp)$ acts on

$$S_{\hat{\Gamma}(\wp)}(\mathbb{C}) = \bar{G}(\mathbb{Q}) \setminus X \times \bar{G}(\mathbb{A}_f)/\hat{\Gamma}(\wp).$$

This action can be described in the following way : $GL_2(O_\wp) \hookrightarrow \bar{G}(\mathbb{A}_\mathbb{Q})$ by $\alpha \mapsto (1, ..., \alpha, 1, ..., 1)$, α at the $\wp$ component. Using the isomorphism $GL_2(O/\wp) \cong GL_2(O_\wp)/(\hat{\Gamma}(\wp))_\wp$, the action of an element $g \in GL_2(O_\wp)$ is given by the left multiplication at the $\wp$ component.

We fix a continuous representation

$$\varphi : \Gamma_E \to GL_2(O/\wp).$$

Let L be the finite Galois extension of $\mathbb{Q}$ defined by $L := (\overline{\mathbb{Q}})^{\ker(\varphi)}$.

Let

$$S^{'} = S_{\hat{\Gamma}(\wp)} \times_{\mathrm{Spec}(E)} \mathrm{Spec}(L).$$

The group $GL_2(O/\wp)$ acts on $S_{\hat{\Gamma}(\wp)}$. Since $\varphi : \mathrm{Gal}(L/E) \hookrightarrow GL_2(O/\wp)$, the group $\mathrm{Gal}(L/E)$ acts on $S_{\hat{\Gamma}(\wp)}$. We denote this action of $\mathrm{Gal}(L/E)$ on $S_{\hat{\Gamma}(\wp)}$ by $\varphi^{'}$. The Galois group $\mathrm{Gal}(L/E)$ has a natural action on $\mathrm{Spec}(L)$ and we can descend via the quotient process $S^{'}$ to $S^{'}_{\hat{\Gamma}(\wp)}/\mathrm{Spec}(E)$ using the diagonal action

$$\mathrm{Gal}(L/E) \ni \sigma \to \varphi^{'}(\sigma) \otimes \sigma$$

on $S^{'}$. Thus, we obtain a quasi-projective variety $S^{'}_{\hat{\Gamma}(\wp)}/\mathrm{Spec}(E)$. This is the twisted quaternionic Shimura surface that we mentioned in the title.

3 Zeta functions of twisted quaternionic Shimura surfaces

From now on, if π is an automorphic representation of $\bar{G}(\mathbb{A}_\mathbb{Q})$, we denote the automorphic representation of $GL_2(\mathbb{A}_F)$, obtained from π by Jacquet-Langlands correspondence (usually denoted $JL(\pi)$) by the same symbol π.

If l is a prime number, we fix an isomorphism $j : \overline{\mathbb{Q}}_l \to \mathbb{C}$, and from now on we identify these two fields. If π is an cuspidal automorphic representation of weight 2 of $GL(2)/F$, then there exists ([T]) a λ-adic representation for $\lambda \nmid \mathbf{n}$ ($\mathbf{n}$ is the level of π)

$$\rho_{\pi,\lambda} : \Gamma_F \to GL_2(O_\lambda) \hookrightarrow GL_2(\overline{\mathbb{Q}}_l) \cong GL_2(\mathbb{C}),$$

which satisfies $L(s-1/2,\pi)=L(s,\rho_{\pi,\lambda})$ and is unramified outside the primes dividing $\mathbf{n}l$. Here O is the coefficients ring of π and λ is a prime ideal of O above some prime number l. In order to simplify the notations we denote by ρ_π the representation $\rho_{\pi,\lambda}$.

Assume that the local and the global Haar measures have been fixed on $\bar{G}(\mathbb{A}_f)$. We assume that $K=\Pi_{v<\infty}K_v$ where K_v is open compact in $G(F_v)$ and $K_v=GL_2(O_v)$ for almost all v, where O_v is the ring of integers of F_v. For $g\in\bar{G}(\mathbb{A}_f)$, let

$$f_g=\mathrm{char}(KgK)/\mathrm{meas}(K)$$

where $\mathrm{char}(KgK)$ is the characteristic function of KgK. Let $\mathbb{H}_K$ be the Hecke algebra generated by the f_g under the convolution. If $\pi=\pi_\infty\otimes\pi_f$ is an automorphic representation of $\bar{G}(\mathbb{A}_\mathbb{Q})$, we denote by π_f^K the space of K invariants of K in π_f. The Hecke algebra $\mathbb{H}_K$ acts on π_f^K.

We have a Galois-equivariant decomposition

$$H^2_{et}(S_K,\bar{\mathbb{Q}}_l)=IH^2_{et}(S_K,\bar{\mathbb{Q}}_l)\oplus W,$$

where

$$IH^2_{et}(S_K,\bar{\mathbb{Q}}_l)\cong\mathrm{Im}(H^2_c(X,\bar{\mathbb{Q}}_l)\to H^2_{et}(S_K,\bar{\mathbb{Q}}_l)),$$

is the *intersection cohomology* and $H^2_c(X,\bar{\mathbb{Q}}_l)$ denotes the compactly supported cohomology and

$$W=\{x\in H^2_{et}(S_K,\bar{\mathbb{Q}}_l)|\ x \text{ has support in } \tilde{\mathrm{S}}^\infty_\mathrm{K}\}.$$

The Galois module W has a decomposition

$$W\cong\oplus_{s\in\tilde{S}^\infty_K}(\bar{\mathbb{Q}}_l(-1))^{\oplus m(s)},$$

where $m(s)$ denotes for each s the number of irreducible components of the resolution $\tilde{S}^\infty_K(s)$ of s.

We have an action of the Hecke algebra $\mathbb{H}_K$ and an action of the Galois group Γ_E on the intersection cohomology $IH^2_{et}(S_K,\bar{\mathbb{Q}}_l)$ and these two actions commute. We say that the representation π is *cohomologycal* if $H^2(g,K_\infty,\pi_\infty)\neq 0$, where g is the Lie algebra of K_∞ (the cohomology is taken with respect to (g,K_∞)-module associated to π_∞).

Proposition 3.1. *The double representation of $\Gamma_E\times\mathbb{H}_K$ on the intersection cohomology $IH^2_{et}(S_K,\bar{\mathbb{Q}}_l)$ is isomorphic to*

$$\oplus_\pi\rho(\pi)\otimes\pi_f^K,$$

where $\rho(\pi)$ is a representation of the Galois group Γ_E. The above sum is over weight 2 cohomologycal automorphic representations π of $\bar{G}(\mathbb{A}_\mathbb{Q})$ and the $\mathbb{H}_K$-representations π_f^K are irreducible and mutually inequivalent, i.e. the decomposition is isotypic with respect to the action of $\mathbb{H}_K$.

The irreducible automorphic representations that appear in Proposition 3.1 are one-dimensional or cuspidal and infinite-dimensional. If π is one-dimensional then $\rho(\pi)$ has dimension two and if π is infinite-dimensional, then $\rho(\pi)$ has dimension four.

We fix an isomorphism $j : \bar{\mathbb{Q}}_l \to \mathbb{C}$ and define the L-function

$$L^2(s, S_{\hat{\Gamma}(\wp)}) := \prod_q \det(1 - Nq^{-s} j(\rho(\pi)(\mathrm{Frob}_q)) | H^2_{et}(S_{\hat{\Gamma}(\wp)}, \bar{\mathbb{Q}}_l)^{I_q})^{-1},$$

where Frob_q is a geometric Frobenius element at a finite place q of E and I_q is the inertia group at q.

We consider the injective limit:

$$V := \varinjlim_K IH^2_{et}(S_K, \bar{\mathbb{Q}}_l) \cong \varinjlim_K \oplus_\pi V(\pi_\infty) \otimes_{\bar{\mathbb{Q}}_l} \pi_f^K,$$

where $V(\pi_\infty)$ is the $\bar{\mathbb{Q}}_l$-space that corresponds to $\rho(\pi)$ (see Proposition 3.1 for notations).

Using the strong multiplicity one for $\bar{G}$, we get that the π-component $V(\pi)$ of V is isomorphic to $\rho(\pi) \otimes \pi_f$ as $\Gamma_E \times \mathbb{H}$-module. Taking the $\hat{\Gamma}(\wp)$-fixed vectors we deduce that $V(\pi)^{\hat{\Gamma}(\wp)}$ is isomorphic to $\rho(\pi) \otimes \pi_f^{\hat{\Gamma}(\wp)}$ as $\Gamma_E \times GL_2(O/\wp O)$-module. Since the varieties $S_{\hat{\Gamma}(\wp)}$ and $S'_{\hat{\Gamma}(\wp)}$ become isomorphic over $\bar{\mathbb{Q}}$, we have the isomorphism $IH^2_{et}(S_{\hat{\Gamma}(\wp)}, \bar{\mathbb{Q}}_l) \cong IH^2_{et}(S'_{\hat{\Gamma}(\wp)}, \bar{\mathbb{Q}}_l)$. The actions of Γ_E on these cohomologies which give the expression of the zeta functions of these varieties are different. If we consider the component $V'(\pi)$ that corresponds to π of $IH^2_{et}(S'_{\hat{\Gamma}(\wp)}, \bar{\mathbb{Q}}_l)$ (see the decomposition of Proposition 3.1), we get that $V'(\pi)$ is isomorphic to $\rho(\pi) \otimes (\pi_f^{\hat{\Gamma}(\wp)} \circ \varphi)$ as Γ_E-module. We denote also by W' the Galois module obtained from W. Hence we conclude the following result (this is a particular case of Theorem 1.1 from [V]):

Theorem 3.2. *The L-function $L'^2(s, S'_{\hat{\Gamma}(\wp)})$ which comes from the intersection cohomology part of $H^2_{et}(S_{\hat{\Gamma}(\wp)}, \bar{\mathbb{Q}}_l)$ is given by the formula:*

$$L'^2(s, S'_{\hat{\Gamma}(\wp)}) = \prod_\pi L(s, \rho(\pi) \otimes (\pi_f^{\hat{\Gamma}(\wp)} \circ \varphi)),$$

where the product is taken over cohomologycal automorphic representations π of $\bar{G}(\mathbb{A}_{\mathbb{Q}})$ of weight 2, such that $\pi_f^{\hat{\Gamma}(\wp)} \neq 0$.

For the part of the L-function of $S'_{\hat{\Gamma}(\wp)}$ which comes from the Galois module W' it is easy to prove the equality between the pole at $s = 2$ and the dimension of the space of Tate classes since the Galois module W' is monomial (see section 6.3 for definition and details).

4 Base change

We know the following result (Theorem 2.2 of [V]):

Theorem 4.1. *If F is a totally real field, π is a cuspidal automorphic representation of weight 2 of $GL(2)/F$ and F_1 is a solvable extension of a totally real field containing F, then there exists a Galois extension F_2 of $\mathbb{Q}$ containing F_1 and there exists a prime λ of the field coefficients of π, such that $\rho_{\pi,\lambda}|_{\Gamma_{F_2}}$ is modular i.e. there exists a cuspidal automorphic representation π_1 of $GL(2)/F_2$ and a prime β of the field of coefficients of π_1 such that $\rho_{\pi,\lambda}|_{\Gamma_{F_2}} \cong \rho_{\pi_1,\beta}$.*

In this section we fix an automorphic representation π as in Theorem 4.1 and we denote $\omega := \pi_f^{\hat{\Gamma}(\wp)} \circ \varphi$. We assume in the rest of the paper that the field $L := \bar{\mathbb{Q}}^{\ker(\varphi)}$ is a solvable extension of a totally real field. Thus the field $K := \bar{\mathbb{Q}}^{\ker(\omega)}$ is a solvable extension of a totally real field.

Let k be a solvable extension of a totally real field which contains E. From Theorem 4.1 we deduce that there exists a Galois extension F_2 of $\mathbb{Q}$ containing Kk, a prime λ of the field coefficients of π and a cuspidal automorphic representation π_1 of $GL(2)/F_2$ and a prime β of the field of coefficients of π_1 such that $\rho_{\pi,\lambda}|_{\Gamma_{F_2}} \cong \rho_{\pi_1,\beta}$.

By Brauer's Theorem (see [SE], Theorems 16 and 19), we can find some subfields $F_i \subset F_2$ such that $\mathrm{Gal}(F_2/F_i)$ are solvable, some characters $\chi_i : \mathrm{Gal}(F_2/F_i) \to \bar{\mathbb{Q}}^\times$ and some integers m_i, such that the representation

$$\omega|_{\Gamma_k} : \mathrm{Gal}(F_2/k) \to \mathrm{Gal}(Kk/k) \to GL_N(\bar{\mathbb{Q}}_l),$$

can be written as $\omega|_{\Gamma_k} = \sum_{i=1}^{i=k} m_i \mathrm{Ind}_{\Gamma_{F_i}}^{\Gamma_k} \chi_i$ (a virtual sum). Then

$$L(s, (\rho(\pi) \otimes \omega)|_{\Gamma_k}) = \prod_{i=1}^{i=k} L(s, \rho(\pi)|_{\Gamma_k} \otimes \mathrm{Ind}_{\Gamma_{F_i}}^{\Gamma_k} \chi_i)^{m_i} =$$

$$\prod_{i=1}^{i=k} L(s, \mathrm{Ind}_{\Gamma_{F_i}}^{\Gamma_k} (\rho(\pi)|_{\Gamma_{F_i}} \otimes \chi_i))^{m_i} = \prod_{i=1}^{i=k} L(s, \rho(\pi)|_{\Gamma_{F_i}} \otimes \chi_i)^{m_i}.$$

Since $\rho_{\pi,\lambda}|_{\Gamma_{F_2}}$ is modular and $\mathrm{Gal}(F_2/F_i)$ is solvable, from Langlands base change one can deduce that $\rho_{\pi,\lambda}|_{\Gamma_{F_i}}$ is modular. We give a short proof of this fact. If the representation π is of CM type (see §5 for the definition), then the existence of the base change of π to an arbitrary extension k/F is well known and thus, in this case we are done. We assume now that π is non-CM. Then from Proposition 5.1 below the representation π_1 in non-CM. By induction we can assume that $\mathrm{Gal}(F_2/F_i)$ is cyclic of prime order. We denote by θ a generator of $\mathrm{Gal}(F_2/F_i)$. Then we know (see for example Proposition 2.3.1 [RA]) that π_1 is a base change of an automorphic representation π' of $GL(2)/F_i$ iff $\pi_1 \cong \pi_1 \circ \theta$. By strong multiplicity one for $GL(2)$, this is equivalent to $\rho_{\pi_1} \cong \rho_{\pi_1}^\theta$, where $\rho_{\pi_1}^\theta(\gamma) = \rho_{\pi_1}(\theta\gamma\theta^{-1})$. Since $\rho_{\pi,\lambda}|_{\Gamma_{F_i}} \cong \rho_{\pi,\lambda}^\theta|_{\Gamma_{F_i}}$ by restriction to Γ_{F_2}, we obtain

$\rho_{\pi_1} \cong \rho_{\pi_1}^{\theta}$. Hence π_1 is a base change of an automorphic representation $\pi^{'}$ of $GL(2)/F_i$. Since π_1 is non-CM, from Proposition 5.1 below, we know that the representation ρ_{π_1} is irreducible. The restrictions of $\rho_{\pi'}$ and $\rho_{\pi,\lambda}|_{\Gamma_{F_i}}$ to Γ_{F_2} are equal to the irreducible representation ρ_{π_1} and since Γ_{F_2} is a normal subgroup of Γ_{F_i} one could prove that $\rho_{\pi,\lambda}|_{\Gamma_{F_i}} \cong \rho_{\pi'} \otimes \chi \cong \rho_{\pi' \otimes \chi}$ where χ is a Galois character corresponding to F_2/F_i. Therefore $\rho_{\pi,\lambda}|_{\Gamma_{F_i}}$ is modular.

5 Known results

It is known that (see for example [HLR] Proposition 4.5.4):

Proposition 5.1. *If π is a cuspidal automorphic representation of $GL(2)/F$, where F is a totally real field. Then one of the following two statements holds:*
(i) $\rho_\pi|_{\Gamma_L}$ is irreducible for each finite extension L/F.
(ii) There exists a quadratic extension L/F and an algebraic Hecke character ψ of L such that $\rho_\pi \cong Ind(\psi)$.

We say that a representation ρ of a group G is *dihedral* if there exists a normal subgroup N of index 2 in G and a character $\chi : N \to \mathbb{C}^\times$ such that $\rho = \mathrm{Ind}_N^G \chi$.

We say that an automorphic representation π of $GL(2)/L$ for some number field L is of CM *type* if there exists some quadratic Galois character $\eta : I_L/L^\times \to \bar{\mathbb{Q}}_l^\times$, with $\eta \neq 1$ such that $\pi \cong \pi \otimes \eta$. If π is an automorphic representation of $GL(2)/L$, then π is of CM type if and only if ρ_π is a dihedral representation.

We know the following result (Theorem 2.1 of [MP]):

Proposition 5.2. *The tensor product of two 2 dimensional irreducible complex representations of a group is reducible only if either both representations are dihedral or they are the twist of each other by a character.*

We know (Lemma 4.2 of [MP]):

Proposition 5.3. *Let π_1 and π_2 be two cuspidal non-CM representations of $GL(2)/F$, where F is a totally real field. Suppose that π_1 and π_2 are twist of each other over an extension of F, then π_1 and π_2 are twist of each other over F.*

We know (Proposition 4.1 of [MP]):

Proposition 5.4. *Suppose that π is a cuspidal, non-CM automorphic representation of $GL(2)/K$ for some finite extension $K/\mathbb{Q}$. Suppose that K is a quadratic extension of k and τ is the automorphism of K over k. If $\pi^\tau \cong \pi \otimes \chi$ for a Hecke character χ of K, then χ is trivial when restricted to the ideles of k.*

We know (Corrolary 2.6 of [MP]):

Proposition 5.5. *Let ρ be a 2-dimensional irreducible representation of a group G. Then $Sym^2(\rho)$ is reducible if and only if ρ is dihedral.*

We know (see Theorem M and Theorem 2.2.7 of [RA]):

Proposition 5.6. *Let π_1 and π_2 are cuspidal, non-CM automorphic representations of weight 2 of $GL(2)/K$ for some finite extension $K/\mathbb{Q}$. Then $\pi_1 \otimes \pi_2$ is a cuspidal automorphic representation of $GL(4)/K$ iff π_1 is not isomorphic to $\tilde{\pi}_2 \otimes \chi$ for some Hecke character χ, where $\tilde{\pi}_2$ is the dual representation. If $\pi_1 \otimes \pi_2$ is cuspidal, then $L(s, \pi_1 \otimes \pi_2)$ is analytic and does not vanish at $s = 1$. If $\pi_1 \cong \tilde{\pi}_2$, then $L(s, \pi_1 \otimes \pi_2)$ has an unique pole of order 1 at $s = 1$.*

We know (see the Main Theorem of [JG]):

Proposition 5.7. *Let π be a cuspidal, non-CM automorphic representations of weight 2 of $GL(2)/K$ for some finite extension $K/\mathbb{Q}$. Then $Sym^2\pi$ is a cuspidal automorphic representation of $GL(3)/K$ and the L-function $L(s, Sym^2\pi)$ is analytic and does not vanish at $s = 1$.*

6 Tate's conjecture for twisted quaternionic Shimura surfaces

Assume that k be a solvable extension of a totally real field which contains E and π is an automorphic representation of $GL(2)/F$ that appears in Theorem 3.2. The representation π is one-dimensional or cuspidal and infinite-dimensional. Let $V(\pi)^{'} = V(\pi_\infty) \otimes \pi_f^{\hat{\Gamma}(\wp)}$ the space considered in §3 just before Theorem 3.2.

We recall that in §4 we denoted $\omega =: \pi_f^{\hat{\Gamma}(\wp)} \circ \varphi$ and we assumed that the field $L := \bar{\mathbb{Q}}^{\ker\varphi}$ is a solvable extension of a totally real field.

We denote by $\rho(\pi)^{ss}$ the semisimplification of $\rho(\pi)$ and define:

$$\mathbf{V}(\pi, k) = \{x \in V(\pi)^{'} | (\rho(\pi) \otimes \omega)(a)x = \xi_l^{-1}(a)x, \ for \ all \ a \in \Gamma_k\}$$

and

$$\mathbf{V}(\pi, k)^{'} = \{x \in V(\pi)^{'} | (\rho(\pi)^{ss} \otimes \omega)(a)x = \xi_l^{-1}(a)x, \ for \ all \ a \in \Gamma_k\},$$

where ξ_l is the l-adic cyclotomic character. The elements of $\mathbf{V}(\pi, k)$ are called *Tate classes* and the elements of $\mathbf{V}(\pi, k)^{'}$ are called *simisimple Tate classes*.

Since at all but a finite number of finite places of E, the representations $\rho(\pi)|_{\Gamma_{F_i}}$ and $\rho(\pi)^{ss}|_{\Gamma_{F_i}}$ yield the same local L-factors, the order of the pole at $s = 2$ of $L(s, (\rho(\pi) \otimes \omega)|_{\Gamma_k}) = \prod_{i=1}^{i=k} L(s, \rho(\pi)|_{\Gamma_{F_i}} \otimes \chi_i)^{m_i}$. is equal to the order of the pole at $s = 2$ of $L(s, (\rho(\pi)^{ss} \otimes \omega)|_{\Gamma_k}) = \prod_{i=1}^{i=k} L(s, \rho(\pi)^{ss}|_{\Gamma_{F_i}} \otimes \chi_i)^{m_i}$.

We will prove the following result:

Theorem 6.1. *If k is a solvable extension of a totally real field which contains E, then the order of the pole of the L-function $L(s, (\rho(\pi)^{ss} \otimes \omega)|_{\Gamma_k})$ at $s = 2$ is equal to* $\dim_{\bar{\mathbb{Q}}_l} \mathbf{V}(\pi, k)^{'}$.

We assume for simplicity that $S_\infty - S^{'}_\infty = \{1, \tau\}$, where 1 is the trivial embedding of F in $\bar{\mathbb{Q}}$. We denote by the same symbol τ the extension of τ to $\bar{\mathbb{Q}}$. Consider

$$S = \Gamma_F \cup \Gamma_F \tau.$$

The stabilizer of S is Γ_E. It is easy to check that the stabilizer of S is equal to $(\Gamma_F\tau \cap \tau^{-1}\Gamma_F) \cup (\Gamma_F \cap \tau^{-1}\Gamma_F\tau)$. Thus we get

$$\Gamma_E = (\Gamma_F\tau \cap \tau^{-1}\Gamma_F) \cup (\Gamma_F \cap \tau^{-1}\Gamma_F\tau).$$

We distinguish two cases:

i) $\Gamma_F\tau \cap \tau^{-1}\Gamma_F = \emptyset$. Then, $\Gamma_E = \Gamma_F \cap \tau^{-1}\Gamma_F\tau$. Thus,

$$F \subset E \subset F^{gal}$$

where F^{gal} is the Galois closure of F.

If π is infinite-dimensional cuspidal automorphic representation, we denote for simplicity $\rho_\pi := \rho_{\pi,\lambda}$. Then we have (see for example [V] 2.3):

$$\rho(\pi)^{ss} \cong \rho_\pi|_{\Gamma_E} \otimes \rho_\pi|^{\tau}_{\Gamma_E},$$

where

$$\rho_\pi|^{\tau}_{\Gamma_E}(\gamma) = \rho_\pi|_{\Gamma_E}(\tau\gamma\tau^{-1}).$$

If π is one-dimensional, then $\pi(g) = \rho_\pi(N(g))|N(g)|^{1/2}$, where N is the reduced norm map and $|\ |$ denotes the ideles norm and ρ_π is a Hecke character. We denote also by ρ_π the λ-adic representation associated to ρ_π. Then

$$\rho(\pi)^{ss} \cong \rho_\pi|_{\Gamma_E} \otimes \rho_\pi|^{\tau}_{\Gamma_E}.$$

ii) $\Gamma_F\tau \cap \tau^{-1}\Gamma_F \neq \emptyset$. Let $\Gamma_{E_1} = \Gamma_F \cap \tau^{-1}\Gamma_F\tau$. Thus

$$F \subset E_1 \subset F^{gal}.$$

Since it is obvious now that $\Gamma_{E_1} \subset \Gamma_E$, $[\Gamma_E : \Gamma_{E_1}] = 2$ and $\Gamma_E \not\subseteq \Gamma_F$, we get $[E_1 : E] = 2$ and $F \not\subseteq E$.

If π is infinite-dimensional cuspidal automorphic, then we know that (see for example [V] 2.3) $\rho(\pi)^{ss}$ is a subrepresentation of

$$\mathrm{Ind}^{\Gamma_E}_{\Gamma_{E_1}}(\rho_\pi|_{\Gamma_{E_1}} \otimes \rho_\pi|^{\tau}_{\Gamma_{E_1}}),$$

which verifies

$$\rho(\pi)^{ss}|_{\Gamma_{E_1}} = \rho_\pi|_{\Gamma_{E_1}} \otimes \rho_\pi|^{\tau}_{\Gamma_{E_1}}.$$

If π is one-dimensional, then $\pi(g) = \rho_\pi(N(g))|N(g)|^{1/2}$ and we have (see for example [G], Proposition 2.7)

$$\rho(\pi)^{ss} \cong \rho_\pi|_{\Gamma_{E_1}}|_{I_E} \oplus \rho_\pi|_{\Gamma_{E_1}}|_{I_E} \cdot \omega_{E_1/E},$$

where $\omega_{E_1/E}$ is the quadratic character corresponding to E_1/E.

6.1 Non-CM semisimple Tate classes case i)

In this section we consider the case i) described above and assume that our automorphic representation π of $GL(2)/F$ is cuspidal non-CM. Thus $F \subset E \subset F^{gal}$ and

$$\rho(\pi)^{ss} \cong \rho_\pi|_{\Gamma_E} \otimes \rho_\pi|_{\Gamma_E}^\tau.$$

Let k be a solvable extension of a totally real field which contains E.

Assume that $\mathbf{V}(\pi, k)' \neq 0$. Let $x \in \mathbf{V}(\pi, k)'$, with $x \neq 0$. Recall that we denoted $K := \bar{\mathbb{Q}}^{\ker(\omega)}$ and by our assumption K is a solvable extension of a totally real field. Thus $(\rho_\pi \otimes \rho_\pi^\tau)(a)x = \xi_l^{-1}(a)x$ for $a \in \Gamma_{Kk}$. Applying the Propositions 5.1, 5.2 and 5.3, we get that $\pi^\tau \cong \pi \otimes \chi$ for some Hecke character χ of F. Therefore:

$$\rho(\pi)^{ss} \cong \rho_\pi|_{\Gamma_E} \otimes \rho_\pi|_{\Gamma_E} \otimes \chi \cong \mathrm{Sym}^2(\rho_\pi|_{\Gamma_E}) \cdot \chi \oplus \wedge^2(\rho_\pi|_{\Gamma_E}) \cdot \chi.$$

Then

$$\rho(\pi)^{ss} \otimes \omega \cong (\mathrm{Sym}^2(\rho_\pi|_{\Gamma_E}) \cdot \chi \otimes \omega) \oplus \wedge^2(\rho_\pi|_{\Gamma_E}) \cdot \chi \otimes \omega.$$

Since π is non-CM from Proposition 5.1, we know that the representation $\rho_\pi|_{\Gamma_{kK}}$ is irreducible and from Proposition 5.5, we deduce that $\mathrm{Sym}^2\rho_\pi|_{\Gamma_{kK}}$ is irreducible and thus the first factor of the above sum has no vector on which Γ_k acts by ξ_l^{-1}.

Since $\omega|_{\Gamma_k} = \sum_{i=1}^{i=k} m_i \mathrm{Ind}_{\Gamma_{F_i}}^{\Gamma_k} \chi_i$, we obtain the following result:

Proposition 6.2. *If π is cuspidal non-CM, then in case i) the dimension of $\mathbf{V}(\pi, k)'$ is equal to the multiplicity of ξ_l^{-1} in*

$$\sum_{i=1}^{i=k} m_i(\wedge^2(\rho_\pi) \cdot \chi\chi_i)|_{\Gamma_{F_i}} (\textit{virtual sum}).$$

In this proposition as in the rest of the paper by the multiplicity of ξ_l^{-1} in $\sum_{i=1}^{i=k} m_i(\wedge^2(\rho_\pi) \cdot \chi\chi_i)|_{\Gamma_{F_i}}$ we understand the sum of the m_i such that $\xi_l^{-1}|_{\Gamma_{F_i}}$ is isomorphic to $(\wedge^2(\rho_\pi) \cdot \chi\chi_i)|_{\Gamma_{F_i}}$.

From §4 we know that:

$$L(s, (\rho(\pi)^{ss} \otimes \omega)|_{\Gamma_k}) = \prod_{i=1}^{i=k} L(s, \rho(\pi)^{ss}|_{\Gamma_{F_i}} \otimes \chi_i)^{m_i}.$$

Since from §4 we know that the representation $\rho_\pi|_{\Gamma_{F_i}}$ is automorphic, we can find an automorphic representation π_i of $GL(2)/F_i$ such that $\rho_\pi|_{\Gamma_{F_i}} \cong \rho_{\pi_i}$. The representation π is non-CM and from Proposition 5.1, we get that the representation π_i is non-CM. Then

$$L(s, \rho(\pi)^{ss}|_{\Gamma_{F_i}} \otimes \chi_i) = L(s, \rho_{\pi_i} \otimes \rho_{\pi_i}^\tau \otimes \chi_i) = L(s-1, \pi_i \otimes \pi_i^\tau \otimes \chi_i).$$

Since the representations π_i and π_i^τ are cuspidal non-CM, from Proposition 5.6, we know that the L-function $L(s-1, \pi_i \otimes \pi_i^\tau \otimes \chi_i)$ has a pole of exact order 1 at $s = 2$ if and only if $\pi_i^\tau \otimes \chi_i \cong \tilde{\pi}_i \cong \pi_i \otimes \omega_{\pi_i}^{-1}$, where ω_i is the central character of π_i. Otherwise the L-function $L(s-1, \pi_i \otimes \pi_i^\tau \otimes \chi_i)$ is analytic and does not vanish at $s = 2$.

We assume now that $\pi_i^\tau \otimes \chi_i \cong \tilde{\pi}_i \cong \pi_i \otimes \omega_{\pi_i}^{-1}$. Then from Proposition 5.3 we deduce that π and π^τ are twist of each other over F. Thus $\pi^\tau \cong \pi \otimes \chi$ for some character χ.

Hence

$$\rho(\pi)^{ss} \cong \mathrm{Sym}^2(\rho_\pi|_{\Gamma_E}) \cdot \chi \oplus \wedge^2(\rho_\pi|_{\Gamma_E}) \cdot \chi.$$

and

$$L(s, (\rho(\pi)^{ss} \otimes \omega)|_{\Gamma_k}) = \prod_{i=1}^{i=k} L(s, \rho(\pi)^{ss}|_{\Gamma_{F_i}} \otimes \chi_i)^{m_i} =$$

$$\prod_{i=1}^{i=k} L(s, \mathrm{Sym}^2 \rho_{\pi_i} \otimes \chi\chi_i)^{m_i} \prod_{i=1}^{i=k} L(s, \wedge^2(\rho_{\pi_i}) \cdot \chi\chi_i)^{m_i}.$$

Since for each i the representation π_i is cuspidal non-CM, from Proposition 5.7, we know that the representation $\mathrm{Sym}^2\pi_i$ is cuspidal automorphic and the L-function $L(s, \mathrm{Sym}^2\rho_{\pi_i} \otimes \chi_i\chi)$ is analytic and does not vanish at $s = 2$.

We deduce that the pole of L-function

$$L(s, (\rho(\pi)^{ss} \otimes \omega)|_{\Gamma_k})$$

at $s = 2$ is equal to the pole of

$$\prod_{i=1}^{i=k} L(s, \wedge^2(\rho_{\pi_i}) \cdot \chi\chi_i)^{m_i}$$

at $s = 2$ which is clear equal to the multiplicity of ξ_l^{-1} in

$$\sum_{i=1}^{i=k} m_i(\wedge^2(\rho_\pi) \cdot \chi\chi_i)|_{\Gamma_{F_i}}.$$

From Proposition 6.2 we obtain that Theorem 6.1 is true in case i) and the representation π is cuspidal non-CM.

6.2 Non-CM semisimple Tate classes case ii)

In this section we consider the case ii) described above and assume that the representation π is cuspidal non-CM. Thus $F \subset E_1 \subset F^{gal}$, $[E_1 : E] = 2$, $F \not\subseteq E$ and $\rho(\pi)^{ss}$ is a subrepresentation of

$$\mathrm{Ind}_{\Gamma_{E_1}}^{\Gamma_E}(\rho_\pi|_{\Gamma_{E_1}} \otimes \rho_\pi|_{\Gamma_{E_1}}^\tau),$$

which verifies

$$\rho(\pi)^{ss}|_{\Gamma_{E_1}} \cong \rho_\pi|_{\Gamma_{E_1}} \otimes \rho_\pi|^{\tau}_{\Gamma_{E_1}}.$$

Define $F_1 := F \cap E$. Then $[F : F_1] = 2$ and τ is the identity embedding when restricted to F_1.

Let k be a solvable extension of a totally real field which contains E.

Assume that $\mathbf{V}(\pi, k)^{'} \neq 0$. Let $x \in \mathbf{V}(\pi, k)^{'}$, with $x \neq 0$. Thus $(\rho_\pi \otimes \rho_\pi^\tau)(a)x = \xi_l^{-1}(a)x$ for $a \in \Gamma_{KE_1k}$. Applying the Propositions 5.1, 5.2 and 5.3, we get that $\pi^\tau \cong \pi \otimes \alpha$ for some character α of I_F. Hence, from Proposition 5.4 we know that α is a Hecke character of I_F which is trivial on I_{F_1}. Therefore α can be written as $\alpha = \chi^\tau/\chi$ for some character χ. Hence

$$(\pi \otimes \chi^{-1})^\tau \cong \pi \otimes \chi^{-1}.$$

So $\pi \cong \pi_{0/F} \otimes \chi$, where $\pi_{0/F}$ is the base change to F of some automorphic representation π_0 of $GL(2)/F_1$.

Then from the proprieties of $\rho(\pi)^{ss}$ (see for example [MP]) we have:

$$\rho(\pi)^{ss} \cong (\mathrm{Sym}^2\rho_{\pi_0} \oplus \omega_{\pi_0} \cdot \omega_{F/F_1})|_{\Gamma_E} \otimes \chi|_{I_{F_1}}|_{\Gamma_E},$$

where ω_{π_0} is the central character of π_0 and ω_{F/F_1} is the quadratic character that corresponds to F/F_1.

Thus we get

$$\rho(\pi)^{ss} \otimes \omega \cong (\mathrm{Sym}^2\rho_{\pi_0} \otimes \chi|_{I_{F_1}} \otimes \omega)|_{\Gamma_E} \oplus (\omega_{\pi_0} \cdot \omega_{F/F_1} \cdot \chi|_{I_{F_1}} \otimes \omega)|_{\Gamma_E}.$$

Since π is non-CM, the representation $\pi_{0/F}$ is non-CM and from Proposition 5.1, we know that the representation $\rho_{\pi_0}|_{\Gamma_{kK}}$ is irreducible and from Proposition 5.5, we deduce that $\mathrm{Sym}^2\rho_{\pi_0}|_{\Gamma_{kK}}$ is irreducible and thus the first factor of the above sum has no vector on which Γ_k acts by ξ_l^{-1}.

Since $\omega|_{\Gamma_k} = \sum_{i=1}^{i=k} m_i \mathrm{Ind}_{\Gamma_{F_i}}^{\Gamma_k} \chi_i$, we obtain the following result:

Proposition 6.3. *If π is cuspidal non-CM, then in case ii) the dimension of $\mathbf{V}(\pi, k)^{'}$ is equal to the multiplicity of ξ_l^{-1} in*

$$\sum_{i=1}^{i=k} m_i(\omega_{\pi_0} \cdot \omega_{F/F_1} \cdot \chi|_{I_{F_1}}\chi_i)|_{\Gamma_{F_i}}.$$

From §4 we know that:

$$L(s, (\rho(\pi)^{ss} \otimes \omega)|_{\Gamma_k}) = \prod_{i=1}^{i=k} L(s, \rho(\pi)^{ss}|_{\Gamma_{F_i}} \otimes \chi_i)^{m_i}.$$

Let's consider one field F_i that appear in the above product. We distinguish two cases:

a) $F \subseteq F_i$. Since the representation $\rho_\pi|_{\Gamma_{F_i}}$ is automorphic, we can find an automorphic representation π_i of $GL(2)/F_i$ such that $\rho_\pi|_{\Gamma_{F_i}} \cong \rho_{\pi_i}$. The representation π is non-CM and from Proposition 5.1 we get that the representation π_i is non-CM. Then

$$L(s, \rho(\pi)^{ss}|_{\Gamma_{F_i}} \otimes \chi_i) = L(s, \rho_{\pi_i} \otimes \rho_{\pi_i}^\tau \otimes \chi_i) = L(s-1, \pi_i \otimes \pi_i^\tau \otimes \chi_i).$$

Since the representations π_i and π_i^τ are cuspidal non-CM, from Proposition 5.6, we know that the L-function $L(s-1, \pi_i \otimes \pi_i^\tau \otimes \chi_i)$ has a pole of exact order 1 at $s = 2$ if and only if $\pi_i^\tau \otimes \chi_i \cong \tilde{\pi}_i \cong \pi_i \otimes \omega_{\pi_i}^{-1}$. Otherwise the L-function $L(s-1, \pi_i \otimes \pi_i^\tau \otimes \chi_i)$ is analytic and does not vanish at $s = 2$.

We assume now that $\pi_i^\tau \otimes \chi_i \cong \tilde{\pi}_i \cong \pi_i \otimes \omega_{\pi_i}^{-1}$. Then from Proposition 5.3 we know that π and π^τ are twist of each other over F and as above, we have that $\pi \cong \pi_{0/F} \otimes \chi$ for some automorphic representation π_0 of $GL(2)/F_1$ and some Hecke character χ. From the existence of π_i, we deduce that there exists an automorphic representation $\pi_{0,i}$ of $GL(2)/F_i$ such that $\rho_{\pi_0}|_{\Gamma_{F_i}} \cong \rho_{\pi_{0,i}}$.

Hence

$$\rho(\pi)^{ss} \cong (\mathrm{Sym}^2 \rho_{\pi_0} \oplus \omega_{\pi_0} \cdot \omega_{F/F_1})|_{\Gamma_E} \otimes \chi|_{I_{F_1}}|_{\Gamma_E}$$

and

$$L(s, (\rho(\pi)^{ss} \otimes \omega)|_{\Gamma_k}) = \prod_{i=1}^{i=k} L(s, \rho(\pi)^{ss}|_{\Gamma_{F_i}} \otimes \chi_i)^{m_i} =$$

$$\prod_{i=1}^{i=k} L(s, \mathrm{Sym}^2 \rho_{\pi_{0,i}} \otimes \chi_i \chi|_{I_{F_1}}|_{\Gamma_{F_i}})^{m_i} \prod_{i=1}^{i=k} L(s, (\omega_{\pi_0} \cdot \omega_{F/F_1} \cdot \chi|_{I_{F_1}} \cdot \chi_i)|_{\Gamma_{F_i}})^{m_i}.$$

Since representation π is non-CM, the representation π_0 is non-CM and from Proposition 5.1 we have that the representation $\pi_{0,i}$ is non-CM. Then, from Proposition 5.7, we know that the representation $\mathrm{Sym}^2 \pi_{0,i}$ is cuspidal automorphic and the L-function $L(s, \mathrm{Sym}^2 \rho_{\pi_{0,i}} \otimes \chi_i \chi|_{I_{F_1}}|_{\Gamma_{F_i}})$ is analytic and does not vanish at $s = 2$.

b) $F \not\subseteq F_i$. Since the representation $\rho_\pi|_{\Gamma_{FF_i}}$ is automorphic, we can find an automorphic representation $\pi_i^{'}$ of $GL(2)/FF_i$ such that $\rho_\pi|_{\Gamma_{FF_i}} \cong \rho_{\pi_i^{'}}$ and because the representation π is non-CM, we get that the representation $\pi_i^{'}$ is non-CM. Then

$$L(s, \rho(\pi)^{ss}|_{\Gamma_{F_i}} \otimes \chi_i) L(s, \rho(\pi)^{ss}|_{\Gamma_{F_i}} \otimes \chi_i \cdot \omega_{FF_i/F_i}) =$$

$$L(s-1, \pi_i^{'} \otimes \pi_i^{'\tau} \otimes \chi_i^{'}) = L(s, \rho_{\pi_i^{'}} \otimes \rho_{\pi_i^{'}}^\tau \otimes \chi_i^{'}),$$

where ω_{FF_i/F_i} is the quadratic character of FF_i/F_i and $\chi_i^{'} = \chi_i|_{\Gamma_{FF_i}}$.

Since the representations $\pi_i^{'}$ and $\pi_i^{'\tau}$ are cuspidal non-CM the L-function $L(s-1, \pi_i^{'} \otimes \pi_i^{'\tau} \otimes \chi_i^{'})$ has a pole of order 1 at $s = 2$ if and only if $\pi_i^{'\tau} \otimes \chi_i^{'} \cong \tilde{\pi}_i^{'} \cong \pi_i^{'} \otimes \omega_{\pi_i^{'}}^{-1}$. Otherwise the L-function $L(s-1, \pi_i \otimes \pi_i^\tau \otimes \chi_i)$ is analytic and does not vanish at $s = 2$. Also the L-functions $L(s, \rho(\pi)^{ss}|_{\Gamma_{F_i}} \otimes \chi_i)$ and

$L(s,\rho(\pi)^{ss}|_{\Gamma_{F_i}} \otimes \chi_i \cdot \omega_{FF_i/F_i})$ have a pole of order at most 1 at $s=2$ and do not vanish $s=2$ (for details see [HLR] Propositions 3.11, 3.12 and 3.13).

We assume now that the L-function $L(s-1,\pi_i^{'}\otimes\pi_i^{'\tau}\otimes\chi_i^{'})$ has a pole of order 1 at $s=2$. Then, as in a) we have that $\pi\cong\pi_{0/F}\otimes\chi$ for some automorphic representation π_0 of $GL(2)/F_1$ and some character χ. From the existence of $\pi_i^{'}$, we deduce that there exists an automorphic representation $\pi_{0,i}$ of $GL(2)/F_i$ such that $\rho_{\pi_0}|_{\Gamma_{F_i}}\cong\rho_{\pi_{0,i}}$.

Thus

$$L(s,(\rho(\pi)^{ss}\otimes\omega)|_{\Gamma_k}) = \prod_{i=1}^{i=k} L(s,\rho(\pi)|_{\Gamma_{F_i}}\otimes\chi_i)^{m_i} =$$

$$\prod_{i=1}^{i=k} L(s,\mathrm{Sym}^2\rho_{\pi_{0,i}}\otimes\chi_i\chi|_{I_{F_1}}|_{\Gamma_{F_i}})^{m_i}\prod_{i=1}^{i=k} L(s,(\omega_{\pi_0}\cdot\omega_{F/F_1}\cdot\chi|_{I_{F_1}}\cdot\chi_i)|_{\Gamma_{F_i}})^{m_i}.$$

Since for each i the representation $\pi_{0,i}$ is cuspidal non-CM, we know that the representation $\mathrm{Sym}^2\pi_{0,i}$ is cuspidal automorphic and that the L-function $L(s,\mathrm{Sym}^2\rho_{\pi_{0,i}}\otimes\chi_i\chi|_{I_{F_1}}|_{\Gamma_{F_i}})$ is analytic and does not vanish at $s=2$.

From the cases a) and b), we get that the pole of L-function

$$L(s,(\rho(\pi)^{ss}\otimes\omega)|_{\Gamma_k})$$

at $s=2$ is equal to the pole of

$$\prod_{i=1}^{i=k} L(s,(\omega_{\pi_0}\cdot\omega_{F/F_1}\cdot\chi|_{I_{F_1}}\cdot\chi_i)|_{\Gamma_{F_i}})^{m_i}$$

at $s=2$ which is clear equal to the multiplicity of ξ_l^{-1} in

$$\sum_{i=1}^{i=k} m_i(\omega_{\pi_0}\cdot\omega_{F/F_1}\cdot\chi|_{I_{F_1}}\cdot\chi_i)|_{\Gamma_{F_i}}.$$

Now from Proposition 6.3 we obtain Theorem 6.1 in case ii) and π is cuspidal non-CM.

6.3 CM and one dimensional semisimple Tate classes

We assume first that our representation π is cuspidal of CM type. Thus there exists a quadratic extension M/F and an algebraic Hecke character Ω of weight one of M such that $\rho_\pi=\mathrm{Ind}_{\Gamma_M}^{\Gamma_F}\Omega$.

In the case i), from §6 we know that:

$$\rho(\pi)^{ss}\cong\rho_\pi|_{\Gamma_E}\otimes\rho_\pi^\tau|_{\Gamma_E}.$$

In the case ii), from the proprieties of $\rho(\pi)^{ss}$ we have that (see for example [MR] 6.3):

$$\Lambda^2(\mathrm{Ind}_{\Gamma_M}^{\Gamma_{F_1}}\Omega)|_{\Gamma_E} \cong \rho(\pi)^{ss} \oplus \mathrm{Ind}_{\Gamma_{E_1}}^{\Gamma_E}(\omega_\pi|_{\Gamma_{E_1}}),$$

where ω_π is the central character of π.

From these identities we get that $\rho(\pi)^{ss}|_{\Gamma_k}$ is a virtual sum of monomial representations of Γ_k. Here a monomial representation of Γ_k is a representation which is induced from a one-dimensional representation of an open subgroup.

With the notation from §4 we have:

$$L(s,(\rho(\pi)^{ss}\otimes\omega)|_{\Gamma_k}) = \prod_{i=1}^{i=k} L(s,\rho(\pi)^{ss}|_{\Gamma_{F_i}}\otimes\chi_i)^{m_i}.$$

Since $\rho(\pi)^{ss}|_{\Gamma_{F_i}}$ is sum of monomial representations, it is easy to see that the pole of $L(s,(\rho(\pi)^{ss}\otimes\omega)|_{\Gamma_k})$ at $s=2$ is equal to the dimension of the space of semisimple Tate classes $\mathbf{V}(\pi,k)^{'}$.

The same argument works when π is one-dimensional, since in this case, we know from the beginning of §6 that $\rho(\pi)^{ss}|_{\Gamma_k}$ is a sum of one-dimensional representations.

From this section and sections 6.1 and 6.2 where we treated the non-CM case, we obtain Theorem 6.1. Actually Theorem 6.1 is true for an arbitrary extension k of E and a general Artin representation ω if π is one-dimensional or of CM type, because in these cases the base change of π to an arbitrary extension k/F exists.

6.4 Tate classes

Using the notations from the begining of §6, we conjecture that:

Conjecture 6.4. *If k is a solvable extension of a totally real field which contains E, then the order of the pole of the L-function $L(s,(\rho(\pi)\otimes\omega)|_{\Gamma_k})$ at $s=2$ is equal to $\dim_{\bar{\mathbb{Q}}_l}\mathbf{V}(\pi,k)$.*

As we remarked just before Theorem 6.1, the orders of the poles at $s=2$ of $L(s,(\rho(\pi)^{ss}\otimes\omega)|_{\Gamma_k})$ and $L(s,(\rho(\pi)\otimes\omega)|_{\Gamma_k})$ are equal. From Theorem 6.1, we get that in order to prove the Conjecture 6.4 one has to show that

$$\dim_{\bar{\mathbb{Q}}_l}\mathbf{V}(\pi,k) = \dim_{\bar{\mathbb{Q}}_l}\mathbf{V}(\pi,k)^{'}.$$

Using Brauer's induction as in sections 6.1, 6.2 and 6.3, we obtain that this identity is a consequence of the following:

Conjecture 6.5. *If $k^{'}/E$ is a finite Galois extension and χ is a finite order Hecke character of $k^{'}$, then*

$$\dim_{\bar{\mathbb{Q}}_l}((\rho(\pi)\otimes\chi)(1))^{\Gamma_{k'}} = \dim_{\bar{\mathbb{Q}}_l}((\rho(\pi)^{ss}\otimes\chi)(1))^{\Gamma_{k'}}.$$

It is conjectured that the representation $\rho(\pi)$ is semisimple, and thus the above results should be trivially true. If $k^{'}$ is as above, then representation $\rho(\pi)|_{\Gamma_{k'}}$ is irreducible iff $\rho(\pi)^{ss}|_{\Gamma_{k'}}$ is irreducible. Therefore, if $\rho(\pi)^{ss}|_{\Gamma_{k'}}$ is irreducible, the Conjecture 6.5 is true.

Remark 6.6. *The results in this article remain true if we replace the prime ideal* $\wp$ *of* O_F *by an arbitrary ideal* $\mathbf{n}$ *of* O_F *such that each prime* $\wp|\mathbf{n}$ *is unramified in* D*.*

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