ZETA FUNCTIONS OF TWISTED QUATERNIONIC SHIMURA VARIETIES

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June 9, 2005

1 Introduction

In this article we compute the zeta function for twisted quaternionic Shimura varieties and prove the meromorphic continuation of the zeta function under certain assumptions. In [BL], the authors computed the zeta function for quaternionic Shimura varieties associated to a totally indefinite quaternion algebra D over a totally real field F i.e. all the infinite places of F are unramified in D. In his book [R], Reimann generalized the result in [BL] to indefinite quaternion algebras D.

Let F be a totally real field, $O := O_F$ the ring of integers of F and D an indefinite quaternion algebra, central over F. Let G be the algebraic group over F defined by the multiplicative group $D^\times$ of D and $\bar{G} = \mathrm{Res}_{F/\mathbb{Q}}(G)$. We consider a prime ideal $\wp$ of O_F, such that $G(F_\wp)$ is isomorphic to $GL_2(F_\wp)$. Let $S_{\bar{G},\hat{\Gamma}(\wp)} = S_{\hat{\Gamma}(\wp)}$ be the canonical model of the quaternionic Shimura variety associated to the adelic principal congruence subgroup $\hat{\Gamma}(\wp)$ of $\bar{G}(\mathbb{A}_f)$ of level $\wp$, where $\mathbb{A}_f$ is the finite adeles ring of $\mathbb{Q}$. Then $S_{\hat{\Gamma}(\wp)}$ is a quasi-projective variety defined over a finite extension $E/\mathbb{Q}$ called the canonical field of definition.

The variety $S_{\hat{\Gamma}(\wp)}$ has a natural action of $GL_2(O/\wp)$ (see section 1.2). If K is a number field, we denote $\Gamma_K := \mathrm{Gal}(\bar{\mathbb{Q}}/K)$. Consider a continuous Galois representation $\varphi : \Gamma_E \to GL_2(O/\wp)$ and let $S'_{\hat{\Gamma}(\wp)}$ be the variety defined over E obtained from $S_{\hat{\Gamma}(\wp)}$ via twisting by φ composed with the natural action of $GL_2(O/\wp)$ on $S_{\hat{\Gamma}(\wp)}$ (see section 1.2 for details).

From [R] Corollary 11.8 (see also Proposition 1.7 below) we know that the local factors of the semisimple zeta function $L^{ss}(s, S_{\hat{\Gamma}(\wp)})$ of $S_{\hat{\Gamma}(\wp)}$ are given by the formula (see 1.1 for notations):

$$L_p^{ss}(s, S_{\hat{\Gamma}(\wp)}) = \prod_\pi L_p^{ss}(s - d'/2, \pi, r)^{m(\pi_\infty)m(\pi_f^{\hat{\Gamma}(\wp)})}.$$

Here the product is taken over cuspidal holomorphic automorphic representations π of $\bar{G}(\mathbb{A}_{\mathbb{Q}})$ of weight 2, $d^{'}$ is the dimension of $S_{\hat{\Gamma}(\wp)}$, $m(\pi_f^{\hat{\Gamma}(\wp)})$ is the dimension of $\pi_f^{\hat{\Gamma}(\wp)}$ ($\pi_f^{\hat{\Gamma}(\wp)}$ denotes the subspace of invariants of $\hat{\Gamma}(\wp)$ in π_f) , r is a well specified representation of the Langlands group ${}^L\bar{G}$ associated to $\bar{G}$ (see section 1.1 for the definition of r) and $m(\pi_\infty)$ will be defined in section 1.1.

In this paper we prove the following result (see section 1.1 for notations):

Theorem 1.1. *The local factors of the semisimple zeta function $L^{ss}(s, S^{'}_{\hat{\Gamma}(\wp)})$ of $S^{'}_{\hat{\Gamma}(\wp)}$ are given by the formula:*

$$L_p^{ss}(s, S^{'}_{\hat{\Gamma}(\wp)}) = \prod_{\pi} L_p^{ss}(s - d^{'}/2, \pi, r \otimes (\pi_f^{\hat{\Gamma}(\wp)} \circ \varphi))^{m(\pi_\infty)},$$

where the product is taken over cuspidal holomorphic automorphic representations π of $\bar{G}(\mathbb{A}_{\mathbb{Q}})$ of weight 2, such that $\pi_f^{\hat{\Gamma}(\wp)} \neq 0$.

If $d^{'} = 1$ or 2 and the field $L := \bar{\mathbb{Q}}^{Ker(\varphi)}$ is a solvable extension of a totally real field, then zeta function $L^{ss}(s, S^{'}_{\hat{\Gamma}(\wp)})$ can be meromorphically continued to the whole complex plane and verifies a functional equation.

The first part of this theorem is proved in section 1.3 by taking the injective limit of the double representations of $\Gamma_E \times \mathbb{H}_K$ on the étale cohomology of Shimura varieties S_K that appear in Proposition 1.5 below and using some linear algebra. Here K is an open compact subgroup $\bar{G}(\mathbb{A}_f)$ and $\mathbb{H}_K$ is the Hecke algebra of convolutions of bi-K-invariant compactly supported functions on $\bar{G}(\mathbb{A}_f)$. In our argument the strong multiplicity one for $\bar{G}$ is important.

The second part of Theorem 1.1 regarding the meromorphic continuation of the zeta function $L^{ss}(s, S^{'}_{\hat{\Gamma}(\wp)})$ is proved in section 2. We prove (see Theorem 2.17) that if $d^{'} = 1$ or 2 and π is a representation as in the product of Theorem 1.1 and ω is an Artin representation of Γ_E such that the field $K := \bar{\mathbb{Q}}^{\mathrm{Ker}(\omega)}$ is a solvable extension of a totally real field, then the L-function $L^{ss}(s, \pi, r \otimes \omega)$ can be meromorphically continued to the whole complex plane and verifies a functional equation. We prove also the meromorphic continuation and functional equation of $L^{ss}(s, S^{'}_{\hat{\Gamma}(\wp)})$ when $d^{'} \geq 3$ and the field $L := \bar{\mathbb{Q}}^{\mathrm{Ker}(\varphi)}$ is a solvable extension of a totally real field, if we assume that some other Langlands L-functions can be meromorphically continued to the whole complex plane and verify a functional equation (see Lemma 2.1 and Remark 2.18 for details). We remark that in order to prove the meromorphic continuation of $L^{ss}(s, \pi, r \otimes \omega)$ we use a different technique then in [V], since the positivity of the density of the ordinary rational primes for Hilbert modular forms is not known. This result is known for classical modular forms over $\mathbb{Q}$ and it was used in [V]. To prove the meromorphic continuation we follow some ideas from [T2] and Fujiwara's deformation theory [F].

We remark that when $D = M_2(F)$ the Shimura variety is not proper and in this case we use the l-adic intersection cohomologies of the Baily-Borel compactification of the Shimura variety.

In a recent preprint, Blasius (see [B]) computed the zeta function of quaternionic Shimura varieties at all places. Using his result, the zeta function of twisted quaternionic Shimura varieties is computed at all places.

In this paper all the automorphic representations π are cuspidal holomorphic. If π is an automorphic representation of $\bar{G}(\mathbb{A}_{\mathbb{Q}})$, we denote the cuspidal representation of $GL_2(\mathbb{A}_F)$ ($\mathbb{A}_F$ is the adeles ring of F), obtained from π by Jacquet-Langlands correspondence (usually denoted $JL(\pi)$) by the same symbol π.

1.1 Zeta function for quaternionic Shimura varieties

In this section we shall expose the computation of the zeta function for quaternionic Shimura varieties following closely [RT].

Let v be a place of a field L and L_v the completion of L at v. If v is a finite, we denote by O_v the completion of the ring of integers O_L at v. Let Nv be the cardinality of the residue field k_v of O_v. Let $\mathbb{A}_f = \mathbb{Z}_f \otimes \mathbb{Q}$ be the finite adeles ring of $\mathbb{Q}$ and $\mathbb{A} = \mathbb{R} \times \mathbb{A}_f$ the adeles ring of $\mathbb{Q}$. Denote the adeles ring of F by $\mathbb{A}_F$.

Consider a totally real number field F of degree d over $\mathbb{Q}$ and D be a quaternion algebra over F. Thus D is a central simple algebra of rank 4 over F. We denote by S_∞ the set of infinite places of F and we identify S_∞ as a $\Gamma_{\mathbb{Q}}$-set with $\Gamma_F \setminus \Gamma_{\mathbb{Q}}$. Let S'_∞ be the subset of S_∞ at which D is ramified. Let d' =the cardinal of $S_\infty - S'_\infty$. We assume $d' > 0$, i.e. D is indefinite over F.

Let G be the algebraic group over F defined by the multiplicative group $D^\times$ of D. By restricting the scalars, we obtain the algebraic group $\bar{G} = \mathrm{Res}_{F/\mathbb{Q}}(G)$ over $\mathbb{Q}$ defined by the propriety: $\bar{G}(A) = G(A \otimes_{\mathbb{Q}} F)$ for all $\mathbb{Q}$-algebras A. The Langlands group associated to $\bar{G}$ is defined by the semidirect product:

$$^L\bar{G} = ^L\bar{G}^0 \rtimes \Gamma_{\mathbb{Q}}$$

where $^L\bar{G}^0$ is the product of d copies of $GL_2(\mathbb{C})$ indexed by elements $\sigma \in \Gamma_F \setminus \Gamma_{\mathbb{Q}}$ and $\Gamma_{\mathbb{Q}}$ acts on $^L\bar{G}^0$ by permuting the factors in the natural way. It is easy to see that $\bar{G}(\mathbb{R})$ is isomorphic to $GL_2(\mathbb{R})^{d'} \times \mathbf{H}^{*(d-d')}$, where $\mathbf{H}$ is the algebra of quaternions over $\mathbb{R}$.

We now define holomorphic cuspidal representations of $\bar{G}$ of weight $k = (k_v)(v \in S_\infty)$ for $k_v = k \geqq 2$ an integer. Identify $\mathbb{C}^\times$ with a subgroup of $GL_2(\mathbb{R})$ through the map

$$(a + bi) \mapsto \left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right).$$

We consider the following representations:

i) For $k \geqq 2$, let π_k be the unique discrete series representation of $GL_2(\mathbb{R})$ such that under the action of $\mathbb{C}^\times$, π_k is the direct sum of eigenspaces for the characters $\lambda^{-k}(\lambda\bar{\lambda})^{1/2}(\bar{\lambda}/\lambda)^{(n-k)/2}$ for $n \in \mathbb{Z}$ such that $|n| \geqq k$ and $n \equiv k(\mathrm{mod}2)$.

ii) For $k \geqq 2$, let $\tilde{\pi}_k$ be the unique irreducible representation of the multiplicative group of the quaternions over $\mathbb{R}$ of dimension $(k-1)$ such that the center $\mathbb{R}^\times$ acts by the character $x \mapsto x^{-k}|x|$.

An automorphic representation π of $\bar{G}$ has the form $\pi = \otimes \pi_v$, where the restricted tensor product is taken over all places v of F and π_v is a representation of $G(F_v)$.

Definition 1.2. *A cuspidal representation $\pi = \otimes \pi_v$ of $\bar{G}$ is called holomorphic discrete series of weight $k \geqq 2$ (k is an integer) if:*

$$\pi_v \sim \begin{cases} \pi_k & \text{for } v \in S_\infty - S_\infty^{'}; \\ \tilde{\pi}_k & \text{for } v \in S_\infty^{'}. \end{cases}$$

The local L-factors associated to representations of $\bar{G}(\mathbb{A})$ are defined as follows. Let v be a finite place of F. An irreducible representation π_v of $GL_2(F_v)$ is called unramified if it contains a nonzero vector which is fixed under $GL_2(O_v)$. One can define a semi-simple conjugacy class $\{g(\pi_v)\}$ in $GL_2(\mathbb{C})$ for each unramified π_v as follows. For $s_1, s_2 \in \mathbb{C}^\times$, let $\psi(s_1, s_2)$ be the character:

$$\psi(s_1, s_2) : \left(\begin{smallmatrix} a & c \\ 0 & d \end{smallmatrix}\right) \mapsto s_1^{val(a)} s_2^{val(d)}$$

of the Borel subgroup $\{\left(\begin{smallmatrix} a & c \\ 0 & d \end{smallmatrix}\right) \in GL_2(F_v)\}$. It is well known that for each unramified π_v, there exists $(s_1, s_2) \in (\mathbb{C}^\times)^2$, unique up to order, such that π_v is the unique unramified constituent of the representation unitarily induced from $\psi(s_1, s_2)$. Set

$$g(\pi_v) = \left(\begin{smallmatrix} s_1 & 0 \\ 0 & s_2 \end{smallmatrix}\right).$$

The local L-factor $L(s, \pi_v)$ associated to π_v is

$$L(s, \pi_v) = \det(1 - Nv^{-s} g(\pi_v))^{-1}.$$

For p a rational prime, $\bar{G}(\mathbb{Q}_p) = \prod_{v|p} G(F_v)$. A representation $\pi_p = \otimes_{v|p} \pi_v$ of $\bar{G}(\mathbb{Q}_p)$, where π_v is a representation of $G(F_v)$, is called unramified if p is unramified in F and for all $v|p$, $G(F_v)$ is isomorphic to $GL_2(F_v)$ and π_v is unramified. If π_v is unramified we associate an element $g(\pi_p) \in {}^L\bar{G}$ to π_p as follows. For $\sigma \in \Gamma_F \setminus \Gamma_\mathbb{Q}$, set

$$g(\pi_p)_\sigma = \begin{pmatrix} s_1^{1/e_v} & 0 \\ 0 & s_2^{1/e_v} \end{pmatrix}$$

where $g(\pi_v) = \left(\begin{smallmatrix} s_1 & 0 \\ 0 & s_2 \end{smallmatrix}\right)$, $e_v = [F_v : \mathbb{Q}_p]$, and σ corresponds to v. Define:

$$g(\pi_p) = (g(\pi_p)_\sigma) \rtimes \phi_p \in {}^L\bar{G},$$

where $\phi_p \in \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ is a geometric Frobenius.

For every representation $r : {}^L\bar{G} \to GL_n(\mathbb{C})$, the L-function $L(s, \pi, r)$ is defined for a representation $\pi = \bigotimes \pi_p = \bigotimes \pi_v$ of $\bar{G}(\mathbb{A})$, where π_p is a representation of $\bar{G}(\mathbb{Q}_p)$ and π_v denotes a representation of $G(F_v)$. We recall that $L(s, \pi, r) = \prod_p L(s, \pi_p, r)$ and for almost all p, π_p is unramified and

$$L(s, \pi_p, r) = \det(1 - p^{-s} r(g(\pi_p)))^{-1}.$$

The standard L-function $L(s,\pi)$ is an Euler product $\prod_v L(s,\pi_v)$, where:

$$L(s,\pi_v) = \det(1 - Nv^{-s} g(\pi_v))^{-1}$$

if π_v is unramified.

Let ${}^L\bar{T}^0$ be the subgroup of ${}^L\bar{G}^0$ of elements (t_σ) such that t_σ is diagonal for all σ and let ν be the character of ${}^L\bar{T}^0$ defined by

$$\nu((t_\sigma)) = \prod \nu_\sigma(t_\sigma),$$

$$\nu_\sigma(\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)) = \begin{cases} a & \text{for } \sigma \in S_\infty - S'_\infty; \\ 1 & \text{for } \sigma \in S'_\infty. \end{cases}$$

The subfield E of $\bar{\mathbb{Q}}$ such that Γ_E is the stabilizer of the subset $S'_\infty \subset \Gamma_F \backslash \Gamma_{\mathbb{Q}}$ is called the canonical field of definition, and Γ_E stabilizes the character ν.

Let r^0 be the finite dimensional representation of ${}^L\bar{G}^0$ whose highest weight with respect to the standard Borel subgroup is ν. Since Γ_E stabilizes ν, there is a unique extension of r^0 to ${}^L\bar{G}^0 \rtimes \Gamma_E$, also denoted by r^0, such that Γ_E acts as the identity on the ν-weight space. Set

$$r = \mathrm{Ind}_{{}^L\bar{G}^0 \rtimes \Gamma_E}^{{}^L\bar{G}} r^0.$$

The dimension of r is $2^{d'}[E:\mathbb{Q}]$.

If $\omega : \Gamma_E \to GL_m(\mathbb{C})$ is an Artin representation then we denote by the same symbol the representation of ${}^L\bar{G}^0 \rtimes \Gamma_E$ that extends ω and restricts to the trivial representation on ${}^L\bar{G}^0$. We define $L(s,\pi, r\otimes\omega) = \prod_p L(s,\pi_p, r\otimes\omega)$, where if π_p is unramified and $\omega(I_{\mathbf{p}}) = 1$ for all prime ideals $\mathbf{p}|p$ in E ($I_{\mathbf{p}} \subseteq \Gamma_E$ is the inertia group at $\mathbf{p}$), then

$$L(s,\pi_p, r\otimes\omega) = \det(1 - p^{-s}\mathrm{Ind}_{{}^L\bar{G}^0 \rtimes \Gamma_E}^{{}^L\bar{G}}(r^0\otimes\omega)(g(\pi_p)))^{-1}.$$

The embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$ defines a bijection between primes $\mathbf{p}$ of E dividing p and the set of representatives $\{\sigma\}$ of the double coset space

$$\Gamma_E \backslash \Gamma_{\mathbb{Q}} / \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p).$$

For such σ, set $G_{\mathbf{p}} = \sigma^{-1}\mathrm{Gal}(\bar{\mathbb{Q}}/E)\sigma \cap \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ where σ corresponds to $\mathbf{p}$. The restriction of r to ${}^L\bar{G}^0 \rtimes \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ is isomorphic to

$$\oplus_{\mathbf{p}|p} r_{\mathbf{p}}$$

where $r_{\mathbf{p}}$ is the representation of ${}^L\bar{G}^0 \rtimes \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ obtained by induction from a representation $r^0 \circ \mathrm{Ad}\sigma$ of ${}^L\bar{G}^0 \rtimes G_{\mathbf{p}}$.

For $v \in S_\infty - S'_\infty$, we fix an isomorphism of $G(F_v)$ with $GL_2(\mathbb{R})$. We have $\bar{G}(\mathbb{R}) = \prod_{v\in S_\infty} G(F_v)$. Let $J = (J_v) \in \bar{G}(\mathbb{R})$, where

$$J_v = \begin{cases} 1 & \text{for } v \in S'_\infty; \\ 1/\sqrt{2}\left(\begin{smallmatrix} 1 & 1 \\ -1 & 1 \end{smallmatrix}\right) & \text{for } v \in S_\infty - S'_\infty. \end{cases}$$

Let K_∞ be the centralizer of J in $\bar{G}(\mathbb{R})$. Set

$$X = \bar{G}(\mathbb{R})/K_\infty.$$

It is well known that X is complex analitically isomorphic to $(h_\pm)^{d'}$ where $h_\pm = \mathbb{C} - \mathbb{R}$. For each open compact subgroup $K \subseteq \bar{G}(\mathbb{A}_f)$ set

$$S_K(\mathbb{C}) = \bar{G}(\mathbb{Q}) \setminus X \times \bar{G}(\mathbb{A}_f)/K.$$

For K sufficiently small, $S_K(\mathbb{C})$ is a complex manifold which is the set of complex points of a quasi projective variety. It is known that $S_K(\mathbb{C})$ has a canonical model over E which is denoted by S_K. Actually this model can be defined over $O_E[1/N]$ for some integer N, where O_E is the ring of integers of E. The dimension of S_K is d'.

The center $\bar{Z}$ of $\bar{G}$ is isomorphic to $\mathrm{Res}_{F/\mathbb{Q}}(G_m)$. Let $\bar{Z}_0(\mathbb{Q})$ be the kernel of the norm map $\mathrm{Res}_{F/\mathbb{Q}}(G_m) \to G_m$. Let M be a finite extension of $\mathbb{Q}$ in $\bar{\mathbb{Q}}$, let $\hat{O}_M = \mathbb{Z}_f \otimes O_M$ and let ξ be an irreducible algebraic representation of $\bar{G}$, defined over a finite extension M of $\mathbb{Q}$, on an M-vector space $V(M)$ whose kernel contains $\bar{Z}_0(\mathbb{Q})$. Then $\bar{G}(\mathbb{A}_f)$ acts on $V(M) \otimes \mathbb{A}_f$. Choose an open compact $O_{\hat{M}}$-submodule $V(\hat{O}_M)$ of $V(M) \otimes \mathbb{A}_f$ which is stable under K and put

$$V(O_M) = V(M) \cap V(\hat{O}_M).$$

For any O_M-module A, let $V(A) = V(O_M) \otimes_{O_M} A$. Let $g = (g_\infty, g_f) \in \bar{G}(\mathbb{A})$ and set:

$$gV(O_M) = V(M) \cap \xi(g_f)V(\hat{O}_M).$$

Then $h \in \bar{G}(\mathbb{Q})$ acts on

$$\bigcup_{g \in \bar{G}(\mathbb{A})} (gV(O_M) \times \bar{G}(\mathbb{R})gK/K_\infty K)$$

by sending (v, y) to $(\xi(h)v, hy)$. For K sufficiently small, $\gamma \in \bar{G}(\mathbb{Q})$ has a fixed point in $X \times \bar{G}(\mathbb{A}_f)/K$ only if $\gamma \in \bar{Z}_0(\mathbb{Q})$. Since $\bar{Z}_0(\mathbb{Q})$ acts trivially on $V(O_M)$, the quotient of the above union by the left action of $\bar{G}(\mathbb{Q})$ defines a locally constant sheaf $F_\xi^K(O_M)$.

Fix a prime number l and let λ be the place of M defined by $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_l$. For each positive integer n let $O_\lambda(n)$ be the quotient of O_λ by the n^{th} power of its maximal ideal and let K_n be the open normal subgroup of K such that K_n acts trivially on $V(O_\lambda(n))$. Then S_{K_n} is a finite Galois *étale* covering of S_K with Galois group K/K_n. Then

$$V(O_\lambda(n)) \times_{K/K_n} S_{K_n}$$

defines a sheaf $F_\xi^K(O_\lambda(n))$ in the *étale* topology over S_K. The inverse limit over n defines an O_λ-sheaf $F_\xi^K(O_\lambda)$ and an M_λ-sheaf

$$F_\xi^K(M_\lambda) = F_\xi^K(O_\lambda) \otimes \mathbb{Q}.$$

Let ν be the central character of ξ and let $\bar{Z}^0(\mathbb{R})$ be the connected component of $\bar{Z}(\mathbb{R})$. Let $L^2(\xi)$ be the space of measurable functions ϕ on $\bar{G}(\mathbb{Q})\setminus\bar{G}(\mathbb{A})$ such that:
i)$\phi(zg) = \nu^{-1}(z)\phi(g)$ for $z \in \bar{Z}^0(\mathbb{R})$,
ii)$|\nu(\det(g))|^{1/2}\phi(g)$ is square-integrable on $\bar{Z}^0(\mathbb{R})\bar{G}(\mathbb{Q}) \setminus \bar{G}(\mathbb{A})$.
Let $L^2_0(\xi)$ be the space of cusp forms in $L^2(\xi)$ and let $A_0(\xi)$ be the set of cuspidal automorphic representations which occur in $L^2_0(\xi)$. Each $\pi \in A_0(\xi)$ can be written as a tensor product $\pi = \pi_f \otimes \pi_\infty$ where $\pi_\infty = \otimes_{v\in S_\infty}\pi_v$ is a representation of $\bar{G}(\mathbb{R})$ and $\pi_f = \otimes_{v<\infty}\pi_v$ is a representation of $\bar{G}(\mathbb{A}_f)$.

Consider ξ as a complex representation of $\bar{G}(\mathbb{R})$ and let $\mathrm{Coh}(\xi)$ be the set of $\pi \in A_0(\xi)$ such that the relative Lie algebra cohomology $H^*(g, K_\infty, \pi_\infty \otimes \xi) \neq 0$ (the cohomology is taken with respect to (g, K_∞)-module associated to π_∞). Denote the space of K-invariant vectors in π_f by π_f^K.

Proposition 1.3. *The de Rham cohomology $H^*_{DR}(S_K(\mathbb{C}), F^K_\xi(\mathbb{C}))$ is isomorphic to*

$$\oplus_{\pi\in Coh(\xi)} H^*(g, K_\infty, \pi_\infty \otimes \xi) \otimes \pi_f^K.$$

The center $\bar{Z}(\mathbb{A})$ will be identify with I_F, the ideles of F. Let $\alpha = \prod \alpha_v$ be the normalized absolute value on I_F.

For $k \geqq 2$, let $\xi_k^{'}$ be the representation of $GL(2)$ of the dimension $(k-1)$ on the $(k-2)^{\mathrm{nd}}$ symmetric power of the standard representation of $GL(2)$. Fix a finite extension $M/\mathbb{Q}$ over which $\bar{G}$ splits and let ξ_k be the representation $\xi_k^{'} \otimes ... \otimes \xi_k^{'}$ (d times) of $\bar{G}/M$ obtained by picking an isomorphism of $\bar{G}/M$ with $GL(2) \times ... \times GL(2)$ (d times). Since $\bar{Z}_0(\mathbb{Q})$ acts trivially through ξ_k, ξ_k defines locally constant sheaf $F^K_{\xi_k}$ on S_K.

We treat ξ_k as a representation of $\bar{G}(\mathbb{R})$. The center of $\bar{G}(\mathbb{R})$ is $(F \otimes \mathbb{R})^\times$ and the central character of ξ_k is Nm^{k-2} where $\mathrm{Nm} : (F \otimes \mathbb{R})^\times \to \mathbb{R}^\times$ is the norm map. For $v \in S_\infty - S_\infty^{'}$ the weights of $\mathbb{C}^\times \subset G(F_v)$ in ξ_k are $\lambda^{k-2}(\lambda/\bar{\lambda})^{-j}$ for $j = 0, 1, ..., k-2$.

Proposition 1.4. *Let $\pi = \pi_f \otimes \pi_\infty \in Coh(\xi_k)$. Then for $k \geqq 2$:*
a) If π_∞ is infinite dimensional, then $H^*(g, K_\infty, \pi_\infty \otimes \xi_k)$ is zero unless $\pi_v \sim \alpha_v^{1/2} \otimes \pi_k$ (resp. $\pi_v \sim \alpha_v^{1/2} \otimes \tilde{\pi}_k$) for $v \in S_\infty - S_\infty^{'}$ (resp. $v \in S_\infty^{'}$). In this case,

$$dim H^q(g, K_\infty, \pi_\infty \otimes \xi_k) = \begin{cases} 2^{d'} & \text{for } q = d^{'}; \\ 0 & \text{for } q \neq d^{'}. \end{cases}$$

b) If π_∞ is one dimensional, then $H^*(g, K_\infty, \pi_\infty \otimes \xi_k)$ is zero unless $k = 2$. In this case,

$$dim H^q(g, K_\infty, \pi_\infty \otimes \xi_k) = \begin{cases} \binom{d'}{q'} & \text{for } q = 2q^{'}; \\ 0 & \text{for } q \text{ odd}. \end{cases}$$

Assume that the local and the global Haar measures have been fixed. For $g \in \bar{G}(\mathbb{A}_f)$, let

$$f_g = \mathrm{char}(KgK)/\mathrm{meas}(K)$$

where $\mathrm{char}(KgK)$ is the characteristic function of KgK. For a algebra A, let $\mathbb{H}_K(A)$ be the A-algebra generated by the f_g under the convolution and denote $\mathbb{H}_K(\mathbb{C})$ by $\mathbb{H}_K$. We assume that $K = \Pi_{v<\infty} K_v$ where K_v is open compact in $G(F_v)$ and $K_v = G(O_v)$ for almost all v. Then $\mathbb{H}_K$ acts on the decomposition from Proposition 1 through its action on the factors π_f^K and the $\mathbb{H}_K$-representations π_f^K for $\pi \in \mathrm{Coh}(\xi_k)$ are irreducible and mutually inequivalent, i.e. the decomposition from Proposition 1.3 is isotypic with respect to the action of $\mathbb{H}_K$.

Proposition 1.5. *There exist:*
i) A finite extension L/M contained in $\bar{\mathbb{Q}}$
ii) for each $\pi \in Coh(\xi_k)$ such that $\pi_f^K \neq 0$
a) an L-representation $\rho(\pi, K)$ of $\mathbb{H}_K(L)$ such that $\rho(\pi, K) \otimes \mathbb{C}$ is isomorphic to π_f^K
b) an L_λ representation $\sigma^q(\pi)$ of Γ_E, where λ is the place of L defined by $L \to \bar{\mathbb{Q}}_l$, of dimension equal to dim $H^q(g, K_\infty, \pi_\infty \otimes \xi_k)$ such that the double representation of $\Gamma_E \times \mathbb{H}_K(L_\lambda)$ on $H^q_{et}(S_K, F^K_{\xi_k}(M_\lambda)) \otimes L_\lambda$ is isomorphic to the direct sum

$$\oplus_{\pi\in Coh(\xi_k),\pi_f^K\neq 0}\sigma^q(\pi)\otimes(\rho(\pi,K)\otimes L_\lambda)$$

Let $\mathbb{H}_p$ denote the Hecke algebra of $\bar{G}(\mathbb{Q}_p)$ of compactly supported bi-K_p-invariant functions. By the Satake isomorphism, there is, for each positive integer j, a unique function $f^j_{\mathbf{p}} \in \mathbb{H}_p$ such for all unramified representations π_p of $\bar{G}(\mathbb{Q}_p)$:

$$\mathrm{Tr}(\pi_p(f^j_{\mathbf{p}})) = p^{d' j e_{\mathbf{p}}/2}\mathrm{Tr}(r_{\mathbf{p}}(g(\pi_p)^{je_{\mathbf{p}}})).$$

It is known that there exists a function $f_\infty = \prod_{v\in S_\infty} f_v$ in $\mathbb{C}^\infty(\bar{G}(\mathbb{R}))$ whose support is compact modulo $\bar{Z}(\mathbb{R})$ such that for all Cartan subgroups $\bar{T}(\mathbb{R})$ of $\bar{G}(\mathbb{R})$ and regular elements $\gamma \in \bar{T}(\mathbb{R})$:

$$\int_{\bar{T}(\mathbb{R})\backslash\bar{G}(\mathbb{R})} f_\infty(g^{-1}\gamma g)dg = \begin{cases} m(\bar{Z}(\mathbb{R})\setminus\bar{T}(\mathbb{R}))^{-1}\mathrm{Tr}(\xi_k(\gamma)) & \bar{T}(\mathbb{R}) \text{ elliptic} \\ 0 \text{ if } \bar{T}(\mathbb{R}) \text{ is not elliptic.} \end{cases}$$

For $g \in \bar{G}(\mathbb{A}_f)$, let $T(g)$ denote the operator on $H^*_{et}(S_K, F^K_{\xi_k}(M_\lambda))$ corresponding to f_g. We assume that the extension L of the previous proposition is sufficiently large and denote by $F_{\xi_k}(L_\lambda)$ the sheaf $F_{\xi_k}(M_\lambda) \otimes L_\lambda$ where λ is a place of L induced by $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_l$.

Let $\pi = \pi_f \otimes \pi_\infty \in A_0(\xi_k)$. It is easy to see that $\mathrm{Tr}(\pi_\infty(f_\infty)) = 0$ if $\pi \notin \mathrm{Coh}(\xi_k)$ and if $\pi \in \mathrm{Coh}(\xi_k)$ then:

$$\mathrm{Tr}(\pi_\infty(f_\infty)) = \begin{cases} (-1)^{d'} & \text{if } \pi_\infty \text{ is infinite-dimensional;} \\ 1 & \text{if } k = 2 \text{ and } \pi_\infty \text{ is one-dimensional;} \\ 0 & \text{if } k > 2 \text{ and } \pi_\infty \text{ is one dimensional.} \end{cases}$$

Let $m(\pi_\infty) = \mathrm{Tr}(\pi_\infty(f_\infty))$.

If ϕ is an endomorphism of the cohomology $H^*_{et}(S_K, F^K_{\xi_k}(L_\lambda))$ which acts on each vector space H^i_{et} separately, set:

$$\mathrm{Tr}(\phi|H^*_{et}(S_K, F^K_{\xi_k}(L_\lambda))) = \sum_i (-1)^i \mathrm{Tr}(\phi|H^i_{et}(S_K, F^K_{\xi_k}(L_\lambda))).$$

Let ρ_k denote the representation of $\bar{G}(\mathbb{A})$ on $L^2(\xi_k)$.

Proposition 1.6. *For all sufficiently small open compact subgroups $K \subseteq \bar{G}(\mathbb{A}_f)$, there is a finite set S such that for all rational primes p prime to the elements of S, primes $\mathbf{p}$ of E dividing p, and $g \in \bar{G}(\mathbb{A}^p_S)$:*
a) S_K has good reduction at $\mathbf{p}$
b) For $\phi_{\mathbf{p}} \in \Gamma_E$ a Frobenius element for $\mathbf{p}$:

$$e_{\mathbf{p}}\, Tr(\phi^j_{\mathbf{p}} \times T(g)|H^*_{et}(S_K, F^K_{\xi_k}(L_\lambda))) = Tr(\rho_k(f_\infty \times f_g \times f^j_{\mathbf{p}})).$$

By the strong multiplicity one and multiplicity one theorems for $\bar{G}$, an automorphic representation π is determined by $\pi^p_S = \otimes_{l\neq p, l\notin S}\pi_l$ and the representation of the Hecke algebra $\otimes_{l\neq p,l\notin S}\mathbb{H}_l$ on π^p_S for $\pi \in \mathrm{Coh}(\xi_k)$ are mutually inequivalent. Hence there exists a function f_π of the form:

$$f_\pi = \sum_{j=1}^{t} a_j f_{g_j}, a_j \in L, g_j \in \bar{G}(\mathbb{A}^p_S)$$

such that f_π acts by the scalar one on the K-invariants of π and by zero on the K-invariants of π' for $\pi' \neq \pi$, $\pi' \in \mathrm{Coh}(\xi_k)$. The operator

$$T_\pi = \sum_{j=1}^{t} a_j T(g_j)$$

acts on $H^*_{et}(S_K, F^K_{\xi_k}(L_\lambda))$ by projecting onto the subspace

$$(\oplus_i \sigma^i(\pi)) \otimes (\rho(\pi, K) \otimes L_\lambda).$$

Proposition 1.7. *For $\pi \in Coh(\xi_k)$ and $\mathbf{p}$ as in the previous theorem:*

$$\prod_i det(1 - N\mathbf{p}^{-s}\sigma^i(\pi)(\phi_{\mathbf{p}}))^{(-1)^{i+1}} = det(1 - p^{-s+(d'/2)} r_{\mathbf{p}}(g(\pi_p)))^{-m(\pi_\infty)}.$$

1.2 Twisted quaternionic Shimura variety

Let $\wp$ be a prime ideal of O_F such that $G(F_\wp)$ is isomorphic to $GL_2(F_\wp)$. Consider $\hat{\Gamma}(\wp) =: 1 + \wp O_D \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$. The group $GL_2(O/\wp)$ acts on

$$S_{\hat{\Gamma}(\wp)}(\mathbb{C}) = \bar{G}(\mathbb{Q}) \setminus X \times \bar{G}(\mathbb{A}_f)/\hat{\Gamma}(\wp).$$

This action can be described in the following way : $GL_2(O_\wp) \hookrightarrow \bar{G}(\mathbb{A}_f)$ by $\alpha \mapsto (1, ..., \alpha, 1, ..., 1)$, α at the $\wp$ component. Using the isomorphism $GL_2(O/\wp) \cong GL_2(O_\wp)/(\hat{\Gamma}(\wp))_\wp$, the action of an element $g \in GL_2(O_\wp)$ is given by the left multiplication at the $\wp$ component.

We fix a continuous representation

$$\varphi : \Gamma_E \to GL_2(O/\wp).$$

Let L be the finite Galois extension of $\mathbb{Q}$ defined by $L = (\overline{\mathbb{Q}})^{\ker(\varphi)}$.

Let

$$S^{'} = S_{\hat{\Gamma}(\wp)} \times_{\mathrm{Spec}(E)} \mathrm{Spec}(L).$$

The group $GL_2(O/\wp)$ acts on $S_{\hat{\Gamma}(\wp)}$. Since $\varphi : \mathrm{Gal}(L/E) \hookrightarrow GL_2(O/\wp)$, the group $\mathrm{Gal}(L/E)$ acts on $S_{\hat{\Gamma}(\wp)}$. We denote this action of $\mathrm{Gal}(L/E)$ on $S_{\hat{\Gamma}(\wp)}$ by $\varphi^{'}$. The Galois group $\mathrm{Gal}(L/E)$ has a natural action on $\mathrm{Spec}(L)$ and we can descend via the quotient process $S^{'}$ to $S^{'}_{\hat{\Gamma}(\wp)}/\mathrm{Spec}(E)$ using the diagonal action

$$\mathrm{Gal}(L/E) \ni \sigma \to \varphi^{'}(\sigma) \otimes \sigma$$

on $S^{'}$. Thus, we obtain a quasi-projective variety $S^{'}_{\hat{\Gamma}(\wp)}/\mathrm{Spec}(E)$. This is the twisted quaternionic Shimura variety that we mentioned in the title.

1.3 Computation of the zeta function of twisted quaternionic Shimura variety

We consider the injective limit:

$$V := \varinjlim_K H^q_{et}(S_K, F^K_{\xi_k}(L_\lambda)) \otimes_{L_\lambda} \bar{\mathbb{Q}}_l \cong \varinjlim_K \oplus_\pi U^q_{\bar{\mathbb{Q}}_l}(\pi_f, \xi_k) \otimes_{\bar{\mathbb{Q}}_l} \tilde{\pi}^K_f,$$

where $U^q_{\bar{\mathbb{Q}}_l}(\pi_f, \xi_k)$ is the $\bar{\mathbb{Q}}_l$-space that corresponds to $\sigma^q(\pi)$ and $\tilde{\pi}^K_f$ is the $\bar{\mathbb{Q}}_l$-space that corresponds to $\rho(\pi, K)$ (see Proposition 1.5 for notations).

Using the strong multiplicity one for $\bar{G}$ we get that the π-component $V(\pi)$ of V is isomorphic to $\sigma^q(\pi) \otimes \pi_f$ as $\Gamma_E \times \mathbb{H}$-module. Taking the $\hat{\Gamma}(\wp)$-fixed vectors we deduce that $V(\pi)^{\hat{\Gamma}(\wp)}$ is isomorphic to $\sigma^q(\pi) \otimes \pi_f^{\hat{\Gamma}(\wp)}$ as $\Gamma_E \times GL_2(O/\wp O)$-module. Since the varieties $S_{\hat{\Gamma}(\wp)}$ and $S^{'}_{\hat{\Gamma}(\wp)}$ become isomorphic over $\bar{\mathbb{Q}}$, we have the isomorphism $H^q_{et}(S_{\hat{\Gamma}(\wp)}, F^{\hat{\Gamma}(\wp)}_{\xi_k}(\bar{\mathbb{Q}}_l)) \cong H^q_{et}(S^{'}_{\hat{\Gamma}(\wp)}, F^{\hat{\Gamma}(\wp)}_{\xi_k}(\bar{\mathbb{Q}}_l))$. The actions of Γ_E on these cohomologies which give the expression of the zeta functions of these varieties are different. If we consider the component $V^{'}(\pi)$ that corresponds to π of $H^q_{et}(S^{'}_{\hat{\Gamma}(\wp)}, F^{\hat{\Gamma}(\wp)}_{\xi_k}(\bar{\mathbb{Q}}_l))$ (see the decomposition of Proposition 1.5), we get that $V^{'}(\pi)$ is isomorphic to $\sigma^q(\pi) \otimes (\pi_f^{\hat{\Gamma}(\wp)} \circ \varphi)$ as Γ_E-module. Using Proposition 1.7 we conclude the first part of Theorem 1.1.

2 Meromorphic continuation

We remark that the first part of Theorem 1.1 remains true if we replace $\wp$ by an ideal $\mathbf{n}$ of O_F such that if $\mathbf{q}|\mathbf{n}$, where $\mathbf{q}$ is a prime ideal of O_F, then $G(F_{\mathbf{q}})$ is isomorphic to $GL_2(F_{\mathbf{q}})$. We fix such an ideal $\mathbf{n}$.

In this section we continue meromorphically the zeta function $L(s, S'_{\hat{\Gamma}_{\mathbf{n}}})$ to the whole complex plane and prove the functional equation in some special cases.

Let $\omega = \pi_f^{\hat{\Gamma}(\mathbf{n})} \circ \varphi$. Define K to be the fixed field of $\mathrm{Ker}(\omega)$. We assume that K is a solvable extension of a totally real field.

If l is a prime number, we fix an isomorphism $j : \overline{\mathbb{Q}}_l \to \mathbb{C}$, and from now on we identify these two fields. If π is an cuspidal holomorphic automorphic representation of weight 2 of $GL(2)/F$, then there is ([T1]) a λ-adic representation for $\lambda \nmid \mathbf{n}$ ($\mathbf{n}$ is the level of π)

$$\rho_{\pi,\lambda} : \Gamma_F \to GL_2(O_\lambda) \hookrightarrow GL_2(\overline{\mathbb{Q}}_l) \cong GL_2(\mathbb{C}),$$

which satisfies $L(s-1/2,\pi) = L(s,\rho_{\pi,\lambda})$ and it is unramified outside the primes dividing $\mathbf{n}l$. Here O is the coefficients ring of π and λ is a prime ideal of O above some prime number l.

We say that a representation $\rho : \Gamma_F \to GL_2(k)$, for some finite field k is modular if $\rho \cong \bar{\rho}_{\pi,\lambda}$, for some π and λ, where we denote by $\bar{\rho}_{\pi,\lambda}$ the reduction of $\rho_{\pi,\lambda} : \Gamma_F \to GL_2(O_\lambda)$ modulo λ.

2.1 Definition of the representation $\rho^{ss}(\pi)$

One can find a representation $\rho^{ss}(\pi)$ of Γ_E ([BR] §7.2 and [R]) such that

$$L(s, \rho^{ss}(\pi)) = L^{ss}(s - d'/2, \pi, r).$$

We describe now the representation $\rho^{ss}(\pi)$. Let G be a group and K and H be two subgroups of G. We consider a representation

$$\tau : H \to GL(W)$$

and a double coset $H\sigma K$ such that $d(\sigma) = |H \setminus H\sigma K| < \infty$. We define a representation $\tau_{H\sigma K}$ of K on the $d(\sigma)$-fold tensor product $W^{\otimes d(\sigma)}$. Consider the representatives $\{\sigma_1, \cdots, \sigma_{d(\sigma)}\}$ such that $H\sigma K = \cup H\sigma_j$. If $\gamma \in K$, then there exists $\xi_j \in H$ and an index $\gamma(j)$ such that

$$\sigma_j \gamma = \xi_j \sigma_{\gamma(j)}.$$

We define the representation:

$$\tau_{H\sigma K}(\gamma)(\omega_1 \otimes \cdots \otimes \omega_{d(\sigma)}) = \tau(\xi_1)\omega_{\gamma^{-1}(1)} \otimes \cdots \otimes \tau(\xi_{d(\sigma)})\omega_{\gamma^{-1}(d(\sigma))}.$$

One can check easily that the class of $\tau_{H\sigma K}$ is independent of the choice of the representatives $\sigma_1, \cdots, \sigma_{d(\sigma)}$.

If $S_\infty - S_\infty^{'} = \{\delta_1, \cdots, \delta_{d'}\}$, and denote $S := \bigcup \Gamma_F \delta_i$, we write S as a disjoint union of double cosets

$$S = \cup_{j=1}^{k} \Gamma_F \sigma_j \Gamma_E$$

and we denote by ρ_j the representation of Γ_E defined by $\rho_{\pi,\lambda}$ and the double coset $\Gamma_F \sigma_j \Gamma_E$, then $\rho^{ss}(\pi)$ is isomorphic to $\rho_1 \otimes \cdots \otimes \rho_k$. Thus

$$L^{ss}(s - d^{'}/2, \pi, r) = L(s, \rho^{ss}(\pi)) = L(s, \rho_1 \otimes \cdots \otimes \rho_k)$$

and from Theorem 1.1 we get

$$L^{ss}(s - d^{'}/2, \pi, r \otimes \omega) = L(s, \rho^{ss}(\pi) \otimes \omega) = L(s, \rho_1 \otimes \cdots \otimes \rho_k \otimes \omega).$$

2.2 Base change to a big totally real field for Hilbert modular automorphic representations

The main result of this section is Theorem 2.2.

We prove the following lemma:

Lemma 2.1. *Let ϕ be an l-adic representation of Γ_E. Suppose that there exists a Galois solvable extension of a totally real field $F^{'}$ which contains the field $K := \bar{\mathbb{Q}}^{ker(\omega)}$ and that the L-function $L(s, \phi|_{\Gamma_{F''}} \otimes \chi)$ can be meromorphically continued to the whole complex plane and verifies a functional equation for any subfield $F^{''}$ of $F^{'}$ containing E such that $F^{'}$ is a solvable extension of $F^{''}$ and any continuous character χ of $\Gamma_{F''}$. Then $L(s, \phi \otimes \omega)$ can be meromorphically continued to the whole complex plane and verifies a functional equation.*

Proof:

By Brauer's Theorem (see [SE], Theorems 16 and 19), we can find some subfields $F_i \subset F^{'}$ such that $\mathrm{Gal}(F^{'}/F_i)$ are solvable, some characters $\chi_i : \mathrm{Gal}(F^{'}/F_i) \to \bar{\mathbb{Q}}^\times$ and some integers m_i, such that the representation

$$\omega : \mathrm{Gal}(F^{'}/E) \to \mathrm{Gal}(K/E) \to GL_N(\bar{\mathbb{Q}}_l),$$

can be written as $\omega = \sum_{i=1}^{i=k} m_i \mathrm{Ind}_{\Gamma_{F_i}}^{\Gamma_E} \chi_i$ (a virtual sum). We know that the L-function $L(s, \phi|_{\Gamma_{F_i}} \otimes \chi_i)$ has a meromorphic continuation to the whole complex plane and verifies a functional equation. Then

$$L(s, \phi \otimes \omega) = \prod_{i=1}^{i=k} L(s, \phi \otimes \mathrm{Ind}_{\Gamma_{F_i}}^{\Gamma_E} \chi_i)^{m_i} =$$

$$\prod_{i=1}^{i=k} L(s, \mathrm{Ind}_{\Gamma_{F_i}}^{\Gamma_E} (\phi|_{\Gamma_{F_i}} \otimes \chi_i))^{m_i} = \prod_{i=1}^{i=k} L(s, \phi|_{\Gamma_{F_i}} \otimes \chi_i)^{m_i}$$

which is a product of L-functions that have a meromorphic continuation to the whole complex plane and verify a functional equation. Thus $L(s, \phi \otimes \omega)$ can be meromorphically continued to the whole complex plane under the above conditions. ■

We use the following theorem to prove Theorem 2.17 (see below):

Theorem 2.2. *If F is a totally real field, π is a cuspidal automorphic representation of weight 2 of $GL(2)/F$ and F_2/F is a solvable extension of a totally real field, then there exists a Galois extension F_3 of $\mathbb{Q}$ containing F_2 and there exists a prime λ of the field coefficients of π, such that $\rho_{\pi,\lambda}|_{\Gamma_{F_3}}$ is modular i.e. there exists an automorphic representation π_1 of $GL(2)/F_3$ and a prime β of the field of coefficients of π_1 such that $\rho_{\pi,\lambda}|_{\Gamma_{F_3}} \cong \rho_{\pi_1,\beta}$.*

The proof of this theorem will be given after starting preliminary results and after Proposition 2.12.

For $F = \mathbb{Q}$ this is Proposition 5 of [V]. The proof in [V] uses the positivity of the density of the set of ordinary primes that is known for cuspidal automorphic representations of $GL(2)/\mathbb{Q}$. This fact is not known for cuspidal automorphic representations of $GL(2)/F$ for general F. To prove the theorem for general F one uses the argument below to generalize some results from [T2] and then apply Theorem 2.14 (Theorem R=T in [F]).

Now we start the proof of Theorem 2.2. We say that an automorphic representation π of $GL(2)/F$ has CM if there exists some Galois character $\eta : I_F/F^\times \to \bar{\mathbb{Q}}_l^\times$, with $\eta \neq 1$ such that $\pi \cong \pi \otimes \eta$. It is known (see [G] Theorem 7.11) that if π has CM, then $\rho_{\pi,\lambda}|_{\Gamma_L}$ is modular for every extension L/F and every prime λ of the field of coefficients of π such that $\lambda \nmid \mathbf{n}$, where $\mathbf{n}$ is the level of π. Thus in this case Theorem 2.2 is true.

From now on we suppose that the representation π has no CM. We can associate to π a Hilbert modular newform f of level $\mathbf{n}$. We assume for later use, that the prime l is unramified in F. We consider a prime ideal λ above l of the field of coefficients O_f of f. From Taylor [T1] we know that one can find a prime ideal λ_1 of O_F and a Hilbert modular newform g of level $n\lambda_1$ that is new at λ_1, such that $f \equiv g \bmod \lambda$, in the sense that they have the same Hecke eigenvalues mod λ. Actually one can find a rational prime number $l^{'}$ and a Hilbert modular form g of level $\mathbf{n}l^{'}$ and new at $l^{'}$, such that $f \equiv g \bmod \lambda$ (see the final part of [T1]). The argument in [T1](final part) forces $l^{'} \equiv 1 \bmod l$. We assume this fact from now on.

We define $c(\mathbf{p}, g)$ to be the eigenvalue given by:

$$c(\mathbf{p}, g)g = g|T(\mathbf{p})$$

for $\mathbf{p}$ a prime ideal of O_F and the Hecke operator $T(\mathbf{p})$. We say that g is ordinary at $\lambda^{'}$, for a prime ideal $\lambda^{'}|l^{'}$ of the field of coefficients O_g of g, if $\lambda^{'} \nmid c(\mathbf{p}, g)$ for all $\mathbf{p}|l^{'}$. We say that g is ordinary at $l^{'}$ if g is ordinary at $\lambda^{'}$ for all $\lambda^{'}|l^{'}$. Since g is new at $l^{'}$, the automorphic representation generated by g is Steinberg at all $\lambda^{'}|l^{'}$; so, g is ordinary at $l^{'}$.

It is known that (see [W1], Theorem 2):

Proposition 2.3. *For g as above, we have*

$$\rho_{g,\lambda'}|_{G_\mathbf{p}} \cong \begin{pmatrix} \epsilon_{l'}\delta & * \\ 0 & \delta \end{pmatrix}$$

where $G_{\mathbf{p}}$ is the decomposition group at $\mathbf{p}$, with $\mathbf{p}|N\lambda^{'}$, δ is an unramified character and $\epsilon_{l^{'}}$ is the $l^{'}$-adic cyclotomic character.

Let $\mathbb{F}_{l^{'}}$ be the finite field of cardinal $l^{'}$. We want to prove the following proposition:

Proposition 2.4. *We can choose the above form g and prime number $l^{'}$, such that for all $\lambda^{'}|l^{'}$, the representation $\rho_{g,\lambda^{'}}$ is full i.e. the image of the reduced representation $\bar{\rho}_{g,\lambda^{'}}$ contains $SL_2(\mathbb{F}_{l^{'}})$.*

More exactly, we prove that there exists a number N such that if g is a Hilbert newform of $GL(2)/F$ that is new at $\lambda^{'}$, with $\lambda^{'}|l^{'}$ and the representation $\rho_{g,\lambda^{'}}$ is not full, then $l^{'} < N$. Since we can choose the prime $l^{'}$ as big as we want, if we show this fact, Proposition 2.4 is proved. To prove this fact we use the following theorem:

Theorem 2.5. *(Dickson) If k is a finite field of characteristic p then:*
(i) An irreducible subgroup of $PSL_2(k)$ of order divisible by p is conjugate inside $PGL_2(k)$ to $PGL_2(\mathbb{F}_q)$ or $PSL_2(\mathbb{F}_q)$, for some q a power of p.
(ii) An irreducible subgroup of $PSL_2(k)$ of order not divisible by p is either dihedral or isomorphic to one of the groups A_4, A_5 or S_4.

We denote by pr the canonical projection of $GL_2(k)$ to $PGL_2(k)$.
We distinguish three cases:
1. $pr(\mathrm{im}(\bar{\rho}_{g,\lambda^{'}}))$ is isomorphic to one of the groups A_4, A_5 or S_4. It is proved in [D], §3.2, using the fact that in this cases the elements of $pr(\bar{\rho}_{g,\lambda^{'}}(I_{\mathbf{p}}))$ have order at most 5, where $I_{\mathbf{p}}$ is the inertia group at a prime $\mathbf{p}$ of F above $l^{'}$, that if $dl^{'} > 5[F:\mathbb{Q}]$, where $d = [F:\mathbb{Q}]$, then $pr(\mathrm{im}(\bar{\rho}_{g,\lambda^{'}}))$ is not isomorphic to one of the groups A_4, A_5 or S_4.
2. The representation $\bar{\rho}_{g,\lambda^{'}}$ is reducible. We denote by $\bar{\rho}^{ss}_{g,\lambda^{'}}$ the semisimplification of $\bar{\rho}_{g,\lambda^{'}}$. Then $\bar{\rho}^{ss}_{g,\lambda^{'}} = \phi^{'}_1 \oplus \epsilon_{l^{'}}\phi^{'}_2$, for some characters $\phi^{'}_1, \phi^{'}_2 : \Gamma_F \to k^{\times}$ which are unramified outside $\mathbf{n}$. As in [D] §3.1, one can find two Hecke characters $\phi_1, \phi_2 : \Gamma_F \to O^{\times}$ (here O is the ring of integers of some local field) of conductors dividing $\mathbf{n}$ and of infinity type 0, such that $\bar{\phi_1} = \phi^{'}_1$ and $\bar{\phi_2} = \phi^{'}_2$. There are only finitely many characters ϕ_1 and ϕ_2 of conductors dividing $\mathbf{n}$ and of infinity type 0.

We want to prove now that $\lambda^{'}$ divides the numerator of $L(-1, \phi_1^{-1}\phi_2)$. Let $E(\phi_1,\phi_2)$ be the Einsenstein series associated to the characters ϕ_1 and ϕ_2 and $C(0, E(\phi_1,\phi_2))$ be the constant term of the Einsenstein series $E(\phi_1,\phi_2)$. We have that $C(0, E(\phi_1,\phi_2)) = L(-1, \phi_1^{-1}\phi_2)$.

Let $h := E(\phi_1,\phi_2) - g$. Then $h \equiv C(0, E(\phi_1,\phi_2)) \bmod \lambda^{'}$.

If $\mathbf{m}$ is an ideal of O_F, then we denote by $S^{ord}_{\chi}(\mathbf{m}; A)$ the space of Hilbert modular forms of $GL(2)/F$ of level $\mathbf{m}$ of weight $\chi \geq 2$ that are ordinary at $l^{'}$, with coefficients in some ring A. We know (this is Theorem 4.37 from [H]):

Theorem 2.6. *If* $\mathbf{n}$ *and* $l^{'}$ *are as above and* $\chi \geq 3$*, then*

$$S_{\chi}^{ord}(\mathbf{n} \cap \Gamma_0(l^{'}); \bar{\mathbb{F}}_{l'}) \cong S_{\chi}^{ord}(\mathbf{n}; \bar{\mathbb{F}}_{l'}).$$

Here $S_{\chi}^{ord}(\mathbf{n} \cap \Gamma_0(l^{'}); \bar{\mathbb{F}}_{l'})$ is the space of cusp form of $GL(2)/F$ of level $\mathbf{n}l^{'}$, which are ordinary at $l^{'}$ and with the usual condition $\Gamma_0(l^{'})$ at $l^{'}$ (see [H] for details). There exists a Hilbert modular form $E \in S_{(l^{'}-1)^a}(\mathbf{n} \cap \Gamma_0(l^{'}); W)$ for some positive natural number a, where W is a local ring with residue field $\mathbb{F}_{l'}$, such that $E \equiv 1 \bmod l^{'}$. We get an injection

$$S_2^{ord}(\mathbf{n} \cap \Gamma_0(l^{'}); \bar{\mathbb{F}}_{l'}) \to S_{(l^{'}-1)^a+2}^{ord}(\mathbf{n} \cap \Gamma_0(l^{'}); \bar{\mathbb{F}}_{l'}),$$

given by $f \mapsto fE$. Thus $hE \in S_{(l^{'}-1)^a+2}^{ord}(\mathbf{n} \cap \Gamma_0(l^{'}); \bar{\mathbb{F}}_{l'}) \cong S_{(l^{'}-1)^a+2}^{ord}(\mathbf{n}; \bar{\mathbb{F}}_{l'})$ and $hE \equiv C(0, E(\phi_1, \phi_2)) \bmod \lambda^{'}$.

If $\mathbf{m}$ is an ideal of O_F, then we denote by $M_{\chi}(\mathbf{m}; k)$ the space of Hilbert modular forms of level $\mathbf{m}$ corresponding to some weight $\chi \in \mathbb{X}_k$, where $\mathbb{X}_k$ is the set of all weights of the space of Hilbert modular forms $M(\mathbf{m}; k)$ of level $\mathbf{m}$, having coefficients in some finite field k of characteristic $l^{'}$.

From [AG], Theorem 7.22 we know:

Theorem 2.7. *Consider the ideal of congruences*

$$I := Ker\{\oplus_{\chi \in \mathbb{X}_k} M_{\chi}(\mathbf{n}; k) \to k[[q^{\nu}]]_{\nu \in M}\},$$

where M *is an* O_F*-module depending on the cusp used to get the q-expansion. Then* I *is spanned by*

$$\{h_{\psi} - 1 | \psi \in \mathbb{X}_k(1)^{+}\},$$

where $\mathbb{X}_k(1)^{+}$ *is some subset of* $\mathbb{X}_k$ *and* h_{ψ} *is a modular form of weight* $l^{'} - 1$*.*

Applying this theorem to $hE - C(0, E(\phi_1, \phi_2)) \equiv 0 \bmod \lambda^{'}$, we get that $hE - C(0, E(\phi_1, \phi_2)) = \sum_{i=1}^{i=m} a_i(h_i - 1)$, for some $h_i \in \mathbb{X}_k(1)^{+}$, $i = 1, \cdots, m$. But hE has weight $(l^{'} - 1)^a + 2$ and each h_i has weight $l^{'} - 1$. Since $l^{'} - 1 \nmid (l^{'} - 1)^a + 2$ for $l^{'} > 3$, the equality $hE - C(0, E(\phi_1, \phi_2)) = \sum_{i=1}^{i=m} a_i(h_i - 1)$ is impossible if $l^{'} > 3$ and hE is not 0 and $\lambda^{'} \nmid C(0, E(\phi_1, \phi_2))$. Thus, if $l^{'} > 3$ we get that $\lambda^{'} | C(0, E(\phi_1, \phi_2)) = L(-1, \phi_1^{-1}\phi_2)$. Since the number of ϕ_1 and ϕ_2 is finite, we deduce that there exists a number N_1, such that if $l^{'} > N_1$, then $\bar{\rho}_{g,\lambda'}$ is irreducible.

3. $pr(\mathrm{im}(\bar{\rho}_{g,\lambda'}))$ is dihedral.

We have $\det\rho_{g,\lambda'} = \epsilon_{l'}\chi$, for a finite order character χ of level dividing $\mathbf{n}$. There is only a finite number of characters χ of finite order and of level dividing $\mathbf{n}$. Then as in [D] §3.4 one can find a quadratic CM-extension K of F of discriminant $\Delta_{K/F}$ dividing $\mathbf{n}$, such that all the primes of F above $l^{'}$ split in K and a Hecke character $\phi^{'}$ of K of conductor dividing $\mathbf{n}$ and infinity type Σ (here Σ is the CM-type of K), such that $\bar{\rho}_{g,\lambda'} = \mathrm{Ind}_K^F \bar{\phi}'$. There is only a finite number of Hecke characters $\phi^{'}$ of K of conductor dividing $\mathbf{n}$ and infinity type Σ. Hence the order of $\bar{\phi}'$ is bounded independently of $l^{'}$. If $\mathbf{p}$ is a prime of F above $l^{'}$, then we write $\mathbf{p} = \mathbf{p}_1\mathbf{p}_2$, with $\mathbf{p}_1$ and $\mathbf{p}_2$ two primes in K. From Proposition 2.3 we get that

$$\bar{\rho}_{g,\lambda'}|_{G_{\mathbf{p}_1}} \sim \begin{pmatrix} \bar{\epsilon}_{l'}\bar{\delta} & * \\ 0 & \bar{\delta} \end{pmatrix} \sim \begin{pmatrix} \bar{\phi}'^{c} & 0 \\ 0 & \bar{\phi}' \end{pmatrix},$$

where c is the nontrivial element of the Galois group $\mathrm{Gal}(K/L)$ and $\phi^{'c}$ is the Galois conjugate of $\phi^{'}$. We can suppose that $\bar{\delta} = \bar{\phi}'$ and $\bar{\epsilon}_{l'}\bar{\delta} = \bar{\phi}'^{c}$ as characters of $G_{\mathbf{p}_1}$. As we remarked above, the order of $\bar{\phi}'$ is bounded independently of $l^{'}$. But $\mathrm{im}\epsilon_{l'}|_{G_{\mathbf{p}_1}}$ increases linearly with $l^{'}$ and thus the equality $\bar{\epsilon}_{l'}\bar{\delta}|_{G_{\mathbf{p}_1}} = \bar{\phi}'|^{c}_{G_{\mathbf{p}_1}}$ is impossible when $l^{'}$ is big. We deduce that, there exists a number N_2, such that if $l^{'} > N_2$, then $\bar{\rho}_{g,\lambda'}$ is not dihedral.

From Dickson's theorem and the cases 1, 2 and 3 treated above we deduce Proposition 2.4.

Thus, using Proposition 2.4, we can assume that the image of $\bar{\rho}_{g,\lambda'}$ contains $SL_2(\mathbb{F}_{l'})$.

If M is a totally real field, then by a M-HBAV over a totally real field E we mean a triple (A, i, j), where
1. A/E is an abelian variety of dimension $[M : \mathbb{Q}]$,
2. $i : O_M \hookrightarrow \mathrm{End}(A/E)$ (algebra homomorphism which takes 1 to the identity map),
3. j is an O_M-polarization (see [T2] page 133 for details).

Let F_1 be a totally real extension of F. We can take F_1 to be the maximal totally real subfield of F_2 from Theorem 2.2. We assume this fact from now on. Hence F_2 is a solvable extension of F_1. Since by our assumption the image of $\bar{\rho}_{g,\lambda'}$ contains $SL_2(\mathbb{F}_{l'})$ and F_1 is a totally real field, we know form [V], Proposition 5 that the image of $\bar{\rho}_{g,\lambda'}|_{\Gamma_{F_1}}$ contains $SL_2(\mathbb{F}_{l'})$ and thus the representation $\bar{\rho}_{g,\lambda'}|_{\Gamma_{F_1}}$ is irreducible. Since $\det\bar{\rho}_{g,\lambda'} = \epsilon_{l'}$, we get that the representation $\rho_{l'} := \bar{\rho}_{g,\lambda'}|_{\Gamma_{F_1}} : \Gamma_{F_1} \to GL_2(k)$ (here k is a finite field of characteristic $l^{'}$) verifies the proprieties from Taylor's paper [T2] §1:
i) is a continuous irreducible representation,
ii) for every place v of F_1 above $l^{'}$ we have

$$\rho_{l'}|_{G_v} \sim \begin{pmatrix} \epsilon_{l'}\chi_v^{-1} & * \\ 0 & \chi_v \end{pmatrix}$$

where G_v is the decomposition group above $l^{'}$ and χ_v is unramified,
iii) for every complex conjugation c we have $\det\rho_{l'}(c) = -1$.

Having a representation of this form, Taylor finds (see [T2] page 136) a totally real field M_1, a totally real extension E/F_1 and a prime $p \neq l^{'}$ such that each place above $l^{'}$ and p in F_1 splits completely in E and a M_1-HBAV $(A, i, j)/E$, such that the representation of Γ_E on $A[\lambda^{''}]$ ($\lambda^{''}$ is a place of M_1

above $l^{'}$) is equivalent to $\bar{\rho}_{g,\lambda'}|_{\Gamma_E}$ and the representation of Γ_E on $A[\mathbf{p}]$ ($\mathbf{p}$ is a place of M_1 above p) is equivalent to $\mathrm{Ind}_{\Gamma_L}^{\Gamma_{F_1}}\psi|_{\Gamma_E}$ (for some quadratic extension L of F_1 and some character ψ) and thus, the representation $\bar{\rho}_{g,\lambda'}|_{\Gamma_E}$ is modular. Since $f \equiv g \bmod \lambda$, we obtain that $\bar{\rho}_{f,\lambda}|_{\Gamma_E}$ is modular.

We want to use Taylor argument to find a totally real extension E_2/F_1 such that l are unramified in E_2, and $\bar{\rho}_{f,\lambda}|_{\Gamma_{E_2}}$ is modular, and then apply the result of [F] (see Theorem 2.14 below) to prove Theorem 2.2. In order to prove this fact, one can modify Taylor's argument from [T2] in the following way. Taylor used the following theorem of Moret-Bailly [M]:

Theorem 2.8. *Let S be a finite set of places, K a number field and K_S/K a unique maximal extension inside a given algebraic closure of K, in which all the places of S split completely. If $X/Spec(K)$ is a geometrically irreducible smooth quasi-projective scheme and $X(K_v)$ is non-empty, then $X(K_S)$ is Zariski dense in X.*

We want to apply this theorem for S the set of places of F_1 above l, $l^{'}$,∞ and another prime p that is considered in [T2].

Let $N_0 = \mathbb{Q}(\zeta, \sqrt{1-4l'})$, where ζ is a root of unity of order $\#k^{\times}$. Then $l^{'}$ is unramified in N_0 and each prime of N_0 above $l^{'}$ has residue field k. Let λ_0 be a prime of N_0 above $l^{'}$ and we fix an isomorphism $O_{N_0}/\lambda_0 \cong k$.

In [T2] § 1 an odd prime $p \neq l$ or $l^{'}$ is chosen which has the proprieties:

1. at all primes w of F_1 above p, the representation $\rho_{l'}$ is unramified and $\rho_{l'}(\mathrm{Frob}_w)$ has distinct eigenvalues,

2. p splits completely in the Hilbert class field of N_0.

Since p splits completely in the Hilbert class field of N_0, for each place w of F_1 above p one can choose $\alpha_w^{'} \in \mathbb{Z}[(1+\sqrt{1-4l'})/2]$ with norm p. Then an element $\alpha_w = \zeta^{a_w}\alpha_w^{'}$ is defined, where a_w is chosen such that α_w is congruent modulo λ_0 to an eigenvalue of $\rho_{l'}(\mathrm{Frob}_w)$.

We remaind that $l^{'} \equiv 1 \bmod l$. Thus $1-4l^{'} \equiv -3 \bmod l$ and if we choose l such that $(\frac{-3}{l}) = 1$, then l splits in N_0. We assume that l splits in N_0 from now on.

In [T2] § 1, a character ψ is defined which verifies some proprieties concerning the two primes p and $l^{'}$. We define a character ψ concerning the three primes p, $l^{'}$ and l. One can choose a quadratic extension L of F_1, a prime $\wp_0$ of N_0 above p and a continuous character $\psi : \Gamma_L \to (N_{0,\wp_0})^{\times}$, such that:

1. L is a totally imaginary field that is not contained in F_1 adjoin a primitive pth root of unity or a lth root of unity,

2. each place v of F_1 above $l^{'}$ splits as $v_1v_1^c$ in L and $\psi|_{W_{L_{v_1}}} = \tilde{\chi}_v$ in $(\overline{O_{N_0}/\wp_0})$,

3. each place w of F_1 above p splits as $w_1w_1^c$ in L and $\psi|_{G_{w_1}}$ is unramified and takes arithmetic Frobenius to a lift of $\alpha_w \in O_{N_0}/\wp_0$,

4. each place u of F_1 above l splits as $u_1u_1^c$ in L,

5. $\mathrm{det}\mathrm{Ind}_{\Gamma_L}^{\Gamma_{F_1}}\psi = \epsilon_p$,

where $W_{L_{v_1}}$ is the Weil group of L_{v_1}, $\tilde{\chi}_v$ is the Teichmuller lift of χ_v.

We consider $\bar{\psi} : \Gamma_L \to (\overline{O_{N_0}/\wp_0})^\times$ the reduction of ψ. We choose a Galois CM extension N/N_0 such that

i) the primes above $l^{'}$ split in N/N_0,

ii) the primes above p are unramified in N/N_0,

iii) the primes above l are unramified in N/N_0,

iv) there is a prime $\wp$ above $\wp_0$ such that $\bar{\psi}$ has image in $O_N/\wp$.

Let $\lambda^{'}$ be a prime of O_N above λ_0 and λ_1 a prime of O_N above l. By global class class field theory we regard ψ as a character of $\mathbb{A}_L^\times$. For $x \in \mathbb{A}_L^\times$, we put $\psi^{'}(x) := \psi(x)x_p^{-k}x_\infty^k$, where k is the infinity type of ψ and x_p and x_∞ are the components of x at p and ∞ respectively. Then $\psi^{'}$ is a Hecke character of L. We put $\psi_1(x) := \psi^{'}(x)x_l^k x_\infty^{-k}$, where x_l is the component of x at l. By global class field theory we obtain a character $\psi_1 : \Gamma_L \to (N_{\lambda_1})^\times$. We consider $\bar{\psi_1} : \Gamma_L \to (\overline{O_N/\lambda_1})^\times$ the reduction of ψ_1. If necessary we can extend the CM field N keeping the above proprieties i), ii) and iii), such that $\bar{\psi_1}$ has image in O_N/λ_1.

We denote by M the maximal totally real subfield of N. In order to simplify the notations from now on we denote by the same symbols the places of M below $\lambda^{'}$, $\wp$ and λ_1 respectively.

The following proposition is a generalization of Lemma 1.2. from [T2].

Proposition 2.9. *There exists a totally real finite abelian extension $F^{'}/F_1$ unramified at l, such that for each prime v of $F^{'}$ above $l^{'}$ one can find a M-HBAV (A_v, i_v, j_v) over $F_v^{'}$ such that:*

1. A_v has potentially good reduction or potentially multiplicative reduction,

2. the action of G_v on $A_v[\lambda^{'}]$ is equivalent to $\rho_{l'}|_{G_v}$,

3. the action of G_v on $A_v[\wp]$ is equivalent to $\bar{\psi}_{v_1} \oplus \bar{\psi}_{v_1^c}$,

4. the action of G_v on $A_v[\lambda_1]$ is equivalent to $\bar{\psi}_{1v_1} \oplus \bar{\psi}_{1v_1^c}$.

Proof:

The representation $\rho_{l'}|_{G_v}$ can be described by a class in

$$\bar{q} \in H^1(G_v, k(\epsilon)) \cong F_{1v}^\times/(F_{1v}^\times)^{l^{'}} \otimes_{\mathbb{F}_{l'}} k \cong F_{1v}^\times \otimes_{\mathbb{Z}} \delta_M^{-1}/\lambda^{'}\delta_M^{-1},$$

where δ_M is the different of M.

We can choose an element

$$q_0 \in F_{1v}^\times \otimes_{\mathbb{Z}} \lambda_1\wp\delta_M^{-1} \subset F_{1v}^\times \otimes_{\mathbb{Z}} \delta_M^{-1}$$

that reduces to

$$\bar{q} \in F_{1v}^\times \otimes_{\mathbb{Z}} \delta_M^{-1}/\lambda^{'}\delta_M^{-1}.$$

We denote $q = q_0q_1$, where the element $q_1 \in F_{1v}^\times \otimes_{\mathbb{Z}} \lambda^{'}\wp\lambda_1\delta_M^{-1}$ is chosen such that $\mathrm{tr}_{M/\mathbb{Q}}(av(q_1q_0)) > 0$ for all totally positive elements $a \in O_M$. Then as in section 2 of [RAP] there is a M-HBAV $(A_v, i_v, j_v)/F_{1v}$ such that $A_v(\bar{F}_{1v}) \cong ((\bar{F}_{1v})^\times \otimes \delta_M^{-1})/O_M q$ as a $O_M[G_v]$-module.

The representation $\rho_{l'}|_{G_v}$ on $A_v[\lambda_1]$ is equivalent to $\begin{pmatrix} \epsilon_l \chi & * \\ 0 & \chi \end{pmatrix}$ for some character χ. Twisting this representation by χ, we may assume that the representation $\rho_{l'}|_{G_v}$ on $A_v[\lambda_1]$ is equivalent to $\begin{pmatrix} \epsilon_l & * \\ 0 & 1 \end{pmatrix}$. Replacing the above q by q^r where $r = plN$ for some N, such that $r \equiv 1 \mod l^{'}$, we may assume that the above representation is equivalent to $\begin{pmatrix} \epsilon_l & 0 \\ 0 & 1 \end{pmatrix}$. After a finite extension to a totally real abelian field $F^{'}/F_1$ unramified at l, we may assume that the representation is trivial and the proposition is proved.■

One can prove two similar results as in Proposition 2.9, but with $l^{'}$ replaced by l and p, respectively which are generalizations of Lemma 1.3 of [T2].

Proposition 2.10. *There exists a totally real finite abelian extension $F_1^{'}/F_1$ unramified at l, such that for each prime w above p of $F_1^{'}$, one can find a M-HBAV (A_w, i_w, j_w) over $F_{1w}^{'}$ such that:*
1. A_w has potentially good reduction or potentially multiplicative reduction,
2. the action of G_w on $A_w[\lambda^{'}]$ is equivalent to $\rho_{l'}|_{G_w}$,
3. the action of G_w on $A_w[\wp]$ is equivalent to $\bar{\psi}_{w_1} \oplus \bar{\psi}_{w_1^c}$,
4. the action of G_w on $A_w[\lambda_1]$ is equivalent to $\bar{\psi}_{1w_1} \oplus \bar{\psi}_{1w_1^c}$.

Proof:

Since $w \nmid \mathbf{n}l^{'}$, the representations $\rho_{l'}|_{G_w}$ is unramified, so abelian and diagonal. Thus, after a totally real finite abelian extension $F_1^{'}/F_1$ unramified at l, we can assume that

$$\rho_{l'}|_{G_w} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and by Honda-Tate theory (see [T] Lemma 3) there exists a M-HBAV (A_w, i_w, j_w) with the above proprieties. ■

Proposition 2.11. *There exists a totally real finite abelian extension $F_1^{''}/F_1$ unramified at l, such that for each prime u of $F_1^{''}$ above l one can find a M-HBAV (A_u, i_u, j_u) over $F_{1u}^{''}$ such that:*
1. A_u has potentially good reduction or potentially multiplicative reduction,
2. the action of G_u on $A_u[\lambda^{'}]$ is equivalent to $\rho_{l'}|_{G_u}$,
3. the action of G_u on $A_u[\wp]$ is equivalent to $\bar{\psi}_{u_1} \oplus \bar{\psi}_{u_1^c}$,
4. the action of G_u on $A_u[\lambda_1]$ is equivalent to $\bar{\psi}_{1u_1} \oplus \bar{\psi}_{1u_1^c}$.

The proof of this proposition is similar to the proof of proposition 2.10.

We remark that after an extension to a totally real field unramified at l if necessary, we may assume that the totally real fields $F^{'}$, $F_1^{'}$ and $F_1^{''}$ that appear in the propositions 2.9, 2.10 and 2.11 are equal. From now on in order to simplify the notations we assume that these fields are equal to F_1, since this fact does not change the proof of Theorem 2.2.

Taylor proved the following result (see [T2] Lemma 1.4):

Proposition 2.12. *For each infinite place x of F_1 there exists a M-HBAV (A_x, i_x, j_x) over F_{1x}.*

Let $V_{\lambda'}/F_1$ be the two dimensional $O_M/\lambda^{'}$-vector space scheme corresponding to $\rho_{l'}$ and fix an alternating isomorphism $a_{\lambda'}$ of $V_{\lambda'}$ with its Cartier dual. Let V_{λ_1}/F_1 be the two dimensional O_M/λ_1-vector space scheme corresponding to $\mathrm{Ind}_{\Gamma_L}^{\Gamma_{F_1}}\bar{\psi}_1$ and fix an alternating isomorphism a_{λ_1} of V_{λ_1} with its Cartier dual. Also let $V_{\wp}/F_1$ be the two dimensional $O_M/\wp$-vector space scheme corresponding to $\mathrm{Ind}_{\Gamma_L}^{\Gamma_{F_1}}\bar{\psi}$ and fix an alternating isomorphism $a_{\wp}$ of $V_{\wp}$ with its Cartier dual. As in [RAP], section 1 we see that there is a fine moduli space X/F_1 for tuples $(A, i, j, m_{\lambda'}, m_{\lambda_1}, m_{\wp})$, where (A, i, j) is an M-HBAV, $m_{\lambda'} : V_{\lambda'} \to A[\lambda^{'}]$, $m_{\lambda_1} : V_{\lambda_1} \to A[\lambda_1]$ and $m_{\wp} : V_{\wp} \to A[\wp]$ such that $a_{\lambda'}$ corresponds to the $j(1)$-Weil pairing on $A[\lambda^{'}]$, a_{λ_1} corresponds to the $j(1)$-Weil pairing on $A[\lambda_1]$ and $a_{\wp}$ corresponds to the $j(1)$-Weil pairing on $A[\wp]$. Since $\ker(GL_2(O_{M,\lambda_1}) \twoheadrightarrow GL_2(O_M/\lambda_1))$ has no element of finite order other then the identity, the moduli space is fine. As in section 1 of [RAP] we see that X is smooth and one can describe for any infinite place of F_1 the complex manifold $X(F_1 \otimes_{F_{1x}} \mathbb{C})$ as a quotient of the product of $[M : \mathbb{Q}]$ copies of the upper half complex plane and deduce that X is geometrically connected.

From Propositions 2.9, 2.10, 2.11 and 2.12 we deduce that for any place x of F_1 above $l^{'}$, l, p or ∞ we have $X(F_{1x}) \neq 0$.

Then using Theorem 2.8 above (for the details see the beginning of page 136 from [T2]), we deduce that there exists a totally real field M and a totally real extension E_2/F_1, such that each place above l, $l^{'}$ and p in F_1 splits completely in E_2 and a M-HBAV $(A, i, j)/E_2$, such that the representation of Γ_{E_2} on $A[\lambda_1]$ is equivalent to $\mathrm{Ind}_{\Gamma_L}^{\Gamma_{F_1}}\bar{\psi}_1|_{\Gamma_{E_2}}$, the representation of Γ_{E_2} on $A[\lambda^{'}]$ is equivalent to $\bar{\rho}_{g,\lambda'}|_{\Gamma_{E_2}}$ and the representation of Γ_{E_2} on $A[\wp]$ is equivalent to $\mathrm{Ind}_{\Gamma_L}^{\Gamma_{F_1}}\bar{\psi}|_{\Gamma_{E_2}}$. Thus $\bar{\rho}_{g,\lambda'}|_{\Gamma_{E_2}}$ is modular. Since $f \equiv g \bmod \lambda$, we obtain that $\bar{\rho}_{f,\lambda}|_{\Gamma_{E_2}}$ is modular and also we know that l is unramified in E_2. As it is explained in [V], page 10, the field E_2 can be chosen to be Galois over $\mathbb{Q}$.

Definition 2.13. *We say that a Hilbert modular newform f associated to an automorphic representation π of $GL(2)/F$, of weight 2 is Galois-minimal for a given prime l if:*
1. the conductor of π outside l is equal to the prime-to-l Artin conductor of the reduced representation $\bar{\rho}_{\pi,\lambda}$ for some prime λ in the field of coefficients of π above l,
2. for each prime v of O_F, with $v|l$, π_v is spherical or unramified special,
3. its central character has order prime to l.

We remark that if π is an automorphic representation, then we can choose our prime number l, such that π has central character of order prime to l. We assume this fact from now on.

We want to use the following theorem (see [F], R=T Theorem):

Theorem 2.14. *Let $F^{'}$ be a totally real field, with $[F^{'} : \mathbb{Q}]$=even, $l > 3$ a prime number and μ_l the group of l-th roots of unity. An l-adic representation $\rho : \Gamma_{F'} \to GL_2(\bar{\mathbb{Q}}_l)$, with $det\rho = \epsilon_l\chi$ for some finite order character χ, is modular if:*

1. $\bar{\rho} \cong \bar{\rho}_{g_1,\gamma}$ *for some Galois-minimal Hilbert modular newform* g_1 *over* $F^{'}$ *of weight 2 and some* $\gamma | l$,
2. the representation $\bar{\rho}|_{\Gamma_{F^{'}(\mu_l)}}$ *is absolutely irreducible,*
3. l *unramified in* $F^{'}$,
4. $\rho|_{G_{F^{'}_{\ell}}}$ *and* $\rho_{g_1,\gamma}|_{G_{F^{'}_{\ell}}}$ *are representations associated to Barsotti-Tate groups, where* ℓ *is a prime of* $F^{'}$ *above* l.

We want to apply the above theorem to the representation $\rho := \rho_{f,\lambda}|_{\Gamma_{E_2}}$ for the totally real extension E_2 of F_1 defined above or more precisely to a totally real extension of $F^{'}/E_2$ and to the representation $\rho := \rho_{f,\lambda}|_{\Gamma_{F^{'}}}$ (see below).

We remark that by an extension if necessary, we may assume that the field E_2 defined above satisfies $[E_2 : \mathbb{Q}]$ =even.

Since we assumed that our π has no CM, one can choose the prime number l such that the image of $\bar{\rho}_{f,\lambda}$ contains $SL_2(\mathbb{F}_l)$ (see [D] Proposition 3.8, where it is proved that if π has no CM, then for all but a finite number of primes l, the image of $\bar{\rho}_{f,\lambda}$ contains $SL_2(\mathbb{F}_l)$). Since E_2 is totally real, the image of $\rho := \bar{\rho}_{f,\lambda}|_{\Gamma_{E_2}}$ contains $SL_2(\mathbb{F}_l)$ (see [V] Proposition 5). One can prove also that, for such a prime number l, the image of $\bar{\rho}_{f,\lambda}|_{\Gamma_{E_2(\mu_l)}}$ contains $SL_2(\mathbb{F}_l)$ and thus the representation $\bar{\rho}_{f,\lambda}|_{\Gamma_{E_2(\mu_l)}}$ is absolutely irreducible and the condition 2 of Theorem 2.14 is verified.

Now, we want to verify the condition 1 of Theorem 2.14. Thus we want to find a totally real extension $F^{'}/F_1$ and a Galois-minimal modular form g_1 for l of $GL(2)/F^{'}$ and some $\gamma | l$, such that $\bar{\rho}_{f,\lambda}|_{\Gamma_{F^{'}}} \cong \bar{\rho}_{g_1,\gamma}$. We have already found a totally real extension E_2/F_1, such that $\bar{\rho}_{f,\lambda}|_{\Gamma_{E_2}}$ is modular. We have (this is the main theorem of [SW]):

Theorem 2.15. *Let* $\rho : \Gamma_F \to GL_2(\bar{\mathbb{F}}_l)$ *be a representation associated to a Hilbert modular newform* f_1 *of weight 2 and level* n_{f_1}. *Let* $n_{f_1} = n^{(l)}_{f_1} n^{'}_{f_1}$, *where* $n^{'}_{f_1}$ *is prime to* l *and* $n^{(l)}_{f_1}$ *divides a power of* l. *If* ρ *is irreducible, then there exists a finite solvable totally real extension* $F^{'}$ *of* F *in which the primes above* l *split completely such that* $\rho|_{\Gamma_{F^{'}}} \cong \bar{\rho}_{g_1,\gamma}|_{\Gamma_{F^{'}}}$ *for some Hilbert modular newform* g_1 *of weight 2 of* $GL(2)/F^{'}$ *and some prime* γ *above* l *satisfying*

$$n_{g_1} | n^{(l)}_{f_1} \prod_{q \in S} q,$$

where S *is the set of primes of* $F^{'}$ *not dividing* l *at which* $\rho|_{\Gamma_{F^{'}}}$ *is ramified.*

We can find a solvable totally real extension $F^{'}/E_2$ unramified at l, such that $\bar{\rho}_{f,\lambda}|_{\Gamma_{F^{'}}}$ is unramified outside l. Thus, if we apply the above theorem to the representation $\rho := \bar{\rho}_{f,\lambda}|_{\Gamma_{E_2}}$, we find a Hilbert modular newform g_1 of weight 2 of $GL(2)/F^{'}$ of level dividing a power of l and a prime γ above l such that $\bar{\rho}_{f,\lambda}|_{\Gamma_{F^{'}}} \cong \bar{\rho}_{g_1,\gamma}$. Since $f \equiv g \bmod \lambda$ and g has level prime to l, the form g_1 has level 1. Thus, the modular form g_1 is Galois minimal for l and the condition 1 of Theorem 2.14 is verified.

The condition 4 of Theorem 2.14 is also verified (see [T3] Theorem 1.6).

From the proprieties that we imposed on l, we know that the prime l is unramified in $F^{'}$ and hence the condition 2 of Theorem 2.14 is verified. Thus we can choose a prime l and a totally real extension $F^{'}/E_2$, such that all the conditions of Theorem 2.14 are verified for the representation $\rho_{f,\lambda}|_{\Gamma_{F^{'}}}$ and hence we deduce that $\rho_{f,\lambda}|_{\Gamma_{F^{'}}}$ is modular. Using again Langlands base change for solvable extensions as we did several times above, we conclude Theorem 2.2.

2.3 Meromorphic continuation of the zeta functions for curves and surfaces

In this section we prove the second part of Theorem 1.1, which is a consequence of Theorem 2.17 below.

It is known ([RA], Theorem M) that:

Proposition 2.16. *If π_1 and π_2 are two cuspidal automorphic representations of $GL(2)/T$, where T is a number field, then $\pi_1 \otimes \pi_2$ is a cuspidal automorphic representation of $GL(4)/T$.*

Now we prove:

Theorem 2.17. *If $K := \bar{\mathbb{Q}}^{Ker(\omega)}$ is a solvable extension of a totally real field and $d^{'} = 1$ or $d^{'} = 2$ then the L-function $L^{ss}(s - d^{'}/2, \pi, r \otimes \omega) = L(s, \rho^{ss}(\pi) \otimes \omega)$ can be meromorphically continued to the whole complex plane and verifies a functional equation.*

Proof:

It is sufficient to prove that there exists a Galois extension $F^{'}$ of $\mathbb{Q}$ which contains F and K such that $\rho^{ss}(\pi)|_{\Gamma_{F^{'}}}$ verifies the conditions of Lemma 1.

We have two cases:

i) $d^{'} = 1$.

We assume for simplicity that $S_\infty - S^{'}_\infty = \{1\}$, where 1 is the trivial embedding of F in $\bar{\mathbb{Q}}$. In this case $E = F$ and $\rho^{ss}(\pi) \cong \rho_{\pi,\lambda}$. By Theorem 2.2 one can find a Galois solvable extension of a totally real field $F^{'}$ of F which contains K such that $\rho_{\pi,\lambda}|_{\Gamma_{F^{'}}}$ is modular. From Langlands base change for $GL(2)$ for solvable extensions ([L]) we deduce that $\rho^{ss}(\pi)|_{\Gamma_{F^{'}}}$ verifies the conditions of Lemma 2.1.

ii) $d^{'} = 2$.

We assume for simplicity that $S_\infty - S^{'}_\infty = \{1, c\}$, where 1 is the trivial embedding of F in $\bar{\mathbb{Q}}$. We denote by the same symbol c the extension of c to $\bar{\mathbb{Q}}$. Then,

$$S = \Gamma_F \cup \Gamma_F c.$$

The stabilizer of S is Γ_E. It is easy to check that the stabilizer of S is equal to $(\Gamma_F c \cap c^{-1}\Gamma_F) \cup (\Gamma_F \cap c^{-1}\Gamma_F c)$. Thus we get

$$\Gamma_E = (\Gamma_F c \cap c^{-1}\Gamma_F) \cup (\Gamma_F \cap c^{-1}\Gamma_F c).$$

We consider two cases:

i) $\Gamma_F c \cap c^{-1}\Gamma_F = \emptyset$. Then, $\Gamma_E = \Gamma_F \cap c^{-1}\Gamma_F c$. Thus,

$$F \subset E \subset F^{gal}$$

where F^{gal} is the Galois closure of F. Since F^{gal} is a totally real field we get that E is totally real. We have

$$S = \Gamma_F \cup \Gamma_F c = \Gamma_F 1 \Gamma_E \cup \Gamma_F c \Gamma_E.$$

If $\gamma \in \Gamma_E$, then

$$\tau_{\Gamma_F 1 \Gamma_E}(\gamma)(\omega_1) = \rho_{\pi,\lambda}(\gamma)(\omega_1)$$

and

$$\tau_{\Gamma_F c \Gamma_E}(\gamma)(\omega_1) = \rho_{\pi,\lambda}(c\gamma c^{-1})(\omega_1).$$

Thus

$$\rho^{ss}(\pi) \cong \rho_{\pi,\lambda}|_{\Gamma_E} \otimes \rho_{\pi,\lambda}|^c_{\Gamma_E},$$

where

$$\rho_{\pi,\lambda}|^c_{\Gamma_E}(\gamma) = \rho_{\pi,\lambda}|_{\Gamma_E}(c\gamma c^{-1}).$$

By Theorem 2.2 one can find a Galois solvable extension of a totally real field $F^{'}$ which contains F and K such that $\rho_{\pi,\lambda}|_{\Gamma_{F^{'}}}$ is modular. Thus

$$\rho^{ss}(\pi)|_{\Gamma_{F^{'}}} \cong \rho_{\pi,\lambda}|_{\Gamma_{F^{'}}} \otimes \rho_{\pi,\lambda}|^c_{\Gamma_{F^{'}}}$$

is a tensor product of two automorphic representations and from Langlands base change for $GL(2)$ for solvable extensions ([L]) and Proposition 2.16 we deduce that $\rho^{ss}(\pi)|_{\Gamma_{F^{'}}}$ verifies the conditions of Lemma 2.1.

ii) $\Gamma_F c \cap c^{-1}\Gamma_F \neq \emptyset$. Let $\Gamma_{E_1} = \Gamma_F \cap c^{-1}\Gamma_F c$. Thus

$$F \subset E_1 \subset F^{gal}.$$

Since it is obvious now that $\Gamma_{E_1} \subset \Gamma_E$, $[\Gamma_E : \Gamma_{E_1}] = 2$ and $\Gamma_E \not\subseteq \Gamma_F$ we get $[E_1 : E] = 2$ and $F \not\subseteq E$. We have

$$S = \Gamma_F \cup \Gamma_F c = \Gamma_F 1 \Gamma_E.$$

If $\gamma \in \Gamma_{E_1}$ then

$$\tau_{\Gamma_F 1 \Gamma_E}(\gamma)(\omega_1 \otimes \omega_2) = \rho_{\pi,\lambda}(\gamma)\omega_1 \otimes \rho_{\pi,\lambda}(c\gamma c^{-1})\omega_2.$$

If $\gamma \in \Gamma_E - \Gamma_{E_1}$ then

$$\tau_{\Gamma_F 1 \Gamma_E}(\gamma)(\omega_1 \otimes \omega_2) = \rho_{\pi,\lambda}(\gamma c^{-1})\omega_2 \otimes \rho_{\pi,\lambda}(c\gamma)\omega_1.$$

Thus $\rho^{ss}(\pi)$ is a subrepresentation of

$$\mathrm{Ind}_{\Gamma_{E_1}}^{\Gamma_E}(\rho_{\pi,\lambda}|_{\Gamma_{E_1}} \otimes \rho_{\pi,\lambda}|^c_{\Gamma_{E_1}}),$$

which verifies

$$\rho^{ss}(\pi)|_{\Gamma_{E_1}} \cong \rho_{\pi,\lambda}|_{\Gamma_{E_1}} \otimes \rho_{\pi,\lambda}|^c_{\Gamma_{E_1}}.$$

By Theorem 2.2 one can find a Galois solvable extension of a totally real field F' which contains F and KE_1 such that $\rho_{\pi,\lambda}|_{\Gamma_{F'}}$ is modular. We get

$$\rho^{ss}(\pi)|_{\Gamma_{F'}} \cong \rho_{\pi,\lambda}|_{\Gamma_{F'}} \otimes \rho_{\pi,\lambda}|^c_{\Gamma_{F'}}.$$

Hence from Langlands base change for $GL(2)$ for solvable extensions ([L]) and Proposition 2.16 we deduce that $\rho^{ss}(\pi)|_{\Gamma_{F'}}$ verifies the conditions of Lemma 2.1. ■

Remark 2.18. *If one can generalize Proposition 2.16 and prove that the tensor product of any given finite number of cuspidal automorphic representations of $GL(2)/T$, with T a number field, is automorphic, then Theorem 2.17 can be proven for any d' by the same method as above.*

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