FREE TALAGRAND INEQUALITY

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This is a brief summary with additional comment of the talk I gave in the conference "Free Probability Theory," Mar.27–Apr.2, 2005, at Oberwolfach. The materials are mainly taken from a recent joint work with Fumio Hiai [9].

1. Transportation Cost Inequalities in Free Probability Theory

Transportation cost inequalities estimate the 2-Wasserstein metric W_2 by the square root of relative entropy H for a given pair of probabilistic distributions, and it was Talagrand [11] who first obtained such a kind of inequality in 1996. In free probability theory, Biane and Voiculescu [2] introduced the free analog of Wasserstein metrics W_p, $1 \le p < \infty$, and obtained a natural free analog of Talagrand's inequality in the 1-dimensional case, that is,

$$W_2(X,S) \le \sqrt{2\left(-\chi(X) + \frac{1}{2}\tau(X^2) + \frac{1}{2}\log 2\pi\right)} \tag{BV}$$

for any (bounded) self-adjoint random variable X, where S is a standard semicircular element with distribution $\frac{1}{2\pi}\sqrt{4-\lambda^2}d\lambda$ supported on $[-2,2]$. Then, Hiai, Petz and I [7] strengthened it to an expected setup but still in the 1-dimensional case. Quite recently, Hiai and I [9] took up the first step towards the desired multivariate case, and obtained a natural multivariate free analog of Talagrand's inequality or other words a multivariate generalization of the inequality (BV), that is,

$$W_2\left((X_1,\dots,X_n),(S_1,\dots,S_n)\right) \le \sqrt{2\left(-\chi(X_1,\dots,X_n) + \frac{1}{2}\sum_{k=1}^{n}\tau(X_k^2) + \frac{n}{2}\log 2\pi\right)} \tag{HU}$$

for any n-tuple $(X_1,\dots,X_n)$ of (bounded) self-adjoint random variables, where $(S_1,\dots,S_n)$ is the standard semicirclar system, i.e, the n-tuple of freely independent standard semicircular elements. The inequality we actually obtained in [9] is slightly more general than the above (HU), that is, the above inequality (HU) is still valid with replacing the multiple constant 2 appeared in the square root by a suitable one even when the standard semicircular system $(S_1,\dots,S_n)$ is replaced by any n-tuple of freely independent self-adjoint random variables with suitable convexity condition. We also obtained its unitary version. I refer the interested reader to the original article [9] for those details.

2. Method – Random Matrix Approximation

Unlike in [2] the main technical ingredient in both [7] and [9] is the use of so-called random matrix approximation, which means the following pattern: For a given non-commutative random variable, a suitable sequence of random matrices indexed by their matrix sizes is chosen in such a way that its scaling limit realizes the given non-commutative random variable in distribution, and then a result in usual (i.e., classical or commutative) probability theory is shown to "converge" to its right free analog. This pattern was initially from Voiculescu's asymptotic freeness result for several independent self-adjoint Gaussian random matrices (see

[12]), and first used by Voiculescu himself [13] to seek for a free analog of Shannon's entropy for single random variables. Then, Biane [1] used the pattern to obtain a free analog of logarithmic Sobolev inequality for single self-adjoint random variables or measures, and slightly after that Hiai, Petz and I [7][8] (also see [6]) systematically used it to strengthen Biane-Voiculecu's free transportation cost inequality (BV) and obtain the unitary versions of free transportation cost and free logarithmic Sobolev inequalities. Here, I should emphasize that Hiai, Mizuo and Petz's previous work [5] on perturbation theory for free entropy in the 1-dimensional case was of considerable importance behind the works [7][8]. Finally, Ledoux [10] combined the pattern with the so-called Hamilton-Jacobi technique and unified free transportation cost and free logarithmic Sobolev inequalities from free Brunn-Minkowski inequality in the 1-dimensional case. Those works we mentioned so far all treat only the 1-dimensional case, and if the pattern was applied to the multivariate case, one would need to handle matrix integrals with general interaction potentials. Matrix integrals with general intraction potentials are quite difficult objects and there are very few results known at the present moment so that it is natural to think that the pattern cannot be easily applied to the multivariate case. However, we found that the pattern is indeed applicable to getting free transportation cost inequalities with respect to freely independent n-tuples of random variables, see the original article [9] for details.

3. Results related to Free Entropy-like Quantity

Hiai [4] introduced a free analog of pressure function as a certain scaling limit of matrix integrals with multivariable interaction potentials, whose definition apparently came from his joint work [5] with Mizuo and Petz. Following an idea in statistical mechanics Hiai also introduced a free entropy-like quantity for non-commutative distributions as the Legendre transform of the free analog of pressure function, which is different from Voiculescu's free entropy in general, but they coincide for single random variables, freely independent families and R-diagonal pairs. In [9], we also obtained the same free Talagrand's inequality with replacing Voiculescu's (microstates) free entropy χ by the free entropy-like quantity under an additional "equilibrium" condition so that it is far from the expected one. However, the inequality implies, for example, a phase transition result for the free entropy-like quantity, which is non-trivial because the free entropy-like quantity involves matrix integrals with multivariate interaction potentials. There are many questions about the quantity, but all of those seem to be difficult to fix at the present moment. I refer the interested reader to the original article for the precise statements as well as the detailed proofs.

4. Additional Remark

This section is devoted to part of works in progress with Hiai. In the conference, Ledoux, Biane and some others asked me whether or not our proof of multivariate free Talagrand's inequality can be applied even when the standard semicircular system $(S_1, \dots, S_n)$ is replaced by a more general non-commutative distribution like those treated in [3]. Concerning it, I would like to give the following comment: Let Q be a "potential" polynomial in self-adjoint indeterminates $X_1, \dots, X_n$, and assume that Q gives the well-defined probability measure on $(M_N(\mathbf{C})^{sa})^n$

$$\lambda_N^Q(dA_1, \dots, dA_n) := \frac{1}{Z_N(Q)} \exp\left(-\mathrm{Tr}_N(Q(A_1, \dots, A_n))\right) dA_1 \cdots dA_n$$

for each dimension N. Then, define the tracial distribution $\widehat{\lambda_N^Q}$ on the non-commutative polynomials $\mathbf{C}\langle X_1,\dots,X_n\rangle$ in self-adjoint indeterminates $X_1,\dots,X_n$ by

$$\widehat{\lambda_N^Q}(P) := \int_{(M_N(\mathbf{C})^{sa})^n} \frac{1}{N}\mathrm{Tr}_N\left(P(A_1,\dots,A_n)\right)\lambda_N^Q(dA_1,\dots,dA_n)$$

for $P\in\mathbf{C}\langle X_1,\dots,X_n\rangle$. Also, we define the restricted probability measure $\lambda_{N,R}^Q$ on $(M_N(\mathbf{C})_R^{sa})^n$ associated with cut-off constant $R>0$ by

$$\lambda_{N,R}^Q(dA_1,\dots,dA_n) := \frac{1}{Z_{N,R}(Q)}\exp\left(-N\mathrm{Tr}_N(Q(A_1,\dots,A_n)\right)dA_1\cdots dA_n,$$

and the corresponding tracial distribution $\widehat{\lambda_{N,R}^Q}$ on $\mathbf{C}\langle X_1,\dots,X_n\rangle$ (extended to that on the C^*-algebra $\mathcal{A}_R^{(n)} := C[-R,R]^{\star n}$ with $X_k(t)=t$ in the kth free component $C[-R,R]$) by

$$\widehat{\lambda_{N,R}^Q}(P) := \int_{\left(M_N(\mathbf{C})_R^{sa}\right)^n} \frac{1}{N}\mathrm{Tr}_N(P(A_1,\dots,A_n))\lambda_{N,R}^Q(dA_1,\dots,dA_n)$$

for $P\in\mathbb{C}\langle X_1,\dots,X_n\rangle$. We now suppose that

(1) $\tau_Q(P) := \lim_{N\to\infty}\widehat{\lambda_N^Q}(P)$ exists and is finite for each $P\in\mathbf{C}\langle X_1,\dots,X_n\rangle$;
(2) $Z(Q) := \lim_{N\to\infty}\frac{1}{N^2}\log Z_N(Q)+\frac{n}{2}\log N$ exists and is finite;
(3) $(A_1,\dots,A_n)\mapsto \mathrm{Tr}_N\left(Q(A_1,\dots,A_n)\right)-\frac{\rho}{2}\|(A_1,\dots,A_n)\|_2^2$ is convex for all dimensions N with a fixed constant $\rho>0$;
(4) (a "compact support" condition) there is a $R_Q>0$ so that every $R\ge R_Q$ satisfies that

$$\lim_{n\to\infty}\lambda_N^Q\left((M_N(\mathbf{C})_R^{sa})^n\right)=1,\quad \tau_Q(P)=\lim_{N\to\infty}\widehat{\lambda_{N,R}^Q}(P),\ P\in\mathbb{C}\langle X_1,\dots,X_n\rangle.$$

(Remark that $\lambda_N^Q((M_N(\mathbf{C})_R^{sa})^n) = Z_{N,R}(Q)/Z_N(Q)$, which implies that $\lim_{N\to\infty}\frac{1}{N^2}Z_{N,R}(Q)+\frac{n}{2}\log N = Z(Q)$.) The potential polynomials Q treated by Guionnet and Maurel-Segala [3] seem to satisfy those properties (1),(2) and (4). Those four properties gurantee that the method in [9] works for τ_Q, and we can indeed prove that

$$W_2(\tau,\tau_Q)\le\sqrt{\frac{2}{\rho}\left(-\chi(\tau)+\tau(Q)+Z(Q)\right)}$$

for every tracial state τ on $\mathcal{A}_R^{(n)}$ with $R\ge R_Q$, where $\chi(\tau)$ is defined via the GNS representation of $\mathcal{A}_R^{(n)}$ associated with τ. More on this will be discussed elsewhere. In closing, I thank Professor Michael Ledoux for useful discussions, and also thank Professor Alice Guionnet for her wonderful talk on [3] in the conference, both of which gave a motivation to us.

References

[1] Ph. Biane, Logarithmic Sobolev inequalities, matrix models and free entropy, *Acta Math. Sinica* **19**, No. 3 (2003), 1–11.
[2] Ph. Biane and D. Voiculescu, A free probability analogue of the Wasserstein metric on the trace-state space, *Geom. Funct. Anal.* **11** (2001), no. 6, 1125–1138.
[3] A. Guionnet and É. Maurel-Segala, Combinatorial aspects of matrix models, math.PR/0503064.
[4] F. Hiai, Free analog of pressure and its Legendre transform, *Comm. Math. Phys.*, **255** (2005), 229–252.
[5] F. Hiai, M. Mizuo and D. Petz, Free relative entropy for measures and a corresponding perturbation theory, *J. Math. Soc. Japan* **54** (2002), 679–718.
[6] F. Hiai, D. Petz and Y. Ueda, Inequalities related to free entropy derived from random matrix approximation, math.OA/0310453, unpublished.
[7] F. Hiai, D. Petz and Y. Ueda, Free transportation cost inequalities via random matrix approximation, *Probab. Theory Related Fields*, **130** (2004), 199–221.

[8] F. Hiai, D. Petz and Y. Ueda, A free logarithmic Sobolev inequality on the circle, *Canad. Math. Bull.*, to appear.
[9] F. Hiai and Y. Ueda, Free transportation cost inequalities for non-commutative multi-variables, math.OA/0501238.
[10] M. Ledoux, A (one-dimensional) free Brunn-Minkowski inequality, *C. R. Acad. Sci. Paris Sér. I Math.*, to appear.
[11] M. Talagrand, Transportation cost for Gaussian and other product measures, *Geom. Funct. Anal.* **6** (1996), 587–600.
[12] D. Voiculescu, Limit laws for random matrices and free products, *Invent. Math.* **104** (1991), 201–220.
[13] D. Voiculescu, The analogues of entropy and of Fisher's information measure in free probability theory, I, *Comm. Math. Phys* **155** (1993), 71–92.

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