FREE PRODUCT FACTORS AND BICENTRALIZER PROBLEM

YOSHIMICHI UEDA

Abstract. We confirmed that any type III_1 free product factor has the trivial asymptotic bicentralizer. This is not intended to be a published paper.

1. Introduction

In the late 70s Connes asked whether or not the so-called asymptotic bicentralizer of any type III_1 factor is trivial, and proved that its affirmative solution in the injective case implies the uniqueness of injective type III_1 factor. A couple of years later than Connes's effort, Haagerup made a real tour de force [4], where he succeeded in, among others, solving in the affirmative the problem in the injective case; hence he put the final piece in the classification of injective (or hyperfinite) factors. See [4, §1] for this amazing story. However, the original problem still remains open. In fact, we have known that the problem is positive only for a few classes of type III_1 factors; injective factors as mentioned above, full factors with almost periodic states [2, Theorem 4.7, Lemma 4.8], and free Araki–Woods factors [5]. Moreover, any counterexample is not known up to now. The purpose of this note is to comfirm that the problem is still positive for the class of type III_1 'free product factors', which includes all the type III_1 free Araki–Woods factors (see e.g. [6, Remark 9]).

Let M_1, M_2 be two non-trivial (i.e., $\neq \mathbb{C}$) von Neumann algebras with separable preduals, and φ_1, φ_2 be faithful normal states on them, respectively. Then their free product $(M, \varphi) = (M_1, \varphi_1) \star (M_2, \varphi_2)$ (see e.g. [8, §§2.1] for its formulation etc.) always admits the following structure: $M = M_d \oplus M_c$ with finite dimensional M_d and diffuse M_c such that M_d can explicitly be calculated with possibly $M = M_c$, and moreover, such that if $(\dim(M_1), \dim(M_2)) \neq (2,2)$, then M_c must be a full factor of type II_1 or III_λ ($\lambda \neq 0$); otherwise $M_c = L^\infty[0,1] \bar{\otimes} M_2(\mathbb{C})$. Also, the modular actions σ^{φ_i} are not both trivial and also have no common period if and only if M_c is a type III_1 factor. See [8, Theorem 4.1] for these facts. With the notation above, we will prove the following:

Theorem 1. *If M_c is of type III_1, then the asymptotic bicentralizer of any faithful normal state on M_c is trivial.*

The theorem together with [4, Theorem 3.1] shows that *any type III_1 free product factor has a norm-dense set of faithful normal states whose centralizers are irreducible (type II_1) subfactors.* Here we should remind the reader that any finite von Neumann algebra (even any finite dimensional algebra like $M_2(\mathbb{C}) \oplus M_3(\mathbb{C})$) can be the centralizer of a certain free product state (see [9, Proposition 2.1]). The main features of the theorem are that the conclusion holds regardless of whether or not the problem is positive for the given M_i as well as that its proof is very short and uses only two previous results: the Connes–Størmer transitivity [3, Theorem

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4, Corollary 8] and the central decomposition result [8, Corollary 3.2, Theorem 4.1]. However, some part of our original intuition came from [1, Theorem 4.20] due to Ando and Haagerup.

We use the same standard notation as in [8],[9] (especially, see the glossary at the end of the introduction of the former paper). We need the notation on the bicentralizer problem, for which we basically follow [4].

2. Proof

Thanks to [4, Corollary 1.5] it suffices to prove that there exists a faithful normal state χ on M_c such that the asymptotic bicentralizer B_χ is trivial, i.e., $B_\chi = \mathbb{C}1_{M_c}$, where $x \in B_\chi$ if and only if $y_n x - x y_n \longrightarrow 0$ σ-strongly as $n \to \infty$ for every norm-bounded sequence (y_n) in M_c with $\lim_{n\to\infty} \|y_n\chi - \chi y_n\| = 0$. We remark that the inclusion relation $B_\chi \subseteq ((M_c)_\chi)' \cap M_c$ trivially holds for every faithful normal state χ on M_c.

The first part of the proof is reduction to two special cases, where the key is [8, Theorem 4.1]. Assume first that both M_i are atomic, that is, of type I with atomic centers. Then both φ_i must be almost periodic, and hence by [9, Theorem 2.1] the resulting $\varphi_c := \varphi\restriction_{M_c}$ satisfies $\big((M_c)_{\varphi_c}\big)' \cap M_c = \mathbb{C}1_{M_c}$ so that $B_\chi = \mathbb{C}1_{M_c}$ with $\chi := \varphi_c(1_{M_c})^{-1}\varphi_c$; hence we are done. Therefore, by the trick explained in [10, §§2.1] (based on [8, Theorem 4.1]), we may and do assume that at least M_1 is diffuse. By the same trick, we may and do further assume that M_1 is either (a) a type III$_1$ factor or (b) diffuse with no type III$_1$ factor direct summand. In each case, $M = M_c$ holds due to [8, Theorem 3.4] (weaker than Theorem 4.1 there). We will treat cases (a) and (b) separately.

Let us prove the desired assertion in case (a). Take a faithful normal state ψ_1 on M_1 such that $(M_1)_{\psi_1}$ is diffuse. The existence of such a state is guaranteed by a (not so immediate) consequence of the Connes–Størmer transitivity, see [3, Corollary 8]. Set $\psi := \psi_1 \circ E_1$, where $E_1 \colon M \to M_1$ is the φ-preserving conditional expectation (see e.g. [8, Lemma 2.1]). Since $(M_1)_{\psi_1}$ is diffuse, [8, Corollary 3.2] shows that

$$B_\psi \subseteq ((M_1)_{\psi_1})' \cap M = ((M_1)_{\psi_1})' \cap M_1 \subseteq M_1. \tag{1}$$

This observation is the starting point of this proof.

Thanks to the Connes–Størmer transitivity [3, Theorem 4] we can choose a sequence of unitaries v_n in M_1 in such a way that $\lim_{n\to\infty} \|v_n\varphi_1 v_n^* - \psi_1\| = 0$. Since $\varphi = \varphi_1 \circ E_1$, we have $|(v_n\varphi v_n^* - \psi)(x)| = |(v_n\varphi_1 v_n^* - \psi_1)(E_1(x))| \leq \|v_n\varphi_1 v_n^* - \psi_1\|\,\|E_1(x)\|_\infty \leq \|v_n\varphi_1 v_n^* - \psi_1\|\,\|x\|_\infty$ for every $x \in M$; hence $\|v_n\varphi v_n^* - \psi\| = \|v_n\varphi_1 v_n^* - \psi_1\|$. Therefore, we get

$$\lim_{n\to\infty} \|v_n\varphi v_n^* - \psi\| = 0. \tag{2}$$

Here is a lemma.

Lemma 2. *There exist a norm-bounded sequence $w_n = w_n^*$ in M_2 and a constant $\varepsilon > 0$ such that*

(2.1) $\lim_{n\to\infty} \|w_n\varphi_2 - \varphi_2 w_n\| = 0$,
(2.2) $\lim_{n\to\infty} \varphi_2(w_n) = 0$, *and*
(2.3) $w_n^2 \geq \varepsilon 1$ *for all* n.

Proof. If $(M_2)_{\varphi_2} \neq \mathbb{C}1$, then the desired w_n can easily be chosen as $w_n = \varphi_2(e_2)e_1 - \varphi_2(e_1)e_2$ for all n, where e_1, e_2 are non-zero projections in $(M_2)_{\varphi_2}$ with $e_1 + e_2 = 1$. (The $\varepsilon > 0$ can be chosen to be $\min\{\varphi_2(e_1)^2, \varphi_2(e_2)^2\}$ in the case.) Hence we may and do assume that $(M_2)_{\varphi_2} = \mathbb{C}1$; hence M_2 must be a type III$_1$ factor (see e.g. [1, Lemma 5.3] again). It is not difficult to find a faithful normal state ψ_2 on M_2 so that $(M_2)_{\psi_2}$ contains a unitary $a = a^*$

with $\psi_2(a) = 0$. (In fact, one can do by using a $*$-isomorphism $M_2 \cong M_2 \bar{\otimes} M_2(\mathbb{C})$.) Then, by the Connes–Størmer transitivity [3, Theorem 4] again we can choose a sequence of unitaries b_n in M_2 so that $\lim_{n\to\infty} \|b_n\varphi_2 b_n^* - \psi_2\| = 0$. The desired w_n is given by $w_n := b_n^* a b_n \in M_2^u$, the unitary group of M_2. In fact, since $a\psi_2 = \psi_2 a$, we have $\|w_n\varphi_2 - \varphi_2 w_n\| \le \|b_n^* a(b_n\varphi_2 - \psi_2 b_n)\| + \|(b_n^*\psi_2 - \varphi_2 b_n^*)ab_n\| = \|b_n\varphi_2 b_n^* - \psi_2\| + \|\psi_2 - b_n\varphi_2 b_n^*\| \longrightarrow 0$ as $n \to \infty$; implying (2.1). Moreover, $|\varphi_2(w_n)| = |(b_n\varphi_2 b_n^*)(a) - \psi_2(a)| \le \|b_n\varphi_2 b_n^* - \psi_2\| \longrightarrow 0$ as $n \to \infty$; implying (2.2). Finally, the last requirement (2.3) is obvious with $\varepsilon = 1$ since $w_n^2 = 1$ for all n. □

Since $\varphi = \varphi_2 \circ E_2$ with the φ-preserving conditional expectation $E_2 \colon M \to M_2$ as before, one has

$$\lim_{n\to\omega} \|w_n\varphi - \varphi w_n\| = \lim_{n\to\omega} \|w_n\varphi_2 - \varphi_2 w_n\| = 0. \tag{3}$$

Hence

$$\begin{aligned} &\|v_n w_n v_n^*\psi - \psi v_n w_n v_n^*\| \\ &\quad\le \|v_n w_n(v_n^*\psi - \varphi v_n^*)\| + \|v_n(w_n\varphi - \varphi w_n)v_n^*\| + \|(v_n\varphi - \psi v_n)w_n v_n^*\| \\ &\quad\le 2\sup_n \|w_n\|_\infty \|\psi - v_n\varphi v_n^*\| + \|w_n\varphi - \varphi w_n\| \longrightarrow 0 \end{aligned} \tag{4}$$

as $n \to \infty$.

Let $x \in B_\psi$ be chosen arbitrary, and set $y := x - \psi(x)1 \in B_\psi$. By (1) y falls in M_1. We consider x inside the ultraproduct M^ω. (The necessary materials on M^ω are briefly summarized in [8, §§2.2]. The recent treatise [1] seems useful to understand it well.) We claim that the sequences v_n and w_n define elements $v, w \in M^\omega$, respectively. This immediately follows, by definition, from the following consideration: For any norm-bounded sequence z_n in M with $\lim_{n\to\omega} z_n = 0$ in the σ-strong topology, we have, by (2), (3),

$$\begin{aligned} \|z_n v_n^{\pm}\|_\varphi^2 &\le \|v_n\varphi v_n^* - \psi\| \big(\sup_n \|z_n\|_\infty\big)^2 + \|z_n\|_\psi^2 \longrightarrow 0, \\ \|z_n w_n\|_\varphi^2 &\le \|w_n\varphi - \varphi w_n\| \sup_n \|w_n\|_\infty \big(\sup_n \|z_n\|_\infty\big)^2 + |\varphi(w_n^2 z_n^* z_n)|, \\ &\le \|w_n\varphi - \varphi w_n\| \sup_n \|w_n\|_\infty \big(\sup_n \|z_n\|_\infty\big)^2 + \|z_n\|_\varphi \sup_n \|z_n\|_\infty \big(\sup_n \|w_n\|_\infty\big)^2 \\ &\longrightarrow 0 \end{aligned}$$

as $n \to \omega$. Clearly, v is a unitary in M_1^ω and $w = w^* \in M_2^\omega$. (Note that M_1^ω and M_2^ω can naturally be regarded as von Neumann subalgebras of M^ω thanks to the existence of faithful normal conditional expectations.) Here is an easy claim.

Lemma 3. *We have*:

(3.1) $v\varphi^\omega v^* = \psi^\omega$.
(3.2) $\varphi^\omega(w) = 0$.
(3.3) $w^2 \ge \varepsilon 1$ *with* $\varepsilon > 0$ *in Lemma 2.*

Proof. For every $z \in M^\omega$ with a representing sequence (z_n) we have, by (2), $|(\varphi(v_n^* z_n v_n) - \psi(z_n)| = |(v_n\varphi v_n^* - \psi)(z_n)| \le \|v_n\varphi v_n^* - \psi\| \sup_n \|z_n\|_\infty \longrightarrow 0$ as $n \to \infty$. In particular,

$$v\varphi^\omega v^*(z) = \varphi^\omega(v^* z v) = \lim_{n\to\omega} \varphi(v_n^* z_n v_n) = \lim_{n\to\omega} \psi(z_n) = \psi^\omega(z).$$

Hence we have confirmed (3.1). (3.2) and (3.3) immediately follow from (2.2) and (2.3) in Lemma 2, respectively. □

Let us complete the proof in case (a). By (3.1) above, we have

$$\psi^\omega((vwv^*y)^*(yvwv^*)) = (v^*\psi^\omega v)((v^*y^*v)w(v^*yv)w) = \varphi^\omega((v^*yv)^*w(v^*yv)w).$$

By (3.1) again we see that $\varphi^\omega(v^*yv) = (v\varphi^\omega v^*)(y) = \psi^\omega(y) = \psi(y) = 0$. These together with (3.2) above imply that

$$(yvwv^* \mid vwv^*y)_{\psi^\omega} = \psi^\omega((vwv^*y)^*(yvwv^*)) = \varphi^\omega((v^*yv)^*w(v^*yv)w) = 0, \tag{5}$$

since $v^*yv \in M_1^\omega$ and $w \in M_2^\omega$ are $*$-freely independent with respect to φ^ω (see [7, Proposition 4]). So far, we have used only (1), i.e., the fact that $y \in M_1$ rather than $y \in B_\psi$. Now, we are using our original assumption that $y \in B_\psi$. By (4) the sequence $v_n w_n v_n^*$ asymptotically commutes with ψ so that $v_n w_n v_n^* y - y v_n w_n v_n^* \longrightarrow 0$ σ-strongly as $n \to \infty$. Consequently, we get

$$\|vwv^*y\|_{\psi^\omega} \le \|vwv^*y - yvwv^*\|_{\psi^\omega} = \lim_{n\to\omega} \|v_n w_n v_n^* y - y v_n w_n v_n^*\|_\psi = 0,$$

where the first inequality follows from (5). Thus $vwv^*y = 0$ (due to the faithfulness of ψ^ω), and then $y = v^*w^{-1}v^*(vwv^*y) = 0$ since v is a unitary and w is invertible thanks to (3.3) above. Consequently, $x = \psi(x)1 \in \mathbb{C}1$. Hence we have proved $B_\psi = \mathbb{C}1$ in case (a).

We then consider case (b), which is easier than case (a). In fact, $(M_1)_{\varphi_1}$ itself must be diffuse in the case; see the proof of [8, Theorem 3.4]. Thus the ψ in case (a) should be replaced with the original φ so that we do not need the sequence v_n there. Then the proof goes along the exactly same line as in case (a) without the v_n, and the conclusion is $B_\varphi = \mathbb{C}1$. Hence we have completed the proof of Theorem 1.

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Soon after the initial version of this note was circulated to some people, Cyril Houdayer found a more sophisticated and nicer proof of the present result, where he uses his new observation together with the essentially same technical ingredients.

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Graduate School of Mathematics, Kyushu University, Fukuoka, 819-0395, Japan
E-mail address: ueda@math.kyushu-u.ac.jp