QUICK REVIEW ON PROPERTY (X)

YOSHIMICHI UEDA （植田　好道）

Abstract. We will review some materials that are useful to prove the uniqueness of preduals. Those were used crucially in our recent work on the uniqueness of predual of any 'finite' non-commutative H^∞.

1. Introduction

In [12] we established, among other things, the uniqueness of predual of any 'finite' non-commutative H^∞-algebra $H^\infty(M,\tau)$, which was introduced by Bill Arveson modeled after the usual pair $H^\infty(\mathbb{D}) \hookrightarrow L^\infty(\mathbb{T})$ with the aid of operator algebra theory. The class of finite non-commutative H^∞-algebras contains $H^\infty(\mathbb{D})$ as well as its abstract generalizations. Thus [12, Theorem 2] covers any existing generalization of the famous result due to Tsuyoshi Ando [3].

The most key ingredient of our proof of the uniqueness of predual of $H^\infty(M,\tau)$ is to provide a non-commutative analog of Amar–Lederer's peak set result [2] (also see [4]), which we fully explained in [12]. However, our proof of the uniqueness of predual also uses two purely Banach space theoretic techniques – Property (X) due to Godefroy and Talagrand and a very clever trick, both of which we just borrowed from some references without any detailed explanation. Here we will give detailed accounts (for non-experts like us) on those techniques as supplements to [12, Theorem 2].

In closing, we should mention our sincere thanks to Professor Kichi-Suke Saito for giving this opportunity.

2. Godefroy–Talagrand's Property (X)

This section mainly follows Gedefroy and Talagrand's elegant work [6]. The key ingredient behind Godefroy–Talagrand's property (X) is the next proposition.

Proposition 2.1. *Let E and G be Banach spaces with $E^\star = G^\star$. If a sequence $\{x_n\} \subset E^\star$ satisfies*

(i) $x_n \longrightarrow 0$ *in* $\sigma(E^\star, E)$*; and*
(ii) $\sum_{n=1}^\infty |\psi(x_{n+1} - x_n)| < +\infty$ *for all* $\psi \in E^{\star\star}$,

then $x_n \longrightarrow 0$ *in* $\sigma(E^\star, G)$.

Proof. Set $u_0 := x_1$, $u_1 := x_2 - x_1$, and $u_n := x_{n+1} - x_n$, and then by (i)

$$\sum_{k=0}^n u_k = x_{n+1} \longrightarrow 0 \quad \text{in } \sigma(E^\star, E). \tag{1}$$

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For each $n \in \mathbb{N}_0 := \{0,1,2,\dots\}$ we consider the map $T_n : \alpha = (\alpha_k) \in \ell^\infty(\mathbb{N}_0) \mapsto \sum_{k=0}^n \alpha_k u_k \in E^\star$ ($\hookrightarrow E^{\star\star\star}$ via the canonical embedding). Then one has, by (ii),

$$\sup\{|(T_n\alpha)(\phi)| : \|\alpha\|_\infty \le 1,\, n \in \mathbb{N}_0\} \le \sum_{k=0}^\infty |\phi(u_k)| < +\infty$$

for all $\phi \in E^{\star\star}$, and hence the uniform boundedness principle shows that there is $K > 0$ such that

$$\left\|\sum_{k=0}^n \alpha_k u_k\right\|_{E^\star} = \|T_n\alpha\|_{E^{\star\star\star}} \le K \tag{2}$$

for all $n \in \mathbb{N}_0$ and for all $\alpha_k \in \mathbb{C}$ with $|\alpha_k| \le 1$.

Choose an arbitrary free ultrafilter $\omega \in \beta(\mathbb{N}_0) \setminus \mathbb{N}_0$ and put $\xi_\omega := \lim_{n\to\omega} \sum_{k=0}^n u_k$ in $\sigma(E^\star, G)$. Let us choose arbitrary $n_1 < n_2 < \cdots < n_{2l-1} < n_{2l}$. Then, using (2) with

$$\alpha_k = \begin{cases} 1 & n_{2j-1} \le k \le n_{2j},\ j = 1,\dots,l, \\ 0 & \text{otherwise} \end{cases}$$

we get

$$\left\|\sum_{k=n_1}^{n_2} u_k + \sum_{k=n_3}^{n_4} u_k + \cdots + \sum_{k=n_{2l-1}}^{n_{2l}} u_k\right\| \le K.$$

Here we have

$$\begin{aligned}\sum_{k=n_1}^{n_2} u_k + \sum_{k=n_3}^{n_4} u_k + \cdots + \sum_{k=n_{2l-1}}^{n_{2l}} u_k &= \sum_{k=n_1}^{n_2} u_k + \sum_{k=n_3}^{n_4} u_k + \cdots + \left(\sum_{k=0}^{n_{2l}} u_k - \sum_{k=0}^{n_{2l-1}} u_k\right) \\ &\longrightarrow \sum_{k=n_1}^{n_2} u_k + \sum_{k=n_3}^{n_4} u_k + \cdots + \xi_\omega - \sum_{k=0}^{n_{2l-1}} u_k \quad \text{in } \sigma(E^\star, G)\end{aligned}$$

as $n_{2l} \to \omega$ but $n_1, \dots, n_{2l-1}$ are fixed. Then it follows that

$$\left\|\sum_{k=n_1}^{n_2} u_k + \sum_{k=n_3}^{n_4} u_k + \cdots + \xi_\omega - \sum_{k=0}^{n_{2l-1}} u_k\right\| \le K$$

for any fixed $n_1 < n_2 < \cdots < n_{2l-1}$. We also have, by (1),

$$\begin{aligned}&\sum_{k=n_1}^{n_2} u_k + \sum_{k=n_3}^{n_4} u_k + \cdots + \xi_\omega - \sum_{k=0}^{n_{2l-1}} u_k \\ &\longrightarrow \sum_{k=n_1}^{n_2} u_k + \sum_{k=n_3}^{n_4} u_k + \cdots + \xi_\omega - 0 \quad \text{in } \sigma(E^\star, E)\end{aligned}$$

as $n_{2l-1} \to \infty$ but $n_1, \dots, n_{2l-2}$ are fixed. Therefore, we get

$$\left\|\sum_{k=n_1}^{n_2} u_k + \sum_{k=n_3}^{n_4} u_k + \cdots + \sum_{k=n_{2l-3}}^{n_{2l-2}} u_k + \xi_\omega\right\| \le K$$

for any fixed $n_1 < n_2 < \cdots < n_{2l-2}$. Clearly, this procedure can be continued for n_{2l-2}, n_{2l-4} and so on, and we finally get $l \cdot \|\xi_\omega\| = \|l\xi_\omega\| \le K$. Since l can be arbitrarily large, ξ_ω must be zero for any $\omega \in \beta(\mathbb{N}_0) \setminus \mathbb{N}_0$, which means that $\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \sum_{k=0}^n u_k = 0$ in $\sigma(E^\star, G)$. $\square$

Based on the lemma, Godefroy and Talagrand introduced property (X).

Definition 2.1. A Banach space E has property (X) if for any $\psi \in E^{\star\star}$ the following conditions are equivalent:

(a) $\psi \in E$ with the canonical embedding $E \hookrightarrow E^{\star\star}$.
(b) For any sequence $\{x_n\} \subset E^\star$ with the properties
  – $x_n \longrightarrow 0$ in $\sigma(E^\star, E)$,
  – $\sum_{n=1}^\infty |\phi(x_{n+1} - x_n)| < +\infty$ for all $\phi \in E^{\star\star}$,
 one has $\psi(x_n) \longrightarrow 0$.

This definition gives, in some sense, a criterion of w^*-continuity for bounded linear functionals on the dual E^* of a Banach space E with property (X).

Definition 2.2. A Banach space E is said to be the unique predual of its dual $E^\star$ if another Banach space G with $G^\star = E^\star$ must coincide with E inside the dual $E^{\star\star}$ of $E^\star$ $(= G^\star)$ via the canonical embedding.

Corollary 2.2. *If a Banach space E has property* (X)*, then E must be the unique predual of its dual $E^\star$.*

Proof. Assume another Banach space G satisfies $G^\star = E^\star$. Embed $G \hookrightarrow (E^\star)^\star = E^{\star\star}$ by $g(x) := x(g)$ for $x \in E^\star = G^\star$ and $g \in G$. Let $\{x_n\} \subset E^\star$ be chosen in such a way that $x_n \longrightarrow 0$ in $\sigma(E^\star, E)$ and $\sum_{n=1}^\infty |\phi(x_{n+1} - x_n)| < +\infty$ for all $\phi \in E^{\star\star}$. By Proposition 2.1 we get $x_n \longrightarrow 0$ in $\sigma(E^\star, G)$, which shows that $g(x_n) = x_n(g) \longrightarrow 0$ for all $g \in G$. Thus, Property (X) ensures that any g must fall in $E \hookrightarrow E^{\star\star}$, that is, $G \subseteq E$ inside $E^{\star\star}$. If $G \subsetneqq E$ inside $E^{\star\star}$, then by the Hahn–Banach extension theorem there is $x \in E^\star$ such that $x \neq 0$ but $x|_G = 0$. (Indeed, there is $e \in E \setminus G$ by the assumption, and thus $[e] \in E/G$ with $[e] \neq 0$. Then by the Hahn–Banach extension theorem there is $\varphi \in (E/G)^\star$ sending $[e]$ to $\|[e]\| = \inf\{\|e - g\| : g \in G\} \neq 0$. Hence the $x := \varphi \circ Q \in E^\star$ with the quotient map $Q : E \to E/G$ becomes a desired element.) This x is a non-zero element in $G^\star = E^\star$ but it is identically zero on G, a contradiction. Hence $G = E$ inside $E^{\star\star}$. ☐

The next proposition has been known, but we do give one proof, which is a prototype of our proof of the uniqueness of predual of $H^\infty(M, \tau)$.

Proposition 2.3. *Let M be a σ-finite von Neumann algebra and $M_\star$ be its predual. Then, $M_\star$ has property* (X).

Proof. It suffices to show that, if $\varphi \in M^\star$ satisfies $\varphi(x_n) \longrightarrow 0$ for any $\{x_n\} \subset M$ with the properties

- $x_n \longrightarrow 0$ in $\sigma(M, M_\star)$ and
- $\sum_{n=1}^\infty |\phi(x_{n+1} - x_n)| < +\infty$ for all $\phi \in M^\star$,

then φ must fall in $M_\star \hookrightarrow M^\star$. Here we need the following standard facts on von Neumann algebras (see e.g. [9] and [11] for their proofs):

(1) Any $\psi \in M^\star$ can be decomposed into $\psi = \psi_{\mathrm{nor}} + \psi_{\mathrm{sing}}$ with $\psi_{\mathrm{nor}} \in M_\star$ and $\psi_{\mathrm{sing}} \in M^\star \ominus M_\star$, and $\|\psi\| = \|\psi_{\mathrm{nor}}\| + \|\psi_{\mathrm{sing}}\|$ holds. (This is the so-called *non-commutative Lebesgue decomposition* due to Takesaki.) We call $M_\star$ the normal part and $M^\star \setminus M_\star$ the singular part. Remark that the notation here is a little bit different from that in [12].
(2) For any $\psi \in M^\star$ (or $\psi \in M_\star$) there are a unique positive linear functional $|\psi| \in M_\star$ (resp. $|\psi| \in M_\star$) and a unique partial isometry $v \in M^{\star\star}$ (resp. $v \in M_\star$) such that $\langle \psi, x \rangle = \langle |\psi|, xv \rangle$ as well as $\langle |\psi|, x \rangle = \langle \psi, xv^* \rangle$ for $x \in M^{\star\star}$, where $\langle \cdot, \cdot \rangle : M^\star \times M^{\star\star} \to \mathbb{C}$ stands for the canonical pairing. (This is the so-called *polar decomposition*

of linear functionals due to Sakai and also Tomita.) Remark here that the second dual $M^{\star\star}$ becomes a von Neumann algebra, which naturally contains the original M as a subalgebra via the canonical embedding $M \hookrightarrow M^{\star\star}$.

(3) Both the closed subspaces $M_\star$ and $M^\star \ominus M_\star$ of $M^\star$ are closed under the operation $\psi \in M^\star \mapsto |\psi| \in M^\star$. (This follows from the construction of the decomposition in (1) together with (2).)

(4) For a positive linear functional $\psi \in M^\star$ the following are equivalent:
- $\psi \in M^\star \ominus M_\star$.
- For every nonzero projection $e \in M$ there is a non-zero projection $e_0 \in M$ such that $e_0 \le e$ and $\psi(e_0) = 0$.

(This is Takesaki's criterion for 'singularity' of linear functionals.)

(5) Any $\psi \in M^\star$ (or $M_\star$) can be written as a linear combination of four positive linear functionals in $M^\star$ (resp. $M_\star$).

Let us decompose the given φ into $\varphi = \varphi_{\rm nor} + \varphi_{\rm sing}$ as in (1), and what we have to show is $\varphi_{\rm sing} = 0$, i.e., $\varphi = \varphi_{\rm nor} \in M_\star$. For contrary we suppose $\varphi_{\rm sing} \neq 0$. Then, by (2) and (3), $|\varphi_{\rm sing}| \neq 0$ and $|\varphi_{\rm sing}| \in M^\star \ominus M_\star$ still holds. Clearly, the orthogonal families of non-zero projections in $\mathrm{Ker}|\varphi_{\rm sing}|$ forms an inductive set by inclusion, and Zorn's lemma ensures the existence of a maximal family $\{q_k\}$, which is at most countable since M is σ-finite. Put $q_0 := \sum_k q_k$ in M, and then $q_0 = 1$ since $q_0 \neq 1$ clearly contradicts to the above (4). Also, if $\{q_k\}$ is a finite family, then $|\varphi_{\rm sing}|(1) = \sum_k |\varphi_{\rm sing}|(q_k) = 0$, a contradiction. Therefore, $\{q_k\}$ must be a countably infinite family with $\sum_k q_k = 1$ in M. Letting $p_n := 1 - \sum_{k\le n} q_k$ we have $p_n \searrow 0$ in $\sigma(M, M_\star)$ but $|\varphi_{\rm sing}|(p_n) = |\varphi_s|(1)$ for all n. The latter says that p_n converges a non-zero projection $p \in M^{\star\star}$ in $\sigma(M^{\star\star}, M^\star)$ with $\langle|\varphi_{\rm sing}|, p\rangle = \langle|\varphi_{\rm sing}|, 1\rangle$ $(= |\varphi_{\rm sing}|(1))$ since p_n is a decreasing sequence. Let $u \in M$ and $v \in M^{\star\star}$ be the partial isometries for the polar decompositions of $\varphi_{\rm nor}$ and $\varphi_{\rm sing}$, respectively. Then, for $x \in M^{\star\star}$ one has $|\langle\varphi_{\rm sing}, (1-p)x\rangle| = |\langle|\varphi_{\rm sing}|, (1-p)xv\rangle| \le \langle|\varphi_{\rm sing}|, 1-p\rangle^{1/2}\langle|\varphi_{\rm sing}|, v^*x^*xv\rangle^{1/2} = 0$ so that $\langle\varphi_{\rm sing}, x\rangle = \langle\varphi_{\rm sing}, px\rangle$ since $\langle|\psi_{\rm sing}|, p\rangle = \langle|\psi_{\rm sing}|, 1\rangle$. Similarly, for $x \in M^{\star\star}$ one has $|\langle\varphi_{\rm nor}, px\rangle| = |\langle|\varphi_{\rm nor}|, pxu\rangle| \le \langle|\varphi_{\rm nor}|, p\rangle^{1/2}\langle|\varphi_{\rm nor}|, u^*x^*xu\rangle^{1/2}$. Since $|\varphi_{\rm nor}|$ still falls in $M_\star$, $\langle|\varphi_{\rm nor}|, p\rangle = \lim_{n\to\infty} |\varphi_{\rm nor}|(p_n) = 0$ so that $\langle\varphi_{\rm nor}, px\rangle = 0$. Consequently, we get $\langle\varphi, px\rangle = \langle\varphi_{\rm nor} + \varphi_{\rm sing}, px\rangle = \varphi_{\rm sing}(x)$ for $x \in M$.

Let $x \in M$ be arbitrary. Clearly, $p_n x \longrightarrow 0$ in $\sigma(M, M_\star)$. Let $\phi \in M^\star$ be arbitrary, and decompose $y \in M \mapsto \phi(yx)$ into a linear combination of four positive linear functionals $\phi_i \in M^\star$, $i = 1, 2, 3, 4$, thanks to the above (5). Since $\sum_{n=1}^N |\phi_i(p_{n+1} - p_n)| = \sum_{n=1}^N \phi_i(q_{n+1}) = \phi_i(\sum_{n=2}^{N+1} q_n) \le \phi_i(1) < +\infty$ for all $N \in \mathbb{N}$, it follows that $\sum_{n=1}^\infty |\phi(p_{n+1}x - p_n x)| < +\infty$. Therefore, by the assumption here one has $\varphi(p_n x) \longrightarrow 0$. On the other hand, $\varphi(p_n x) = \langle\varphi, p_n x\rangle \longrightarrow \langle\varphi, px\rangle = \varphi_{\rm sing}(x)$ so that $\varphi_{\rm sing} = 0$, a contradiction. □

The heart of the above proof is as follows. Although $\varphi_{\rm nor}$ and $\varphi_{\rm sing}$ are 'orthogonal', we cannot find a projection in M that distinguishes those. (Of course, we can find such a projection in $M^{\star\star}$ since both functionals can be regarded as 'normal' ones on $M^{\star\star}$.) Thus we first construct a projection $p \in M^{\star\star}$ in such a way that it can be 'nicely' approximated by projections in M and p is greater than 'the support of $\varphi_{\rm sing}$' but 'disjoint' from 'the support of $\varphi_{\rm nor}$'. This essentially says that M 'remembers' the decomposition '$M^\star = M_\star \oplus \big(M^\star \ominus M_\star\big)$' of $M^\star$ (the second dual of $M_\star$). This suggests us that such a decomposition of the second dual should be related to property (X) of a Banach space in question. This was quite recently answered affirmatively by Hermann Pfitzner when a Banach space in question is separable, see [8].

Further accounts on the present topics can be found in [5].

3. Addendum – a clever trick due to Pełczyński

The essential idea of our proof of the uniqueness of predual of $H^\infty(M,\tau)$ is similar to that of Proposition 2.3. However, the luck of self-adjointness of our algebra $H^\infty(M,\tau)$ (thus we cannot use the order structure) makes some trouble, which we overcame with a clever trick borrowed from the proof of [7, Proposition 1.c.3]. (The trick is due to Aleksander Pełczyński, see [10, p.637] for this credit, and it was originally used for proving that if a Banach space has Pełczyński's property (u) then so does any closed subspace, see [7] or more recent [1].) Here we will explain it. The situation we deal with is as follows. Let M be a von Neumann algebra and A be its σ-weakly closed (possibly non-self-adjoint) unital subalgebra. Assume that we have two sequences $\{a_n\} \subset A$ and $\{b_n\} \subset M$ such that

(i) both a_n and b_n converge to the same $p \in M^{\star\star}$ in $\sigma(M^{\star\star}, M^\star)$, and
(ii) $\sum_{n=1}^\infty |\phi(b_{n+1} - b_n)| < +\infty$ for all $\phi \in M^\star$.

What we want to do is to replace a_n by a new one with keeping (i) and further satisfying (ii). This can be done by utilizing the above-mentioned clever trick in Banach space theory.

Proposition 3.1. *There is another $\{a'_n\} \subset A$ such that*

(i') *$a'_n \longrightarrow p$ in $\sigma(M^{\star\star}, M^\star)$, and*
(ii') *$\sum_{n=1}^\infty |\phi(a'_{n+1} - a'_n)| < +\infty$ for all $\phi \in M^\star$.*

We need one elementary lemma due to Stanisław Mazur.

Lemma 3.2. *Let E be a normed space and $\{x_n\} \subset E$ be such that $x_n \longrightarrow 0$ in $\sigma(E, E^\star)$. Then, for each $\varepsilon > 0$ and each $m \in \mathbb{N}$ there is a convex combination $y = \sum_{n \geq m} \lambda_n x_n$ with $\|y\| < \varepsilon$.*

Proof. Let C_m be the closed convex hull of $\{x_n\}_{n \geq m}$ in E. It suffice to show $0 \in C_m$. Thus, for contrary, suppose $0 \notin C_m$. Then there is a small open ball B centered at 0 with $C_m \cap B = \emptyset$. The Hahn–Banach separation theorem ensures that there are $\varphi \in E^\star$ and $t \in \mathbb{R}$ such that $\mathrm{Re}\varphi(b) \lneqq t \leqq \mathrm{Re}\varphi(c)$ for all $b \in B$ and $c \in C_m$. This is impossible since $x_n \longrightarrow 0$ in $\sigma(E, E^\star)$ (implying $t \leq 0$) and $0 \in B$ (implying $t \gneqq 0$). Thus $0 \in C_m$, which means the desired assertion. ☐

Proof. (Proposition 3.1) Putting $b_0 := 0$ we have $\sum_{n=1}^\infty |\phi(b_n - b_{n-1})| < +\infty$ for all $\phi \in M^\star$. Set $u_n := a_n - \sum_{k=1}^n b_k - b_{k-1}$, and then $u_n = a_n - b_n \longrightarrow 0$ in $\sigma(M, M^\star)$ by (i). By Lemma 3.2 there are convex combinations $u'_j = \sum_{n=p_{j-1}+1}^{p_j} \lambda_n^{(j)} u_n$ such that $0 = p_0 < p_1 < p_2 < \cdots$ and $\|u'_j\| \leq 2^{-j}$. Then We define $a'_j := \sum_{n=p_{j-1}+1}^{p_j} \lambda_n^{(j)} a_n \in A$ and put $a'_0 := 0$ for convenience. Let us prove that this $\{a'_j\}$ gives a desired sequence.

Since $a_n \longrightarrow p$ in $\sigma(M^{\star\star}, M^\star)$, for any $\varepsilon > 0$ and any $\phi \in M^\star$ there is $n_0 \in \mathbb{N}$ such that $|\langle a_n, \phi\rangle - \langle p, \phi\rangle| < \varepsilon$ for all $n \geq n_0$, where $\langle\cdot,\cdot\rangle : M^{\star\star} \times M^\star \mapsto \mathbb{C}$ is the canonical pairing. If j_0 is chosen so that $p_{j_0-1}+1 \geq n_0$, then one has $|\langle a'_j, \phi\rangle - \langle p, \phi\rangle| \leq \sum_{n=p_{j-1}+1}^{p_j} \lambda_n^{(j)} |\langle a_n, \phi\rangle - \langle p, \phi\rangle| < \varepsilon$ for all $j \geq j_0$. Thus $a'_j \longrightarrow p$ in $\sigma(M^{\star\star}, M^\star)$ as $j \to \infty$.

One has

$$\begin{aligned} a'_{j+1} - a'_j &= u'_{j+1} + \sum_{n=p_j+1}^{p_{j+1}} \lambda_n^{(j+1)}(a_n - u_n) - u'_j - \sum_{n=p_{j-1}+1}^{p_j} \lambda_n^{(j)}(a_n - u_n) \\ &= u'_{j+1} - u'_j + \sum_{n=p_j+1}^{p_{j+1}} \lambda_n^{(j+1)}(\sum_{k=1}^n b_k - b_{k-1}) - \sum_{n=p_{j-1}+1}^{p_j} \lambda_n^{(j)}(\sum_{k=1}^n b_k - b_{k-1}) \\ &= u'_{j+1} - u'_j + \sum_{n=p_{j-1}+1}^{p_{j+1}} \mu_n^{(j)}(b_n - b_{n-1}) \end{aligned}$$

with some $0 \le \mu_n^{(j)} \le 1$. Hence,

$$\begin{aligned}
&\sum_{j=0}^{\infty} |\phi(a'_{j+1} - a'_j)| \\
&\le \sum_{j=0}^{\infty} \|\phi\|\|u'_{j+1}\| + \sum_{j=0}^{\infty} \|\phi\|\|u'_j\| + \sum_{j=1}^{\infty} \sum_{n=p_{j-1}+1}^{p_{j+1}} \mu_n^{(j)} |\phi(b_n - b_{n-1})| \\
&\le 2\sum_{j=0}^{\infty} \|\phi\|\|u'_j\| + \sum_{n=1}^{\infty} |\phi(b_n - b_{n-1})| \\
&\le 4\|\phi\| + \sum_{n=1}^{\infty} |\phi(b_n - b_{n-1})| < +\infty
\end{aligned}$$

by $\|u'_j\| \le 2^{-j}$ and (ii). □

Remark here that the argument presented above uses only the linear structure; hence clearly it can be applied to more general situations.

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Graduate School of Mathematics, Kyushu University, Fukuoka, 810-8560, Japan
E-mail address: ueda@math.kyushu-u.ac.jp