ON ORBITAL FREE ENTROPY DIMENSION

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Abstract. Due to the lack of time, I present an outtake from one of my praivate notes. I believe that this may serve as an introduction to orbital free entropy dimensions.

1. Introduction

In [4] we introduced the notion of orbital free entropy dimension $\delta_{0,\mathrm{orb}}(\mathbf{X}_1,\dots,\mathbf{X}_n)$ for hyperfinite random self-adjoint multi-variables (i.e., tuples of self-adjoint random variables in a fixed tracial W^*-probability space, each of which generates a hyperfinite von Neumann algebra), and showed

$$\delta_0(\mathbf{X}_1 \sqcup \cdots \sqcup \mathbf{X}_n) \leq \delta_{0,\mathrm{orb}}(\mathbf{X}_1,\dots,\mathbf{X}_n) + \sum_{i=1}^{n} \delta_0(\mathbf{X}_i), \tag{1}$$

where δ_0 means Voiculescu's (modified) free entropy dimension (see [8]). Moreover, we could showed that the equality holds true when all $W^*(\mathbf{X}_i)$'s are finite dimensional, and it had been open whether the equality holds in general. Very recently we resolved it affirmatively, which led to the following lower semicontinuity result for δ_0: Let $\mathbf{X}_1,\dots,\mathbf{X}_n$ be hyperfinite self-ajoint multi-variables, and assume, for each $k=1,\dots,n$, that we have a sequence $\mathbf{X}_k^{(m)}$ of hyperfinite multi-variables that converges to $\mathbf{X}_k$ strongly. In this setup, we will see that, if $\mathbf{X}_k^{(m)} \subseteq W^*(\mathbf{X}_k)$ is further assumed for every $m \in \mathbb{N}$ and $k=1,\dots,n$, then

$$\liminf_{m\to\infty} \delta_0(\mathbf{X}_1^{(m)} \sqcup \cdots \sqcup \mathbf{X}_n^{(m)}) \geq \delta_0(\mathbf{X}_1 \sqcup \cdots \sqcup \mathbf{X}_n)$$

holds. This is probably the first semicontinuity result of δ_0 of non-commutative nature. The details of this recent progress will be presented in a revised and expanded version of [4].

Here we would like to give a rather direct and standard proof of the following general upper bound:

$$\delta_{0,\mathrm{orb}}(\mathbf{X}_1,\dots,\mathbf{X}_n) \leq -(n-1)\delta_0\big(W^*(\mathbf{X}_1) \cap \ldots \cap W^*(\mathbf{X}_n)\big). \tag{2}$$

Although this fact itself can be immediately obtained as a simple corollary of our affirmative resoltion mentioned above, the argument presented in this note may have some degree of positive significance as an introduction to the orbital theory of free entropy dimension.

2. Preliminaries

Let $\mathbf{X}_1,\dots,\mathbf{X}_n$ be hyperfinite random self-adjoint multi-variables. For each multi-variable $\mathbf{X}_i = (X_{i1},\dots,X_{ir(i)})$ one can choose a sequence of microstates $\Xi_i(N) = (\xi_{i1}(N),\dots,\xi_{ir(i)}(N))$, $N \in \mathbb{N}$, which means that $\Xi_i(N) \subset M_N(\mathbb{C})^{sa}$, $\|\xi_{ij}(N)\|_\infty \leq \|\mathbf{X}_{ij}\|_\infty$ ($j=1,\dots,r(i)$), and $\Xi_i(N)$ converges, in moments, to $\mathbf{X}_i$. Then for each $m \in \mathbb{N}$ and $\delta > 0$ the orbital microstates $\Gamma_{\mathrm{orb}}(\mathbf{X}_1,\dots,\mathbf{X}_n : \Xi_1(N),\dots,\Xi_n(N); N, m, \delta)$ is defined to be all n-tuples $(U_1,\dots,U_n) \in \mathrm{U}(N)^n$ satisfying $\big(U_1\Xi_1(N)U_1^*,\dots,U_n\Xi_n(N)U_n^*\big) \in \Gamma(\mathbf{X}_1 \sqcup \cdots \sqcup \mathbf{X}_n; N, m, \delta)$, where $U_k\Xi_k(N)U_k^*$

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denotes the l_k-tuple $U_k\xi_{k1}(N)U_k^*,\dots,U_k\xi_{kl_k}(N)U_k^*$. Then the orbital free entropy dimension $\delta_{0,\mathrm{orb}}(\mathbf{X}_1,\dots,\mathbf{X}_n)$ is originally defined by utilizing the orbital free entropy χ_{orb} with Voiculescu's liberation process, but it admits Jung's covering/packing formalism. Namely, letting

$$\mathbb{K}_\varepsilon^{\mathrm{orb}}(\mathbf{X}_1,\dots,\mathbf{X}_n) := \lim_{\substack{m\to\infty\\ \delta\nearrow 0}} \limsup_{N\to\infty} \frac{1}{N^2}\log K_\varepsilon\big(\Gamma_{\mathrm{orb}}(\mathbf{X}_1,\dots,\mathbf{X}_n : \Xi_1(N),\dots,\Xi_n(N); N, m, \delta)\big),$$

$$\mathbb{P}_\varepsilon^{\mathrm{orb}}(\mathbf{X}_1,\dots,\mathbf{X}_n) := \lim_{\substack{m\to\infty\\ \delta\nearrow 0}} \limsup_{N\to\infty} \frac{1}{N^2}\log P_\varepsilon\big(\Gamma_{\mathrm{orb}}(\mathbf{X}_1,\dots,\mathbf{X}_n : \Xi_1(N),\dots,\Xi_n(N); N, m, \delta)\big)$$

we have

$$\delta_{0,\mathrm{orb}}(\mathbf{X}_1,\dots,\mathbf{X}_n) = \limsup_{\varepsilon\searrow 0}\frac{\mathbb{P}_\varepsilon^{\mathrm{orb}}(\mathbf{X}_1,\dots,\mathbf{X}_n)}{|\log\varepsilon|} - n = \limsup_{\varepsilon\searrow 0}\frac{\mathbb{K}_\varepsilon^{\mathrm{orb}}(\mathbf{X}_1,\dots,\mathbf{X}_n)}{|\log\varepsilon|} - n. \tag{3}$$

Here $K_\varepsilon(\mathcal{X})$ and $P_\varepsilon(\mathcal{X})$ for a subset $\mathcal{X}$ in the metric space $\mathrm{U}(N)^n$ equipped with the metric

$$d((U_1,\dots,U_n),(V_1,\dots,V_n)) := \sqrt{\sum_{k=1}^n \|U_k - V_k\|_{\mathrm{tr}_N,2}^2} \tag{4}$$

denote the minimal number of ε-balls that covers $\mathcal{X}$ and the maxiaml number of disjoint ε-balls inside $\mathcal{X}$, respectively. Note that $\delta_{0,\mathrm{orb}}(\mathbf{X}_1,\dots,\mathbf{X}_n)$ is independent of the choices of $\Xi_i(N)$'s, and moreover it depends only on the relative position among the $W^*(\mathbf{X}_k)$'s in the tracial W^*-probability space, that is,

$$W^*(\mathbf{X}_i) = W^*(\mathbf{X}_i'),\ i=1,\dots,n \quad\Longrightarrow\quad \delta_{0,\mathrm{orb}}(\mathbf{X}_1,\dots,\mathbf{X}_n) = \delta_{0,\mathrm{orb}}(\mathbf{X}_1',\dots,\mathbf{X}_n'). \tag{5}$$

The latter fact trivially implies that

$$\mathbf{Y}_i \subset W^*(\mathbf{X}_i),\ i=1,\dots,n \quad\Longrightarrow\quad \delta_{0,\mathrm{orb}}(\mathbf{X}_1,\dots,\mathbf{X}_n) \le \delta_{0,\mathrm{orb}}(\mathbf{Y}_1,\dots,\mathbf{Y}_n). \tag{6}$$

These facts on $\delta_{0,\mathrm{orb}}$ come from the corresponding ones on the orbital free entropy χ_{orb}.

3. Upper estimate of $\delta_{0,\mathrm{orb}}$

Theorem 3.1. *Let $\mathbf{X}$ be a hyperfinite self-adjoint random multi-variable in a tracial W^*-probability space (M,τ). Then we have*

$$\delta_{0,\mathrm{orb}}(\underbrace{\mathbf{X},\dots,\mathbf{X}}_{n\ times}) \le -(n-1)\delta_0(\mathbf{X}).$$

The proof will be devided into several steps, and we begin by looking at the structure of $W^*(\mathbf{X})$.

Let us decompose $W^*(\mathbf{X}) = C_0 \oplus C_1 \oplus \cdots \oplus C_s$ possibly with $s=\infty$ such that C_0 has no minimal projection and each C_r ($r=1,\dots,s$) is isomorphic to $M_{m_r}(\mathbb{C})$. Let p_r be the central support projection of C_r in $W^*(\mathbf{X})$ for $r=1,\dots,n$. We may and do assume that C_0 is abelian, i.e., C_0 is isomorphic to $L^\infty[0,1]$. Choose $X_0 \in C_0 = L^\infty[0,1]$ in $W^*(\mathbf{X})$ as $X_0(t)=t$ in $[0,1]$, and a matrix unit system $\{e_{ij}^{(r)}\}_{i,j=1}^{m_r}$ in C_r ($\cong M_{m_r}(\mathbb{C})$), $1\le r\le s$. For a while we do further assume that $s<\infty$. Set $X_{r1} := e_{11}^{(r)},\dots,X_{rm_r} := e_{m_rm_r}^{(r)}$ and $X_{r0} := \sum_{i=1}^{m_r} e_{ii}^{(r)}$, all of which are self-ajoint. Then the new hyperfinite self-adjoint random multi-variable $\mathbf{X}' = \{p_0,\dots,p_s\}\sqcup\{X_0\}\sqcup\bigsqcup_{r=1}^s\{X_{ri} : i=0,\dots,m_r\}$ clearly satisfies $W^*(\mathbf{X}') = W^*(\mathbf{X})$. Hence it suffices to prove the desired inequality with replacing $\mathbf{X}$ by $\mathbf{X}'$.

For any sufficiently large $N \in \mathbb{N}$ one can choose positive integers $n_0(N)$ and $n_r(N) = m_r m_r(N)$, $r = 1, \dots, s$, in such a way that

$$\sum_{r=0}^{s} n_r(N) = N, \tag{7}$$

$$\lim_{N\to\infty} \frac{n_r(N)}{N} = \tau(p_r), \quad r = 1, \dots, s. \tag{8}$$

Then one can also choose an orthogonal family $\{P_r(N)\}_{r=0}^{s}$ of projections in $M_N(\mathbb{C})$ so that $\sum_{r=0}^{s} P_r(N) = I_N$ and $\mathrm{rank}(P_r(N)) = n_r(N)$ for $r = 0, \dots, s$. For $r = 1, \dots, s$ we observe

$$P_r(N)M_N(\mathbb{C})P_r(N) \cong M_{n_r(N)}(\mathbb{C}) \cong M_{m_r}(\mathbb{C}) \otimes M_{m_r(N)}(\mathbb{C}),$$

and one identification $P_r(N)M_N(\mathbb{C})P_r(N) = M_{m_r}(\mathbb{C}) \otimes M_{m_r(N)}(\mathbb{C})$ is fixed for each r in what follows. Let

$$\xi_0(N) : = \mathrm{Diag}[1/n_0(N), 2/n_0(N), \dots, 1] \in M_{n_0(N)}(\mathbb{C}) = P_0(N)M_N(\mathbb{C})P_0(N), \tag{9}$$

$$\eta_{ij}^{(r)}(N) : = e_{ij}^{(r)} \otimes I_{m_r(N)} \in M_{m_r}(\mathbb{C}) \otimes M_{m_r(N)}(\mathbb{C}) = P_r(N)M_N(\mathbb{C})P_r(N) \tag{10}$$

for $i, j = 1, \dots, m_r$ and $r = 1, \dots, s$. The next lemma is clear, and the details are left to the reader.

Lemma 3.2. *The matricial multi-variables* $\{P_0(N), \dots, P_s(N)\} \sqcup \{\xi_0(N)\} \sqcup \bigsqcup_{r=1}^{s}\{\eta_{ij}^{(r)}(N) : i, j = 1, \dots, m_r\}$ *converges in moments to* $\{p_0, \dots, p_s\} \sqcup \{X_0\} \sqcup \bigsqcup_{r=1}^{s}\{e_{ij}^{(r)} : i, j = 1, \dots, m_r\}$ *as* $N \to \infty$.

For each $r = 1, \dots, s$ set $\xi_{ri}(N) := \eta_{ii}^{(r)}(N)$, $i = 1, \dots, m_r$ and $\xi_{r0}(N) := \sum_{i,j=1}^{m_r} \eta_{ij}^{(r)}(N)$. Then the above lemma says that the matricial multi-variable

$$\Xi(N) := \{P_0(N), \dots, P_s(N)\} \sqcup \{\xi_0(N)\} \sqcup \bigsqcup_{r=1}^{s}\{\xi_{ri}(N) : i = 0, \dots, m_r\}$$

coverges in moments to $\mathbf{X}'$ as $N \to \infty$. Remark here that $\xi_{ri}(N)\xi_{r0}(N)\xi_{rj}(N) = \eta_{ij}^{(r)}(N)$, being exactly a matrix unit in $M_{m_r}(\mathbb{C}) \otimes \mathbb{C}I_{m_r(N)} \subset P_r(N)M_N(\mathbb{C})P_r(N)$ for $i, j = 1, \dots, m_r$ and $r = 1, \dots, s$. Correspondingly, $X_{ri}X_{r0}X_{rj} = e_{ij}^{(r)}$ holds too.

For $\varepsilon > 0$ let $\Omega(N; \varepsilon)$ be the set of all $W \in \mathrm{U}(N)$ such that

$$\|[W, P_r(N)]\|_{\mathrm{tr}_N,2} < \varepsilon \quad (r = 0, \dots, s), \tag{11}$$

$$\|[W, \xi_0(N)]\|_{\mathrm{tr}_N,2} < \varepsilon, \tag{12}$$

$$\|[W, \eta_{ij}^{(r)}(N)]\|_{\mathrm{tr}_N,2} < \varepsilon \quad (i, j = 1, \dots, m_r,\ r = 1, \dots, s). \tag{13}$$

Then we have the following lemma.

Lemma 3.3. *For each* $\varepsilon > 0$ *there are* $m \in \mathbb{N}$ *and* $\delta > 0$ *such that*

$$\Gamma_{\mathrm{orb}}(\mathbf{X}', \dots, \mathbf{X}' : \Xi(N), \dots, \Xi(N); N, m, \delta)$$
$$\subseteq \Psi(N; \varepsilon) := \{(U, UW_1, \dots, UW_{n-1}) : U \in \mathrm{U}(N),\ W_1, \dots, W_{n-1} \in \Omega(N)\}$$

for all sufficiently large $N \in \mathbb{N}$.

Proof. Choose $(U_1, \dots, U_n)$ from the left-hand side. Since

$$U_k\eta_{ij}^{(r)}(N)U_k^* = (U_k\xi_{ri}(N)U_k^*)(U_k\xi_{r0}(N)U_k^*)(U_k\xi_{rj}(N)U_k^*), \quad k = 1, \dots, n,$$

one can easily find $m \in \mathbb{N}$ and $\delta > 0$ so that

$$\left|\mathrm{tr}_N((U_kP_r(N)U_k^* - U_1P_r(N)U_1^*)^2) - \tau((p_r - p_r)^2)\right| < \varepsilon^2 \quad (r = 0, \dots, s);$$

$$\big|\mathrm{tr}_N((U_k\xi_0(N)U_k^* - U_1\xi_0(N)U_1^*)^2) - \tau((X_0-X_0)^2)\big| < \varepsilon^2;$$
$$\big|\mathrm{tr}_N((U_k\eta_{ij}^{(r)}(N)U_k^* - U_1\eta_{ij}^{(r)}(N)U_1^*)^*(U_k\eta_{ij}^{(r)}(N)U_k^* - U_1\eta_{ij}^{(r)}(N)U_1^*)) - \tau((e_{ij}^{(r)}-e_{ij}^{(r)})^*(e_{ij}^{(r)}-e_{ij}^{(r)}))\big| < \varepsilon^2 \quad (i,j=1,\dots,m_r,\ r=1,\dots,s).$$

Then by letting $W_k := U_1^*U_{k+1}$ the assertion immediately follows. □

The next lemma is the main estimate.

Lemma 3.4. *For each $t\in(0,1)$ let $c(t) := \sqrt{s(s+1)+t^{-2}+s(m_*+1)} > 0$ with $m_* := \max\{m_r : r=1,\dots,s\}$. Then, for any sufficiently small $\kappa>0$ there is $\varepsilon\in(0,\kappa)$ so that*

$$K_{2(c(t)+1)\kappa}\big(\Omega(N;\varepsilon)\big) \le \left(\frac{3\sqrt{s+1}}{\kappa}\right)^{2n_0(N)^2t}\left(\frac{2C_{\mathrm{unitary}}\sqrt{s+1}}{\kappa}\right)^{\sum_{r=1}^s m_r(N)^2},$$

where $C_{\mathrm{unitary}}>0$ is a universal constant, independent of any other parameter.

Proof. Let $W\in\Omega(N;\varepsilon)$ be arbitrary. By (11) we have

$$\|P_{r_1}(N)WP_{r_2}(N)\|_{\mathrm{tr}_N,2} \le \Big\|\sum_{r\neq r'} P_{r'}(N)WP_r(N) - P_r(N)WP_{r'}(N)\Big\|_{\mathrm{tr}_N,2} = \|[P_r(N),W]\|_{\mathrm{tr}_N,2} < \varepsilon$$

as long as $r_1\neq r_2$. Also, by (12) we have

$$\|[P_0(N)WP_0(N),\xi_0(N)]\|_{\mathrm{tr}_N,2} \le \Big\|\sum_{r=0}^s P_r(N)W\xi_0(N) - \xi_0(N)WP_r(N)\Big\|_{\mathrm{tr}_N,2} = \|[W,\xi_0(N)]\|_{\mathrm{tr}_N,2} < \varepsilon.$$

In what follows we write $W_{r_1r_2} := P_{r_1}(N)WP_{r_2}(N)$ for $r_1,r_2 = 0,\dots .s$. Then it follows that

$$\|W_{r_1r_2}\|_{\mathrm{tr}_N,2} < \varepsilon \quad \text{as long as } r_1\neq r_2, \tag{14}$$
$$\|[W_{00},\xi_0(N)]\|_{\mathrm{tr}_N,2} < \varepsilon. \tag{15}$$

and, in particular, (14) implies that

$$\Big\|\sum_{r_1\neq r_2} W_{r_1r_2}\Big\|_{\mathrm{tr}_N,2} < \sqrt{s(s-1)}\,\varepsilon. \tag{16}$$

Let $t\in(0,1)$ be also arbitrary. Denote $\lambda_i := \frac{i}{n_0(N)}$, the ith nonzero eigenvalue of $\xi_0(N)$. Then it is plain to see that $S_0 := \{(i,j)\in\{1,\dots,n_0(N)\}^2 : |\lambda_i-\lambda_j|<t\}$ has the cardinality less than $n_0(N)^2t$. Let $S_0^\perp := \{1,\dots,n_0(N)\}^2\setminus S_0$, and decompose $P_0(N)M_N(\mathbb{C})P_0(N) = M_{n_0(N)}(\mathbb{C}) = [E_{ij}^{(0)} : (i,j)\in S_0]\oplus[E_{ij}^{(0)} : (i,j)\in S_0^\perp]$, a direct sum with respect to the Euclidean structure induced from tr_N, where the $E_{ij}^{(0)}$'s are standard matrix units in $M_{n_0(N)}(\mathbb{C}) = P_0(N)M_N(\mathbb{C})P_0(N)$. Write $W_{00} = \sum_{i,j=1}^{n_0(N)} w_{ij}^{(0)}E_{ij}^{(0)}$, and define $W_{00}^{S_0} := \sum_{(i,j)\in S_0} w_{ij}^{(0)}E_{ij}^{(0)}$, $W_{00}^{S_0^\perp} := \sum_{(i,j)\in S_0^\perp} w_{ij}^{(0)}E_{ij}^{(0)}$. Then we estimate

$$\varepsilon^2 > \|[W_{00},\xi_0(N)]\|_{\mathrm{tr}_N,2}^2$$

$$
\begin{aligned}
&= \frac{1}{N}\sum_{i,j=1}^{n_0(N)} |\lambda_i - \lambda_j|^2 |w_{ij}^{(0)}|^2 \\
&\geq \frac{1}{N}\sum_{(i,j)\in S_0^\perp} |\lambda_i - \lambda_j|^2 |w_{ij}^{(0)}|^2 \\
&\geq t^2 \frac{1}{N}\sum_{(i,j)\in S_0^\perp} |w_{ij}^{(0)}|^2 \\
&= t^2 \|W_{00}^{S_0^\perp}\|_{\mathrm{tr}_N,2}^2
\end{aligned}
$$

so that

$$
\|W_{00}^{S_0^\perp}\|_{\mathrm{tr}_N,2} < \varepsilon/t. \tag{17}
$$

Next we will treat with W_{rr} with $r = 1,\dots,s$. As before we identify $P_r(N)M_N(\mathbb{C})P_r(N) = M_{m_r}(\mathbb{C}) \otimes M_{m_r(N)}(\mathbb{C})$, and write $W_{rr} := \sum_{i,j=1}^{m_r} e_{ij}^{(r)} \otimes W_{ij}^{(r)}$ with $W_{ij}^{(r)} \in M_{m_r(N)}(\mathbb{C})$. By (13) we have

$$
\begin{aligned}
\varepsilon^2 &> \|[W, \eta_{ij}^{(r)}]\|_{\mathrm{tr}_N,2}^2 \\
&= \left\| \sum_{r'=0}^{s} P_{r'}(N) W \eta_{ij}^{(r)}(N) - \eta_{ij}^{(r)}(N) W P_{r'}(N) \right\|_{\mathrm{tr}_N,2}^2 \\
&\geq \|P_r(N) W \eta_{ij}^{(r)}(N) - \eta_{ij}^{(r)}(N) W P_r(N)\|_{\mathrm{tr}_N,2}^2 \\
&= \|W_{rr}\eta_{ij}^{(r)}(N) - \eta_{ij}^{(r)}(N) W_{rr}\|_{\mathrm{tr}_N,2}^2 \\
&= \left\| \sum_{k=1}^{m_r} e_{ik}^{(r)} \otimes W_{jk}^{(r)} - e_{kj}^{(r)} \otimes W_{ki}^{(r)} \right\|_{\mathrm{tr}_N,2}^2 \\
&= \sum_{k\neq i} \|e_{kj}^{(r)} \otimes W_{ki}^{(r)}\|_{\mathrm{tr}_N,2}^2 + \sum_{k\neq j} \|e_{ik}^{(r)} \otimes W_{jk}^{(r)}\|_{\mathrm{tr}_N,2}^2 \\
&\quad + \|e_{ij}^{(r)} \otimes (W_{ii}^{(r)} - W_{jj}^{(r)})\|_{\mathrm{tr}_N,2}^2.
\end{aligned}
$$

for every $i, j = 1,\dots,m_r$. Hence, letting $W_0^{(r)} := \frac{1}{m_r}\sum_{i=1}^{m_r} W_{ii}^{(r)}$ we get

$$
\|e_{ij}^{(r)} \otimes W_{ij}^{(r)}\|_{\mathrm{tr}_N,2} < \varepsilon \quad (i \neq j), \tag{18}
$$

$$
\|e_{ii}^{(r)} \otimes \big(W_{ii}^{(r)} - W_0^{(r)}\big)\|_{\mathrm{tr}_N,2} < \varepsilon \quad (i = 1,\dots,m_r). \tag{19}
$$

Let $\hat{W}_{rr} := I_{m_r} \otimes W_0^{(r)}$, which is the image of the trace-preserving conditional expectation of $W_{rr} \in M_{m_r}(\mathbb{C}) \otimes M_{m_r(N)}(\mathbb{C})$ to $\mathbb{C}I_{m_r} \otimes M_{m_r(N)}(\mathbb{C})$. Also let

$$
\begin{aligned}
\hat{W}_{rr}^\perp :&= W_{rr} - \hat{W}_{rr} \\
&= \sum_{i=1}^{m_r} e_{ii}^{(r)} \otimes \big(W_{ii}^{(r)} - W_0^{(r)}\big) + \sum_{i\neq j} e_{ij}^{(r)} \otimes W_{ij}^{(r)}.
\end{aligned}
$$

Then we have

$$
\|\hat{W}_{rr}\|_\infty \leq \|W_{rr}\|_\infty \leq 1, \tag{20}
$$

$$
\|\hat{W}_{rr}^\perp\|_\infty \leq \|W_{rr}\|_\infty + \|\hat{W}_{rr}\|_\infty \leq 2, \tag{21}
$$

and also by (18),(19)

$$\|\hat{W}_{rr}^{\perp}\|_{\mathrm{tr}_N,2} = \Big\{ \sum_{i=1}^{m_r} \|e_{ii}^{(r)} \otimes W_{ii}^{(r)} - e_{ii}^{(r)} \otimes W_0^{(r)}\|_{\mathrm{tr}_N,2}^2 + \sum_{i\neq j} \|e_{ij}^{(r)} \otimes W_{ij}^{(r)}\|_{\mathrm{tr}_N,2}^2 \Big\}^{1/2}$$
$$< m_r\varepsilon. \tag{22}$$

Here we prove that W_{rr} and also $\hat{W}_{rr}$ is almost a unitary inside $\mathbb{C}I_{m_r} \otimes M_{m_r(N)}(\mathbb{C})$. By (14) we have

$$\begin{aligned}
&\|P_r(N) - W_{rr}W_{rr}^*\|_{\mathrm{tr}_N,2} \\
&\leq \|P_r(N) - P_r(N)WW^*P_r(N)\|_{\mathrm{tr}_N,2} + \|P_r(N)WW^*P_r(N) - W_{rr}W_{rr}^*\|_{\mathrm{tr}_N,2} \\
&= \Big\| \sum_{r'=0}^{s} W_{rr'}W_{rr'}^* - W_{rr}W_{rr}^* \Big\|_{\mathrm{tr}_N,2} = \Big\| \sum_{r\neq r'} W_{rr'}W_{rr'}^* \Big\|_{\mathrm{tr}_N,2} \\
&\leq \sum_{r\neq r'} \|P_r(N)WP_{r'}(N)W_{rr'}^*\|_{\mathrm{tr}_N,2} \\
&\leq \sum_{r\neq r'} \|W_{rr'}^*\|_{\mathrm{tr}_N,2} < s\varepsilon.
\end{aligned} \tag{23}$$

Similarly we have

$$\begin{aligned}
&\|W_{rr}^*W_{rr} - W_{rr}W_{rr}^*\|_{\mathrm{tr}_N,2} \\
&\leq \|P_r(N)W^*WP_r(N) - W_{rr}^*W_{rr}\|_{\mathrm{tr}_N,2} + \|P_r(N)WW^*P_r(N) - W_{rr}W_{rr}^*\|_{\mathrm{tr}_N,2} \\
&\qquad + \|P_r(N)(W^*W - WW^*)P_r(N)\|_{\mathrm{tr}_N,2} \\
&= \Big\| \sum_{r'\neq r} W_{r'r}^*W_{r'r} \Big\|_{\mathrm{tr}_N,2} + \Big\| \sum_{r'\neq r} W_{rr'}W_{rr'}^* \Big\|_{\mathrm{tr}_N,2} < 2s\varepsilon.
\end{aligned} \tag{24}$$

Then, by (23) and (20)–(22) we get

$$\begin{aligned}
&\|P_r(N) - \hat{W}_{rr}\hat{W}_{rr}^*\|_{\mathrm{tr}_N,2} \\
&\leq \|P_r(N) - W_{rr}W_{rr}^*\|_{\mathrm{tr}_N,2} + \|W_{rr}W_{rr}^* - \hat{W}_{rr}\hat{W}_{rr}^*\|_{\mathrm{tr}_N,2} \\
&< s\varepsilon + \|\hat{W}_{rr}\hat{W}_{rr}^{\perp *} + \hat{W}_{rr}^{\perp}\hat{W}_{rr}^* + \hat{W}_{rr}^{\perp}\hat{W}_{rr}^{\perp *}\|_{\mathrm{tr}_N,2} \\
&< s\varepsilon + 4\|\hat{W}_{rr}^{\perp}\|_{\mathrm{tr}_N,2} < (s + 4m_r)\varepsilon,
\end{aligned} \tag{25}$$

and also, by (23) and (20)–(22) as before,

$$\begin{aligned}
&\|\hat{W}_{rr}^*\hat{W}_{rr} - \hat{W}_{rr}\hat{W}_{rr}^*\|_{\mathrm{tr}_N,2} \\
&\leq \|\hat{W}_{rr}^*\hat{W}_{rr} - W_{rr}^*W_{rr}\|_{\mathrm{tr}_N,2} + \|W_{rr}^*W_{rr} - W_{rr}W_{rr}^*\|_{\mathrm{tr}_N,2} \\
&\qquad + \|W_{rr}W_{rr}^* - \hat{W}_{rr}\hat{W}_{rr}^*\|_{\mathrm{tr}_N,2} \\
&< \|\hat{W}_{rr}^*\hat{W}_{rr}^{\perp} + \hat{W}_{rr}^{\perp *}\hat{W}_{rr} + \hat{W}_{rr}^{\perp *}\hat{W}_{rr}^{\perp}\|_{\mathrm{tr}_N,2} + 2s\varepsilon \\
&\qquad + \|\hat{W}_{rr}\hat{W}_{rr}^{\perp *} + \hat{W}_{rr}^{\perp}\hat{W}_{rr}^* + \hat{W}_{rr}^{\perp}\hat{W}_{rr}^{\perp *}\|_{\mathrm{tr}_N,2} \\
&< 4\|\hat{W}_{rr}^{\perp}\|_{\mathrm{tr}_N,2} \times 2 + 2s\varepsilon \\
&< 2(s + 4m_r)\varepsilon
\end{aligned} \tag{26}$$

Hence $\hat{W}_{rr}$ is almost a unitary in $\mathbb{C}I_{m_r} \otimes M_{m_r(N)}(\mathbb{C})$ with unit $P_r(N) = I_{m_r} \otimes I_{m_r(N)}$. Then, [3, Lemma 2.3] shows that for any $\kappa > 0$ one can find $\varepsilon \in (0, \kappa)$ (depending only on κ) in such a way that for each $W \in \Omega(N, \varepsilon)$ there is $\tilde{W}_{rr} \in I_{m_r} \otimes \mathrm{U}(m_r(N))$ such that

$$\|\hat{W}_{rr} - \tilde{W}_{rr}\|_{\mathrm{tr}_N,2} < \kappa. \tag{27}$$

In what follows we fix $\varepsilon \in (0, \kappa)$ as above for a given $\kappa > 0$. For each $W \in \Omega(N; \varepsilon)$ we set

$$\begin{aligned}\tilde{W} &:= W_{00}^{S_0} + \sum_{r=1}^{s} \tilde{W}_{rr} \\ &\in [E_{ij}^{(0)} : (i,j) \in S_0] \oplus \bigoplus_{r=1}^{s} I_{m_r} \otimes \mathrm{U}(m_r(N)).\end{aligned}$$

Then, by (16),(17) and (22),(27) we have

$$\begin{aligned}&\|W - \tilde{W}\|_{\mathrm{tr}_N,2}^2 \\ &= \Big\| \sum_{r_1 \neq r_2} W_{r_1 r_2} \Big\|_{\mathrm{tr}_N,2}^2 + \|W_{00}^{S_0^\perp}\|_{\mathrm{tr}_N,2}^2 + \sum_{r=1}^{s} \|W_{rr} - \tilde{W}_{rr}\|_{\mathrm{tr}_N,2}^2 \\ &< s(s+1)\varepsilon^2 + \frac{\varepsilon^2}{t^2} + \sum_{r=1}^{s} \left(\|\hat{W}_{rr}^\perp\|_{\mathrm{tr}_N,2} + \|\hat{W}_{rr} - \tilde{W}_{rr}\|_{\mathrm{tr}_N,2} \right)^2 \\ &< s(s+1)\varepsilon^2 + \frac{\varepsilon^2}{t^2} + s(m_r \varepsilon + \kappa)^2.\end{aligned}$$

Since $\varepsilon < \kappa$, we get

$$\|W - \tilde{W}\|_{\mathrm{tr}_N,2} < c(t)\kappa,$$

where $c(t) > 0$ is defined as in the statement of this lemma. Consequently, $\Omega(N; \varepsilon)$ is contained in the $c(t)\kappa$-neighborhood of

$$\begin{aligned}&[E_{ij}^{(0)} : (i,j) \in S_0] \oplus \bigoplus_{r=1}^{s} I_{m_r} \otimes \mathrm{U}(m_r(N)) \\ &= \begin{bmatrix} [E_{ij}^{(0)} : (i,j) \in S_0] & & & \\ & I_{m_1} \otimes \mathrm{U}(m_1(N)) & & \\ & & \ddots & \\ & & & I_{m_s} \otimes \mathrm{U}(m_s(N)) \end{bmatrix}\end{aligned}$$

inside $M_N(\mathbb{C})$. Now, let us choose a $\kappa/\sqrt{s+1}$-net of minimal cardinality, whose center points are denoted by $(A_{\lambda_0})_{\lambda_0 \in \Lambda_0}$, and also for each $r = 1, \ldots, s$ choose a $\kappa/\sqrt{s+1}$-net of minimal cardinality, whose center points are denoted by $(V_{\lambda_r})_{\lambda_r \in \Lambda_r}$. It is a standard fact that if $\kappa/\sqrt{s+1} < 1$, then

$$|\Lambda_0| \leq \left(\frac{3\sqrt{s+1}}{\kappa} \right)^{2n_0(N)^2 t} \tag{28}$$

since $|S_0| \leq n_0(N)^2 t$ as remarked before. Also, [7, Theorem 7] shows there is a universal constant $C_{\mathrm{unitary}} > 0$ such that

$$|\Lambda_r| \leq \left(\frac{2C_{\mathrm{unitary}}\sqrt{s+1}}{\kappa} \right)^{m_r(N)^2} \tag{29}$$

since $\max\{\|P_r(N) - I_{m_r}\otimes U\|_{\mathrm{tr}_N,2} : U \in \mathrm{U}(m_r(N))\} = \max\{\|I_{m_r}\otimes(I_{m_r(N)} - U)\|_{\mathrm{tr}_N,2} : U \in \mathrm{U}(m_r(N))\} = \sqrt{\frac{n_r(N)}{N}}\max\{\|I_{m_r(N)} - U\|_{\mathrm{tr}_{m_r(N)},2} : U \in \mathrm{U}(m_r(N))\} \leq 2$, i.e., the diameter of $I_{m_r}\otimes \mathrm{U}(m_r(N)) \cong \mathrm{U}(m_r(N))$ with respect to $\|\cdot\|_{\mathrm{tr}_N,2}$ is less than 2 uniformly in N. It is clear that the κ-balls at

$$V_{(\lambda_0,\dots,\lambda_s)} := A_{\lambda_0} + \sum_{r=1}^s V_{\lambda_r}, \quad (\lambda_0,\dots,\lambda_s) \in \Lambda_0\times\cdots\times\Lambda_s$$

cover $[E_{ij} : (i,j) \in S_0] \oplus \bigoplus_{r=1}^s I_{m_r}\otimes \mathrm{U}(m_r(N))$. Therefore, the $(c(t)+1)\kappa$-balls at the same $V_{(\lambda_0,\dots,\lambda_s)}$'s cover $\Omega(N;\varepsilon)$ inside $M_N(\mathbb{C})$. Note that each such ball clearly contains at least one element in $\Omega(N;\varepsilon)$, say $W_{(\lambda_0,\dots,\lambda_s)} \in \Omega(N;\varepsilon)$, and hence the $2(c(t)+1)\kappa$-balls at $W_{(\lambda_0,\dots,\lambda_s)}$'s clearly cover $\Omega(N;\varepsilon)$ inside $\mathrm{U}(N)$. Hence

$$K_{2(c(t)+1)\kappa}(\Omega(N;\varepsilon)) \leq |\Lambda_0|\times|\Lambda_1|\times\cdots\times|\Lambda_s|,$$

from which the assertion is immediate. □

Completion of the proof of Theorem 3.1. Firstly we complete the proof of the desired inequality when $s<\infty$. Since

$$\|UW - U'W'\|_{\mathrm{tr}_N,2} \leq \|U-U'\|_{\mathrm{tr}_N,2} + \|W-W'\|_{\mathrm{tr}_N,2}$$

the mapping $(U,W_1,\dots,W_{n-1}) \in \mathrm{U}(N)\times\Omega(N;\varepsilon)^{n-1} \mapsto (U,UW_1,\dots,UW_{n-1}) \in \Psi(N;\varepsilon)$ is clearly Lipschitz continuous, and a rough estimate shows the Lipschitz constant is less than $\sqrt{3n-2}$. Thus, by Lemma 3.4 and [7, Theorem 7] we have

$$\begin{aligned}
&K_{2\sqrt{3n^2-2n}(c(t)+1)\kappa}\big(\Gamma_{\mathrm{orb}}(\mathbf{X}',\dots,\mathbf{X}' : \Xi(N),\dots,\Xi(N);N,m,\delta)\big)\\
&\leq K_{2\sqrt{3n^2-2n}(c(t)+1)\kappa}\big(\Psi(N;\varepsilon)\big)\\
&\leq K_{2\sqrt{n}(c(t)+1)\kappa}\big(\mathrm{U}(N)\times\Omega(N;\varepsilon)^{n-1}\big)\\
&\leq K_{2(c(t)+1)\kappa}(\mathrm{U}(N))\times K_{2(c(t)+1)\kappa}(\Omega(N;\varepsilon))^{n-1}\\
&\leq \left(\frac{\frac{C_{\mathrm{unitary}}}{c(t)+1}}{\kappa}\right)^{N^2}\times\left(\left(\frac{3\sqrt{s+1}}{\kappa}\right)^{2n_0(N)^2t}\left(\frac{2C_{\mathrm{unitary}}\sqrt{s+1}}{\kappa}\right)^{\sum_{r=1}^s m_r(N)^2}\right)^{n-1}
\end{aligned}$$

for every sufficiently small $\kappa>0$ and $t\in(0,1)$ together with a corresponding $\varepsilon<\kappa$ in Lemma 3.4 and then $m\in\mathbb{N}$, $\delta>0$ due to Lemma 3.3, where $C_{\mathrm{unitary}}>0$ denotes a universal constant due to [7, Theorem 7] as before. Let $C(t) := \max\left\{\frac{C_{\mathrm{unitary}}}{c(t)+1}, 3\sqrt{s+1}, 2C_{\mathrm{unitary}}\sqrt{s+1}\right\}$, and then it follows that

$$\begin{aligned}
&\mathbb{K}^{\mathrm{orb}}_{2\sqrt{3n^2-2n}(c(t)+1)\kappa}(\mathbf{X}',\dots,\mathbf{X}')\\
&\leq \limsup_{N\to\infty}\left(1+(n-1)\left(\frac{2n_0(N)^2}{N^2}t + \sum_{r=1}^s\frac{m_r(N)^2}{N^2}\right)\right)\times\log\frac{C(t)}{\kappa}\\
&= \left(1+(n-1)\left(2\tau(p_0)^2t + \sum_{r=1}^s\frac{\tau(p_r)^2}{m_r^2}\right)\right)\times\log\frac{C(t)}{\kappa}
\end{aligned}$$

by (8) and $m_r(N) = n_r(N)/m_r$. Then we get

$$\delta_{0,\mathrm{orb}}(\mathbf{X}',\dots,\mathbf{X}') = \limsup_{\kappa\searrow0}\frac{\mathbb{K}^{\mathrm{orb}}_{2\sqrt{3n^2-2n}(c(t)+1)\kappa}(\mathbf{X}',\dots,\mathbf{X}')}{\big|\log(2\sqrt{3n^2-2n}(c(t)+1)\kappa)\big|} - n$$

$$
\begin{aligned}
&\leq 1 + (n-1)\left(2\tau(p_0)^2 t + \sum_{r=1}^{s} \frac{\tau(p_r)^2}{m_r^2}\right) - n \\
&\leq -(n-1)\left(1 - \sum_{r=1}^{s} \frac{\tau(p_r)^2}{m_r^2}\right) + 2(n-1)\tau(p_0)^2 t \\
&= -(n-1)\delta_0(\mathbf{X}') + 2(n-1)\tau(p_0)^2 t,
\end{aligned}
$$

where the last equality is due to Jung's computation of δ_0 ([5]). Since t is arbitrary, we obtain

$$\delta_{0,\mathrm{orb}}(\mathbf{X},\dots,\mathbf{X}) = \delta_{0,\mathrm{orb}}(\mathbf{X}',\dots,\mathbf{X}') \leq -(n-1)\delta_0(\mathbf{X}') = -(n-1)\delta_0(\mathbf{X})$$

whenever $s < \infty$.

We will next get rid of the assumption of $s < \infty$; thus assume that $s = \infty$. For each $s_0 < \infty$ let $C^{s_0} := C_0 \oplus \cdots C_{s_0-1} \oplus \mathbb{C}p^{s_0}$ with $p^{s_0} := \sum_{r=s_0}^{\infty} p_r \searrow 0$ strongly as $s_0 \nearrow \infty$. Clearly $C^{s_0} \subset W^*(\mathbf{X})$, and choose a hyperfinite self-adjoint random multi-variable $\mathbf{X}^{s_0}$ such that $W^*(\mathbf{X}^{s_0}) = C^{s_0}$. Then, by what we have shown above and (6)

$$
\begin{aligned}
\delta_{0,\mathrm{orb}}(\mathbf{X},\dots,\mathbf{X}) &\leq \delta_{0,\mathrm{orb}}(\mathbf{X}^{s_0},\dots,\mathbf{X}^{s_0}) \\
&\leq -(n-1)\left(1 - \sum_{r=1}^{s_0-1} \frac{\tau(p_r)^2}{m_r^2} + \tau(p^{s_0})^2\right) \\
&\longrightarrow -(n-1)\left(1 - \sum_{r=1}^{\infty} \frac{\tau(p_r)^2}{m_r^2}\right) = -(n-1)\delta_0(\mathbf{X})
\end{aligned}
$$

as $s_0 \nearrow \infty$, thanks again to Jung's computation of δ_0 ([5]). □

The following is immediate from the above theorem and (6):

Corollary 3.5. *Let $\mathbf{X}_1,\dots,\mathbf{X}_n$ be hyperfinite self-adjoint multivariables in a tracial W^*-probability space (M,τ). Then one has*

$$\delta_{0,\mathrm{orb}}(\mathbf{X}_1,\dots,\mathbf{X}_n) \leq -(n-1)\,\delta_0\big(W^*(\mathbf{X}_1) \cap \dots \cap W^*(\mathbf{X}_n)\big),$$

where $\delta_0\big(W^*(\mathbf{X}_1)\cap\dots\cap W^*(\mathbf{X}_n)\big)$ means the unique number of $\delta_0(\mathbf{X})$ with $W^*(\mathbf{X}) = W^*(\mathbf{X}_1)\cap \dots \cap W^*(\mathbf{X}_n)$ due to Jung [5].

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