On the Cauchy problem for second-order hyperbolic operators with the coefficients of their principal parts depending only on the time variable

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1. Main results

Notations:

the time variable: $t \in \mathbb{R} \overset{\text{dual}}{\longleftrightarrow} \tau \in \mathbb{R}$

the space variables: $x = (x_1, \cdots, x_n) \in \mathbb{R}^n \overset{\text{dual}}{\longleftrightarrow} \xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n$

$D_t = -i\partial_t,\ D_x = (D_1, \cdots, D_n) = -i(\partial_{x_1}, \cdots, \partial_{x_n})$

Consider hyperbolic operators of second order whose symbols have the form

$$P(t, x, \tau, \xi) = \tau^2 - a(t, \xi) + b_0(t, x)\tau + b(t, x, \xi) + c(t, x),$$

where $a(t, \xi) = \sum_{j,k=1}^n a_{j,k}(t)\xi_j\xi_k,\ b(t, x, \xi) = \sum_{j=1}^n b_j(t, x)\xi_j,\ a_{j,k}(t) \in C^\infty([0, \infty))$ and $b_j(t, x), c(t, x) \in C^\infty([0, \infty) \times \mathbb{R}^n)$, and the Cauchy problem

$$\textbf{(CP)} \qquad \begin{cases} P(t, x, D_t, D_x)u(t, x) = f(t, x) & \text{in } [0, \infty) \times \mathbb{R}^n, \\ D_t^j u(t, x)|_{t=0} = u_j(x) & \text{in } \mathbb{R}^n \quad (j = 0, 1) \end{cases}$$

in the framework of C^∞.

Def: We say that (CP) is C^∞ well-posed if

(E) $\forall f \in C^\infty([0, \infty) \times \mathbb{R}^n),\ \forall u_j \in C^\infty(\mathbb{R}^n)\ (j = 0, 1),\ \exists u \in C^\infty([0, \infty) \times \mathbb{R}^n)$ satisfying (CP). (**Existence**)

(U) If $s > 0,\ u \in C^\infty([0, \infty) \times \mathbb{R}^n),\ D_t^j u(t, x)|_{t=0} = 0$ in $\mathbb{R}^n$ $(j = 0, 1)$ & $\operatorname{supp} P(t, x, D_t, D_x)u \subset \{t \ge s\}$, then $\operatorname{supp} u \subset \{t \ge s\}$. (**Uniqueness**)

Taking account of Lax-Mizohata theorem we assume that

(H) $a(t, \xi) \ge 0$ for $(t, \xi) \in [0, \infty) \times \mathbb{R}^n$

(see S. Mizohata, J. Math. Kyoto Univ. **1** (1961), 109–127). From Ivrii-Petkov's result we can assume without loss of generality that

(F) $a(t, \xi) \not\equiv 0$ in t for $\forall \xi \in \mathbb{R}^n \setminus \{0\}$

(see V. Ya. Ivrii and V. M. Petkov, Russian Math. Surveys **29** (1974), 1–70).

Moreover, we assume that $a(t, \xi)$ satisfies the following condition (A):

(A) $\forall T > 0,\ \exists k_T \in \mathbb{Z}_+ (= \mathbb{N} \cup \{0\})$ s.t.

$$\sum_{k=0}^{k_T} |\partial_t^k a(t, \xi)| \ne 0 \quad \text{for } \forall (t, \xi) \in [0, T] \times S^{n-1}.$$

If the $a_{j,k}(t)$ are real analytic on $[0,\infty)$, then the condition (A) is satisfied. For simplicity we assume that the $a_{j,k}(t)$ are real analytic on $[0,\infty)$, in order to describe the condition (L) below in a simplle form. Let Ω be a neighborhood of $[0,\infty)$ in $\mathbb{C}$ where the $a_{j,k}(t)$ are analytic. Put

$$\mathcal{R}(\xi)=\{(\mathrm{Re}\,\lambda)_+;\ \lambda\in\Omega \text{ and } a(\lambda,\xi)=0\}$$

for $\xi\in\mathbb{R}^n\setminus\{0\}$, where $a_+=\max\{a,0\}$.

Sufficiency:

We assume in "Sufficiency" that

(A)$'$ the $a_{j,k}(t)$ are real analytic (for simplicity),

(B) $\forall K\Subset\mathbb{R}^n$, $\exists\Omega_K$: complex neighborhood of $[0,\infty)$ s.t. $b_j(t,x)$ ($1\le j\le n$) are analytic in Ω_K for $\forall x\in K$,

(L) $\forall T>0$, $\forall x\in\mathbb{R}^n$, $\exists C>0$ s.t.

$$\min_{\tau\in\mathcal{R}(\xi)}|t-\tau|\,|b(t,x,\xi)|\le C\sqrt{a(t,\xi)}\quad\text{for }\forall(t,\xi)\in[0,T]\times S^{n-1}.$$

Thm 1: Under (B) and (L) (CP) is C^∞ well-posed.

Remark: $\mathcal{R}(\xi)$ can be replaced in (L) by $\mathcal{R}'(\xi)$ satisfying

$$\sup_{\xi\in S^{n-1}}\#(\mathcal{R}'(\xi)\cap\{t\le T\})<\infty \text{ for }\forall T>0,$$

where $\#A$ denotes the number of the elements of a set A.

Def: (i) Let f be a function on $\mathbb{R}$. We say that $f(t)$ is a semi-algebraic function if the graph of f is a semi-algebraic set, *i.e.*, the graph of f is a set defined by polynomial equations and inequalities. (ii) Let $t_0\in\mathbb{R}$, U be a neighborhood of t_0 and $f:\ U\to\mathbb{R}$. We say that f is semi-algebraic at t_0 if there is $c>0$ such that $\{(t,y)\in\mathbb{R}^2;\ y=f(t)$ and $|t-t_0|<c\}$ is a semi-algebraic set.

Necessity:

We assume in "Necessity" that (A)$'$ and (B) are satisfied. Let $t_0\ge 0$, $x^0\in\mathbb{R}^n$ and $\xi^0\in S^{n-1}$. If $n\ge 3$, we assume the following condition:

(A)$''_{(t_0,x^0)}$ the $a_{j,k}(t)$ and $b_j(t,x^0)$ ($1\le j\le n$) are semi-analytic at t_0.

The following condition is very similar to the condition (L):

(L)$_{(t_0,x^0,\xi^0)}$ $\exists U$: nbd of t_0, $\exists\Gamma$: conic nbd of ξ^0, $\exists C>0$ s.t.

$$\min_{\tau\in\mathcal{R}(\xi)}|t-\tau|\,|b(t,x^0,\xi)|\le C\sqrt{a(t,\xi)}\quad\text{for }\forall(t,\xi)\in U\times\Gamma.$$

Thm 2: Assume that (A)$'$ and (B) are satisfied. Moreover, we assume that (A)$''_{(t_0,x^0)}$ is satisfied if $n\ge 3$. Then (L)$_{(t_0,x^0,\xi^0)}$ is necessary for C^∞ well-posedness.

Remark: Assume that (A)$'$ and (B) are satisfied, and that (A)$''_{(t_0,x^0)}$ is valid for any $t_0\ge 0$ and $x^0\in\mathbb{R}^n$ if $n\ge 3$. Then (CP) is C^∞ well-posed if and only if (L) is satisfied.

<u>**related results**</u>:

- Colombini-Ishida-Orrú: Ark. Mat. **38** (2000), 223–230.

(CP) is C^∞ well-posed if the coefficients do not depend on x and if (A) and the following condition are satisfied:

$$|b(t,\xi)| \le Ca(t,\xi)^{1/2-1/k} \quad \text{for } (t,\xi) \in [0,\infty) \times S^{n-1}.$$

- Colombini-Nishitani: Osaka J. Math. **41** (2004), 933–947.

They tried to generalize C-I-O's results to the case the lower order terms also depend on x.

In the proof of Thm 1 we adopted some ideas used in C-I-O and C-N.

- W: J. Math. Soc. Japan **62-1** (2010), 95–133.

The proof of Thm 2 is given in this paper.

2. Outline of Proof of Thm 1

We can assume without loss of generality that there is $K \Subset \mathbb{R}^n$ such that $\mathrm{supp}_x\, b_j(t,x)$, $\mathrm{supp}_x\, c(t,x) \subset K$. Let $t_0 \ge 0$, $\mathcal{O}_{t_0}$ be the ring of power series centered at t_0 in one variable and

$$\mathfrak{M}_{t_0} := \{(\beta_1(t),\cdots,\beta_n(t)) \in \mathcal{O}_{t_0}^n;\ \min_{\tau\in\mathcal{R}(\xi)} |t-\tau|\cdot|\sum_{j=1}^n \beta_j(t)\xi_j| \le {}^\exists C\sqrt{a(t,\xi)}$$
$$\text{if } t \text{ belongs to a neighborhood of } t_0 \text{ in } [0,\infty) \text{ and } \xi \in S^{n-1}\}.$$

Since $\mathcal{O}_{t_0}$-submodule of $\mathcal{O}_{t_0}^n$ is finitely generated, there are $\psi_j(t) = (\psi_{j,1}(t),\cdots,\psi_{j,n}(t)) \in \mathfrak{M}_{t_0}$ ($1 \le j \le r_0$) such that

$$\mathfrak{M}_{t_0} = \Big\{\sum_{j=1}^{r_0} c_j(t)\psi_j(t);\ c_j(t) \in \mathcal{O}_{t_0}\ (\ 1 \le j \le r_0)\Big\}.$$

The condition (L) implies that $(b_1(t,x),\cdots,b_n(t,x)) \in \mathfrak{M}_{t_0}$ for each $x \in \mathbb{R}^n$. So there are C^∞ functions $c_j(t,x)$ of (t,x) such that $b(t,x,\xi) = \sum_{j=1}^{r_0} c_j(t,x)\psi_j(t,\xi)$ in a neighborhood of t_0, where $\psi_j(t,\xi) = \sum_{k=1}^n \psi_{j,k}(t)\xi_k$. Let $T > 0$. Then there are $\varphi_j(t) = (\varphi_{j,1}(t),\cdots,\varphi_{j,n}(t)) \in (C^\infty(\mathbb{R}))^n$ and $c_j(t,x) \in C^\infty([0,\infty)\times\mathbb{R}^n)$ ($1 \le j \le r$) such that

$$\min_{\tau\in\mathcal{R}(\xi)} |t-\tau|\cdot|\varphi_j(t,\xi)| \le C\sqrt{a(t,\xi)} \quad (\ (t,\xi) \in [0,T]\times S^{n-1},\ 1 \le j \le r),$$

$$b(t,x,\xi) = \sum_{j=1}^r c_j(t,x)\varphi_j(t,\xi)\ (\ (t,x,\xi) \in [0,T]\times\mathbb{R}^n\times\mathbb{R}^n),$$

where $\varphi_j(t,\xi) = \sum_{k=1}^n \varphi_{j,k}(t)\xi_k$. Put

$$w_\rho(t,\xi) := a(t,\xi) + \langle\xi\rangle^{2\rho}, \quad \rho := \frac{2}{k_0+2},$$
$$W_0(t,\xi) := \frac{\langle\xi\rangle^{2\rho}}{\sqrt{w_\rho(t,\xi)}} + 1, \quad W_j(t,\xi) := \frac{|\varphi_j(t,\xi)|}{\sqrt{w_\rho(t,\xi)}} \quad (\ 1 \le j \le r),$$
$$W(t,\xi) := \sum_{j=0}^r W_j(t,\xi), \quad \Phi(t,\xi) := \int_0^t \Big(W(s,\xi) + \frac{|(\partial_t a)(s,\xi)|}{w_\rho(s,\xi)} \Big)\, ds.$$

Let us introduce the parameter $\varepsilon \in [0,1]$ (to prove finite propagation property). Consider

$$\begin{cases} (P(t,x,D_t,D_x) + \varepsilon\Delta_x)u_\varepsilon(t,x) = f(t,x) & (\ t \ge 0,\ x \in \mathbb{R}^n), \\ u_\varepsilon(0,x) = u_0(x), \quad (D_t u_\varepsilon)(0,x) = u_1(x) & (\ x \in \mathbb{R}^n). \end{cases}$$

We use an Energy form

$$\mathcal{E}_\varepsilon(t;\gamma,A,l) := \int_{\mathbb{R}^n} E_\varepsilon(t,\xi) K(t,\xi;\gamma,A,l)\, đ\xi,$$

where $\gamma > 0$, $A > 0$, $l \in \mathbb{R}$, $t \in [0,T]$ and

$$E_\varepsilon(t,\xi) := |\partial_t v_\varepsilon(t,\xi)|^2 + (a(t,\xi) + \varepsilon|\xi|^2 + \langle\xi\rangle^{2\rho})|v_\varepsilon(t,\xi)|^2,$$
$$K(t,\xi;\gamma,A,l) := \exp[-(\gamma t + A\Phi(t,\xi)) + l\log\langle\xi\rangle],$$
$$v_\varepsilon(t,\xi) := \mathcal{F}_x[u_\varepsilon(t,x)](\xi).$$

A simple calculation gives

$$\begin{aligned} &\partial_t E_\varepsilon(t,\xi) - (\gamma + A\partial_t\Phi(t,\xi))E_\varepsilon(t,\xi) \\ &\le |\hat{f}(t,\xi)|^2/W(t,\xi) - (\gamma + (A-3)W(t,\xi) - 2)|\partial_t v_\varepsilon|^2 \\ &\quad - \{(A-1)(|\partial_t a(t,\xi)| + W(t,\xi)w_\rho(t,\xi)) + \gamma w_\rho(t,\xi)\}|v_\varepsilon|^2 \\ &\quad + |\mathcal{F}_x[b_0(t,x)\partial_t u_\varepsilon(t,x)](\xi)|^2 + |\mathcal{F}_x[b(t,x,D_x)u_\varepsilon](\xi)|^2/W(t,\xi) + |\mathcal{F}_x[c(t,x)u_\varepsilon](\xi)|^2. \end{aligned}$$

Lemma: (i) $\Phi(T,\xi) \le {}^\exists C_T(1 + \log\langle\xi\rangle) \quad (\ \xi \in \mathbb{R}^n)$.
(ii) $\forall\delta > 0$, $\exists c_\delta(T) > 0$ s.t.

$$(1+\delta)^{-1}W(t,\xi) \le W(t,\eta) \le (1+\delta)W(t,\xi) \quad \text{if } t \in [0,T],\ |\xi - \eta| \le c_\delta(T)\langle\xi\rangle^\rho,$$
$$(1+\delta)^{-1}\Phi(t,\xi) \le \Phi(t,\eta) \le (1+\delta)\Phi(t,\xi)$$
$$\text{if } t \in [0,T],\ |\xi-\eta| \le c_\delta(T)\langle\xi\rangle^\rho/(1+\log\langle\xi\rangle).$$

(iii) $\exists C_T > 0$, $\exists l_0 > 0$ s.t. $\exp[\pm\Phi(t,\xi)] \le C_T\langle\xi-\eta\rangle^{l_0}\exp[\pm\Phi(t,\eta)]$.

Using the above lemma, we have

$$\partial_t\mathcal{E}_\varepsilon(t;\gamma,A,l) \le \int |\hat{f}(t,\xi)|^2 W(t,\xi)^{-1} K(t,\xi;\gamma,A,l)\, đ\xi$$

if $A \ge {}^\exists A(b,T)$, $\gamma \ge {}^\exists\gamma(b,b_0,c,T,A,l)$ and $t \in [0,T]$. Then a standard argument proves Thm 1.