

SCATTERING OF ROUGH SOLUTIONS OF THE NONLINEAR KLEIN-GORDON EQUATIONS IN 3D

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ABSTRACT. We prove scattering of solutions below the energy norm of the nonlinear Klein-Gordon equation in 3D with a defocusing power-type nonlinearity that is superconformal and energy subcritical: this result extends those obtained in the energy class [4, 18, 19] and those obtained below the energy norm under the additional assumption of spherical symmetry [25]. In order to do that, we generate an exponential-type decay estimate in H^s , $s < 1$, by means of concentration [1] and a low-high frequency decomposition [2, 7] : this is the starting point to prove scattering. On low frequencies we modify the arguments in [18, 19]; on high frequencies we use the smoothing effect of the solutions to control the error terms: this, combined with an almost conservation law, allows to prove this decay estimate.

1. INTRODUCTION AND THEOREM

In this paper we consider the defocusing nonlinear Klein-Gordon equation on \mathbb{R}^3 :

$$(1.1) \quad \partial_{tt}u - \Delta u + u = -|u|^{p-1}u$$

with data $u(0) = u_0$, $\partial_t u(0) = u_1$ lying in H^s , H^{s-1} respectively.

We are interested in the strong solutions of the defocusing nonlinear Klein-Gordon equation on some interval $[0, T]$ i.e maps u , $\partial_t u$ that lie in $C([0, T], H^s(\mathbb{R}^3))$, $C([0, T], H^{s-1}(\mathbb{R}^3))$ respectively and that satisfy

$$(1.2) \quad u(t) = \cos(t\langle D \rangle)u_0 + \frac{\sin(t\langle D \rangle)}{\langle D \rangle}u_1 - \int_0^t \frac{\sin((t-t')\langle D \rangle)}{\langle D \rangle} |u|^{p-1}(t')u(t') dt'.$$

The defocusing nonlinear Klein-Gordon equation is closely related to the defocusing nonlinear wave equation:

$$(1.3) \quad \partial_{tt}v - \Delta v = -|v|^{p-1}v$$

with data $v(0) := v_0$, $\partial_t v(0) := v_1$. (1.3) enjoys the following scaling property

$$(1.4) \quad v(t, x) \rightarrow \frac{1}{\lambda^{\frac{2}{p-1}}} v\left(\frac{t}{\lambda}, \frac{x}{\lambda}\right), \quad v_0(x) \rightarrow \frac{1}{\lambda^{\frac{2}{p-1}}} v_0\left(\frac{x}{\lambda}\right), \quad v_1(x) \rightarrow \frac{1}{\lambda^{\frac{2}{p-1}+1}} v_1\left(\frac{x}{\lambda}\right).$$

We define the critical exponent $s_c := \frac{3}{2} - \frac{2}{p-1}$. One can check that the $\dot{H}^{s_c} \times \dot{H}^{s_c-1}$ norm of (u_0, u_1) is invariant under the scaling transformation (1.4)¹. (1.1) is known to be locally well-posed in $H^s \times H^{s-1}$, $s \geq s_c$, $p \geq \frac{7}{3}$ by using an iterative argument. If $p = 5$ then $s_c = 1$ and we say that the nonlinearity $|u|^{p-1}u$ is \dot{H}^1 (or energy) critical. If $p = \frac{7}{3}$ then $s_c = 0$ and we say that the nonlinearity is L^2 (or mass) critical. If $p = 3$ then $s_c = \frac{1}{2}$ and we say that the nonlinearity is conformal. If $\frac{7}{3} < p < 5$ then we say that the regime is mass supercritical-energy subcritical. If $3 < p < 5$ then we say that

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¹Here \dot{H}^m denotes the standard homogeneous Sobolev space endowed with the norm $\|f\|_{\dot{H}^m} := \|D^m f\|_{L^2(\mathbb{R}^3)}$

the regime is superconformal and energy subcritical.

It is well-known that smooth solutions of (1.1) have a conserved energy

$$(1.5) \quad \begin{aligned} E(u(t)) := & \frac{1}{2} \int_{\mathbb{R}^3} |\partial_t u(t, x)|^2 dx + \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u(t, x)|^2 dx + \frac{1}{2} \int_{\mathbb{R}^3} |u(t, x)|^2 dx \\ & + \frac{1}{p+1} \int_{\mathbb{R}^3} |u(t, x)|^{p+1} dx. \end{aligned}$$

In fact by standard limit arguments the energy conservation law remains true for solutions $(u, \partial_t u) \in H^s \times H^{s-1}$, $s \geq 1$. Since the lifespan of the local solution depends only on the $H^s \times H^{s-1}$ norm of the initial data (u_0, u_1) (see [13]) for $s > s_c$, then it suffices to find an a priori pointwise in time bound in $H^s \times H^{s-1}$ of the solution $(u, \partial_t u)$ in order to establish global well-posedness.

The long-time behavior in the energy space (i.e with data $(u_0, u_1) \in H^1 \times L^2$) has attracted much attention from the community. The energy captures the evolution in time of the $H^1 \times L^2$ norm of the solutions. Since it is conserved we have global existence of solutions of (1.1) in the energy space for all dimension n and for all exponent p that is mass-supercritical and energy-subcritical, i.e $1 + \frac{4}{n} < p < 1 + \frac{4}{n-2}$. The next stage is to understand the asymptotic behavior of the solutions of (1.1) in the energy space. The scattering, i.e the linear asymptotic behavior, was proved in [3, 4, 9, 18, 19, 20, 21] for all dimension n .

The long-time behavior below energy norm (i.e with data in $H^s \times H^{s-1}$, $s < 1$) has also received much attention from the community. The global existence of solutions of (1.1) has been investigated in [24]. The scattering of solutions of (1.1) with radial data and in dimension 3 has been studied in [25]. More precisely it was proved that the asymptotic behavior for spherical solutions is linear for $3 < p < 5$ and in $H^s \times H^{s-1}$, $\bar{s} := \bar{s}(p) < s < 1$ where

$$\bar{s} := \begin{cases} 1 - \frac{(5-p)(p-3)}{2(p-1)(p-2)}, & 4 \geq p > 3 \\ 1 - \frac{(5-p)^2}{2(p-1)(6-p)}, & 5 > p \geq 4. \end{cases}$$

In this paper we are interested in proving scattering results for general data below the energy norm and in dimension 3. The main result of this paper is the following one:

Theorem 1.1. *Let $5 > p > 3$, $A \gtrsim 1$ ² and $(u_0, u_1) \in H^s \times H^{s-1}$ such that*

$$(1.6) \quad \|(u_0, u_1)\|_{H^s \times H^{s-1}} \leq A.$$

Then there exists $\tilde{s} := \tilde{s}(A, p) < 1$ such that $\tilde{s} \rightarrow 1$ as $A \rightarrow \infty$ and such that the solution of (1.1) with data (u_0, u_1) exists for all time T and scatters as T goes to infinity, i.e there exists $(u_{+,0}, u_{+,1}) \in H^s \times H^{s-1}$ such that

$$\lim_{T \rightarrow \infty} \|(u(T), \partial_t u(T)) - K(T)(u_{+,0}, u_{-,0})\|_{H^s \times H^{s-1}} = 0.$$

Here,

$$K(T) := \begin{pmatrix} \cos(T\langle D \rangle) & \frac{\sin(T\langle D \rangle)}{\langle D \rangle} \\ -\langle D \rangle \sin(T\langle D \rangle) & \cos(T\langle D \rangle) \end{pmatrix}.$$

Remark 1.2. *In fact, as $A \rightarrow \infty$, then there exists a constant $\tilde{\alpha} := \tilde{\alpha}(p) > 1$ such that one can choose \tilde{s} depending on A in the following fashion:*

$$(1.7) \quad \tilde{s} = 1 - \frac{1}{\tilde{\alpha}^{\tilde{\alpha} \dots \tilde{\alpha}}}.$$

Here the height of the tower is $\sim A^{\tilde{\alpha}}$.

²The scattering for small data (i.e $A \ll 1$) is well-known. The proof is also contained in the proof of our theorem.

2. NOTATION

2.1. General notation. We set some general notation that appear throughout the proof.

If $x \in \mathbb{R}$, then $\langle x \rangle := (1 + |x|^2)^{\frac{1}{2}}$, x^+ is a slightly larger number than x , x^{++} is a slightly larger number than x^+ , x^- is a slightly smaller number than x , and x^{--} is a slightly smaller number than x^- .

Let f be a function differentiable in time and smooth in space. We write $F(f)$ for the following function

$$F(f) := |f|^{p-1}f.$$

Given J a time interval, we denote by X_f^J the following number

$$X_f^J(t) := - \int_J \frac{\sin((t-s)\langle D \rangle)}{\langle D \rangle} (f(s)) ds$$

We denote by W the set of wave admissible points, i.e

$$W := \left\{ (q, r), (q, r) \in (2, \infty) \times [2, \infty), \frac{1}{q} + \frac{1}{r} \leq \frac{1}{2} \right\}.$$

Given $m \in [0, 1]$, we say that (q, r) is m -wave admissible if

$$(q, r) \in W \quad \text{and} \quad \frac{1}{q} + \frac{3}{r} = \frac{3}{2} - m$$

and $q \geq 2+$ if $m = 1$. We denote by W_m the set of m -wave admissible points. We denote by \tilde{W} the dual set of W , i.e

$$\tilde{W} := \left\{ (\tilde{x}, \tilde{y}), \exists (x, y) \in W, \frac{1}{x} + \frac{1}{\tilde{x}} = 1, \frac{1}{y} + \frac{1}{\tilde{y}} = 1 \right\}.$$

A graphical representation of these sets is given on Figure 1.

2.2. The multiplier I , the numbers Z , and the mollified energies $E(I f)$. We introduce the multiplier I .

The proof of Theorem 1.1 involves the multiplier I defined as follows:

$$\widehat{I f}(t, \xi) := m(\xi) \hat{f}(t, \xi),$$

where

- $m(\xi) := \eta\left(\frac{\xi}{N}\right)$
- η is a smooth, radial, nonincreasing function in $|\xi|$ such that $\eta(\xi) := 1$, $|\xi| \leq 1$ and $\eta(\xi) := \frac{1}{|\xi|^{1-s}}$, $|\xi| \geq 2$
- $N \gg 1$ is a parameter.

Throughout the paper we choose (N, s) such that

$$(2.1) \quad N^{1-s} \sim 1.$$

We shall explain in Section 4 why this choice of (N, s) is natural.

We introduce some numbers that we constantly use in the proof. Given J a time interval, let

$$Z_{m,s}(J, f) := \sup_{\substack{(q,r)-m \\ \text{wave adm}}} \|\partial_t \langle D \rangle^{-m} I f\|_{L_t^q L_x^r(J)} + \|\langle D \rangle^{1-m} I f\|_{L_t^q L_x^r(J)},$$

$$Z(J, f) := \sup_{m \in [0,1]} Z_{m,s}(J, f),$$

and

$$\begin{aligned} \tilde{Z}(J, f) &:= \|\partial_t \langle D \rangle^{-\frac{1}{2}} If\|_{L_t^4 L_x^4(J)} + \|\langle D \rangle^{1-\frac{1}{2}} If\|_{L_t^{2(p-1)-} L_x^{\frac{6(p-1)}{2p-3}+}(J)} + \|If\|_{L_t^{2(p-1)-} L_x^{2(p-1)+}(J)} \\ &\quad + \|\langle D \rangle^{1-\frac{2}{p}} If\|_{L_t^p L_x^{\frac{2p}{p-2}}(J)} + \|\langle D \rangle^{1-\frac{3p-5}{2p}} If\|_{L_t^p L_x^{2p}(J)} + \|\langle D \rangle^{1-\frac{1}{2}} If\|_{L_t^{4+} L_x^{4-}(J)} \\ &\quad + \|\langle D \rangle^{1-\frac{p-3}{2}} If\|_{L_t^{\frac{4}{p-3}} L_x^{\frac{4}{5-p}}(J)} + \|\langle D \rangle^{1-1} If\|_{L_t^{\frac{4(p-1)}{p-1}} L_x^{\frac{4(p-1)}{p-3}}(J)} + \|\langle D \rangle^{1-s_c} If\|_{L_t^{2(p-1)-} L_x^{2(p-1)+}(J)}. \end{aligned}$$

Remark 2.1. Here $(2(p-1)-, 2(p-1)+)$ is a small variation of $(2(p-1), 2(p-1))$ such that $(2(p-1)-, 2(p-1)+)$ is s_c -wave admissible: see the point A on Figure 1. It is necessary to create this variation to make the proof work. This variation creates other variations of bipoins, such as $(4+, 4-)$. For the sake of simplification, we will not describe how these variations are created: the reader is invited to do that himself. It is recommended that the reader ignores all these variations at first reading.

We define the mollified energy of f to be the following:

$$\begin{aligned} E(If(t)) &:= \frac{1}{2} \int_{\mathbb{R}^3} |\partial_t If(t, x)|^2 dx + \frac{1}{2} \int_{\mathbb{R}^3} |If(t, x)|^2 dx \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^3} |\nabla If(t, x)|^2 dx + \frac{1}{p+1} \int_{\mathbb{R}^3} |If(t, x)|^{p+1} dx. \end{aligned}$$

We also define

$$\begin{aligned} E_c(If(t)) &:= \frac{1}{2} \int_{\mathbb{R}^3} |\partial_t If(t, x)|^2 dx + \frac{1}{2} \int_{\mathbb{R}^3} |If(t, x)|^2 dx \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^3} |\nabla If(t, x)|^2 dx, \end{aligned}$$

$$\begin{aligned} E(If(t), B(x_0, R)) &:= \frac{1}{2} \int_{B(x_0, R)} |\partial_t If(t, x)|^2 dx + \frac{1}{2} \int_{B(x_0, R)} |If(t, x)|^2 dx \\ &\quad + \frac{1}{2} \int_{B(x_0, R)} |\nabla If(t, x)|^2 dx + \frac{1}{p+1} \int_{B(x_0, R)} |If(t, x)|^{p+1} dx, \end{aligned}$$

with $B(x_0, R) := \{x \in \mathbb{R}^3, |x - x_0| \leq R\}$.

2.3. Leibnitz rules. We recall some well-known Leibnitz-type rules. Let $(r, r_1, r_2, \tilde{r}_1, \tilde{r}_2, \tilde{r}_3) \in (1, \infty)^6$ be such that

$$\begin{aligned} \frac{1}{r} &= \frac{p-1}{r_1} + \frac{1}{r_2} \text{ and} \\ \frac{1}{r} &= \frac{p-2}{\tilde{r}_1} + \frac{1}{\tilde{r}_2} + \frac{1}{\tilde{r}_3}. \end{aligned}$$

If $\alpha \geq 1 - s$ then

$$\|\langle D \rangle^\alpha IF(f)\|_{L_x^r} \lesssim \|f\|_{L_x^{r_1}}^{p-1} \|\langle D \rangle^\alpha If\|_{L_x^{r_2}}.$$

If $\alpha \geq 0$ then

$$\begin{aligned} \|\langle D \rangle^\alpha (F(f) - F(g))\|_{L_x^r} &\lesssim \left(\|f\|_{L_x^{r_1}}^{p-1} + \|g\|_{L_x^{r_1}}^{p-1} \right) \|\langle D \rangle^\alpha (f - g)\|_{L_x^{r_2}} \\ &\quad + \left(\|f\|_{L_x^{\tilde{r}_1}}^{p-2} + \|g\|_{L_x^{\tilde{r}_1}}^{p-2} \right) \left(\|\langle D \rangle^\alpha f\|_{L_x^{\tilde{r}_2}} + \|\langle D \rangle^\alpha g\|_{L_x^{\tilde{r}_2}} \right) \|f - g\|_{L_x^{\tilde{r}_3}}. \end{aligned}$$

The proof of the first estimate is a simple modification of that of the composition rule (see e.g [26]). The second estimate comes from the fundamental theorem of calculus $F(f) = F(g) + \int_0^1 F'(f + t(g - f)) \cdot (g - f) dt$ and the product rule (see e.g [26]).

2.4. The Paley-Littlewood decomposition. We constantly use throughout the paper the Paley-Littlewood decomposition.

Let $\phi(\xi)$ be a real, radial, nonincreasing function that is equal to 1 on the unit ball $\{\xi \in \mathbb{R}^3 : |\xi| \leq 1\}$ and that is supported on $\{\xi \in \mathbb{R}^3 : |\xi| \leq 2\}$. Let ψ denote the function $\psi(\xi) := \phi(\xi) - \phi(2\xi)$. If $(M, M_1, M_2) \in 2^{\mathbb{N}}$ are dyadic numbers such that $M_2 \geq M_1$ we define the Paley-Littlewood operators by ³

$$\begin{aligned}\widehat{P_M f}(t, \xi) &:= \psi\left(\frac{\xi}{M}\right) \hat{f}(t, \xi), \quad M > 1 \\ \widehat{P_1 f}(t, \xi) &:= \phi(\xi) \hat{f}(t, \xi) \\ \widehat{P_{\leq M} f}(t, \xi) &:= \phi\left(\frac{\xi}{M}\right) \hat{f}(t, \xi) \\ \widehat{P_{> M} f}(t, \xi) &:= \hat{f}(t, \xi) - \widehat{P_{\leq M} f}(t, \xi) \\ P_{\ll M} f &:= P_{\leq \frac{M}{128}} f \\ P_{\gtrsim M} f &:= P_{> \frac{M}{128}} f \\ P_{M_1 < \cdot \leq M_2} f &:= P_{\leq M_2} f - P_{\leq M_1} f.\end{aligned}$$

Then we have

$$\begin{aligned}f(t) &= \sum_{M \in 2^{\mathbb{N}}} P_M f(t), \\ f(t) &= P_{\ll M} f(t) + P_{\gtrsim M} f(t).\end{aligned}$$

2.5. The numbers α , C and c . All the relevant constants are denoted by the numbers $\alpha := \alpha(p) \gtrsim 1$, $C := C(p) \gtrsim 1$, or $c := c(p) \ll 1$. The constants α are exclusively used in expression involving powers (such as powers of A); otherwise, the constants C and c are used.

Constants C	defined in	Constants c	defined in	Constants α	defined in
$C_{in,1}$	(3.3)	c_1	(5.3)	α_1	(5.3)
$C_{in,2}$	(3.3)	c_2	(5.4)	α_2	(5.6)
C_1	(5.6)	c_3	(5.8)	α_3	(5.8)
C_2	(5.9)	c_4	(5.10)	α_4	(5.9)
C_3	(5.10)	c_5	(5.14)	α_5	(5.9)
C_4	(5.11)	c_6	(5.16)	α_6	(5.10)
C_5	(7.2)	c_7	(7.2)	α_7	(5.11)
C_6	(7.5)	c_8	(7.3)	α_8	(5.12)
		c_9	(7.6)	α_9	(5.14)
		c_{10}	Lemma 8.1	α_{10}	(5.16)
				α_{11}	(7.2)
				α_{12}	(7.2)
				α_{13}	(7.3)
				α_{14}	(7.4)
				α_{15}	(7.5)
				α_{16}	(7.6)

³Here \hat{f} denotes the Fourier transform in space of f

3. PRELIMINARY RESULTS

In this section we recall some results that we constantly use throughout the proof of Theorem 1.1.

Let $J := [a, b]$ be a time interval.

The wave Strichartz estimates (see for example [3, 10, 14, 17]⁴) can be stated as follows:

Proposition 3.1. "Strichartz estimates" Assume that w satisfies the following Klein-Gordon equation on J

$$(3.1) \quad \begin{cases} \partial_{tt}w - \Delta w + w &= G \\ w(a, x) &= w_0(x) \\ \partial_t w(a, x) &= w_1(x). \end{cases}$$

Then, if $m \in [0, 1]$, we have

$$(3.2) \quad \begin{aligned} &\|w\|_{L_t^q L_x^r(J)} + \|\partial_t \langle D \rangle^{-1} w\|_{L_t^q L_x^r(J)} + \|w\|_{L_t^\infty H^m(J)} + \|\partial_t w\|_{L_t^\infty H^{m-1}(J)} \\ &\lesssim \|w_0\|_{H^m} + \|w_1\|_{H^{m-1}} + \|G\|_{L_t^{\tilde{q}} L_x^{\tilde{r}}(J)}, \end{aligned}$$

under the assumptions

$$(q, r) \in W_m, \quad (\tilde{q}, \tilde{r}) \in \tilde{W}, \quad \text{and} \quad \frac{1}{\tilde{q}} + \frac{3}{\tilde{r}} - 2 = \frac{1}{q} + \frac{3}{r}.$$

The next proposition shows that the mollified energy at time zero of the solution u of (1.1) with data (u_0, u_1) satisfying (1.6) is bounded:

Proposition 3.2. "Boundedness of mollified energy at time 0" [25] There exist two constants $C_{in,1}$ and $C_{in,2}$ such that⁵

$$(3.3) \quad E(Iu(0)) \leq C_{in,1} N^{2(1-s)} A^{p+1} \leq C_{in,2} A^{p+1}$$

The next proposition shows that the variation of a solution of (1.1) on a time interval can be estimated. More precisely

Proposition 3.3. "Almost Conservation Law" [25] Let w be a solution of (1.1). Let $t_0 \in J$. Then

$$(3.4) \quad \left| \sup_{t \in J} E(Iw) - E(Iw(t_0)) \right| \lesssim \int_J \int_{\mathbb{R}^3} |\partial_t Iw| |IF(w) - F(Iw)| dx dt \lesssim \frac{Z^{p+1}(J, w)}{N^{\frac{5-p}{2}}}.$$

The last proposition allows to control a weighted norm of w on a time interval; more precisely

Proposition 3.4. "Almost Morawetz-Strauss estimate" [25] Let w be a solution of (1.1). Let $\tilde{x} \in \mathbb{R}^3$. Then

$$(3.5) \quad \int_J \int_{\mathbb{R}^3} \frac{|Iw|^{p+1}}{|x - \tilde{x}|} dx dt \lesssim \sup_{t \in J} E(Iw(t)) + R_1(J, w) + R_2(J, w).$$

with

$$\begin{aligned} R_1(J, w) &:= \int_J \int_{\mathbb{R}^3} \frac{\nabla Iw \cdot (x - \tilde{x})}{|x - \tilde{x}|} (F(Iw) - IF(w)) dx dt \\ \text{and} \quad R_2(J, w) &:= \int_J \int_{\mathbb{R}^3} \frac{Iw}{|x - \tilde{x}|} (F(Iw) - IF(w)) dx dt. \end{aligned}$$

Moreover for $i = 1, 2$ we have

⁴see also [25]

⁵Notice that in (3.3) we have deliberately chosen to keep the term N^{1-s} . Indeed, we will use (3.3) in Section 4 to explain why it is natural to choose $N^{1-s} \sim 1$

$$(3.6) \quad |R_i(J, w)| \lesssim \frac{Z^{p+1}(J, w)}{N^{\frac{5-p}{2}} -}.$$

4. IDEAS

In this section we explain the main ideas of this proof.

In [18, 19], Nakanishi found for some $(q := q(p), r := r(p))$ an upper bound of the $L_t^q L_x^r$ norm of the solution of (1.1) with data in the energy class by a tower-exponential type bound of the energy of the form

$$(4.1) \quad \|u\|_{L_t^q L_x^r(\mathbb{R})} \lesssim E^{E^{\dots^E}},$$

where the height of the tower also depends on the energy. This decay estimate was the preliminary step to prove scattering. A natural question is: is it possible to prove decay estimates of this form (or a modified form) for rougher data? If this is possible, then it might help us to prove scattering of solutions of (1.1), by analogy with the scattering theory for data in the energy space. This paper gives a positive answer to this question, at least for s close enough to one.

Of course, one cannot use the energy conservation law because the energy can be infinite on $H^s \times H^{s-1}$, $s < 1$. Instead we introduce the multiplier I and we work with the mollified energy of u that is finite in these rough spaces and that is almost conserved: this is the I -method (see e.g [7, 12]), inspired by the *Fourier truncation method*, designed in [2]. We aim at proving a decay estimate that is finite in $H^s \times H^{s-1}$. Therefore, by analogy with the energy conservation law, our decay estimate should not only depend on u but also on I . It was proved in [25] that, under the additional assumption of radial symmetry, we can control pretty easily the norm $\|Iu\|_{L_t^{p+2} L_x^{p+2}(\mathbb{R})}$ by combining the “Almost Morawetz-Strauss estimate” (3.5) with a pointwise decay estimate, namely a radial Sobolev inequality. Unfortunately, such a pointwise decay estimate does not exist for general data and we shall establish, by means of concentration [1], a tower-exponential bound of the norm $\|Iu\|_{L_t^{2(p-1)-} L_x^{2(p-1)+}(\mathbb{R})}$. We shall denote this norm by the *target norm*.

In order to prove this bound, the idea is to use the H^1 theory for frequencies smaller or equal to the parameter N and to control for frequencies larger than N all the errors that appear in the process of generating this estimate. The success of the I method depends on one condition: by choosing appropriately a parameter $N \gg 1$, one can make the variation of the mollified energy on an arbitrarily long time interval small compare with its initial size at time zero. Assuming that this condition is satisfied for a moment we can neglect the variation of the mollified energy and we expect to have, by analogy with the bound (4.1) of the $L_t^q L_x^r$ norm of the solutions of (1.1) with data in the energy class, a tower-exponential bound of the form

$$(4.2) \quad \|Iu\|_{L_t^{2(p-1)-} L_x^{2(p-1)+}(\mathbb{R})} \lesssim (C_{in,1} N^{2(1-s)} A^{p+1})^{\dots^{C_{in,1} N^{2(1-s)} A^{p+1}}},$$

where we substitute the energy E in (4.1) for the initial size of the mollified energy, i.e $C_{in,1} N^{2(1-s)} A^{p+1}$ by Proposition 3.2. The variation of the mollified energy is then estimated by iteration of the almost conservation law (see Proposition 3.3) on intervals such that the target norm is small (see Proposition 5.1), using (4.2). One gets, roughly speaking, a variation of the form

$$Var := \frac{(C_{in,1} N^{2(1-s)} A^{p+1})^{\dots^{C_{in,1} N^{2(1-s)} A^{p+1}}}}{N^{\frac{5-p}{2}} -}.$$

In order to satisfy our condition, one must make Var small compare with the initial size of the mollified energy: it is natural to choose (N, s) such that

$$N^{1-s} \sim 1,$$

and $N := N(A)$ a large number depending on A .

Theorem 1.1 is proved in Section 5. The proof is based upon a modification of the method of induction on levels of the conserved energy for data in the energy space that is designed in [1]. Indeed, since the mollified energy is not conserved, we have to modify significantly this method. In particular, we design a relation that allows to control not only the target norm but also the mollified energy of a solution of (1.1), assuming that we control its mollified energy at one time: see the definition of $\mathcal{P}(l)$. Then we prove that this relation is true for large levels of mollified energy at one time by induction, using the small mollified energy at one time theory (see Proposition 5.4). Also, we have to make sure that we can make the mollified energy decrease at one time at a non-decreasing rate, in order to reach the small mollified energy level and apply the small mollified energy theory: this is done by introducing the parameters c_6 and α_{10} in order to make the variation of the mollified energy small enough. The proof of Theorem 1.1 relies upon some propositions that we prove in the other sections. In Section 6, we prove some local bounds: these bounds, combined with Proposition 3.3 (resp. Proposition 3.4), allow to estimate by iteration the variation of the mollified energy (resp. an “Almost Morawetz-Strauss estimate”) on an arbitrarily long-time interval. In Section 7 and Section 8 we modify arguments of [1, 18, 19] to separate the localized mollified energy and prove a perturbation result. Notice that throughout these sections, the multiplier I does not commute with the nonlinearity, and one has to prove some commutator estimates, i.e estimates involving the commutator $I(F(f)) - F(I f)$: these estimates are proved in Appendix A.

5. PROOF OF THEOREM

In this section, we prove Theorem 1.1.

The proof of Theorem 1.1 relies upon several propositions such as local boundedness, separation of the localized mollified energy, perturbation argument, and small mollified energy theory. We shall prove these propositions in Section 6, 7, 8, and 9.

5.1. Propositions.

We consider J , J' , and \bar{J}' three intervals

$$J := [a, b] \subset [0, \infty); \bar{J}' \subset J' \subset J.$$

Let w be a solution of (1.1).

We assume that there exist L_w, X_w such that

$$(5.1) \quad \sup_{t \in J} E(Iw(t)) \leq X_w \lesssim A^{p+1},$$

and

$$(5.2) \quad \|Iw\|_{L_t^{2(p-1)-} L_x^{2(p-1)+}(J)} \leq L_w < \infty.$$

We will prove in Subsection 5.2 that these assumptions are always true.

Under these assumptions, one can find constants α_1 and c_1 such that if

$$(5.3) \quad \frac{\langle L_w \rangle^{\alpha_1} A^{\alpha_1}}{N^{(1-s_c)-}} \leq c_1,$$

then three propositions hold.

The first proposition shows that if \bar{J}' is small in the sense of (5.4), then one can control several norms on this subinterval:

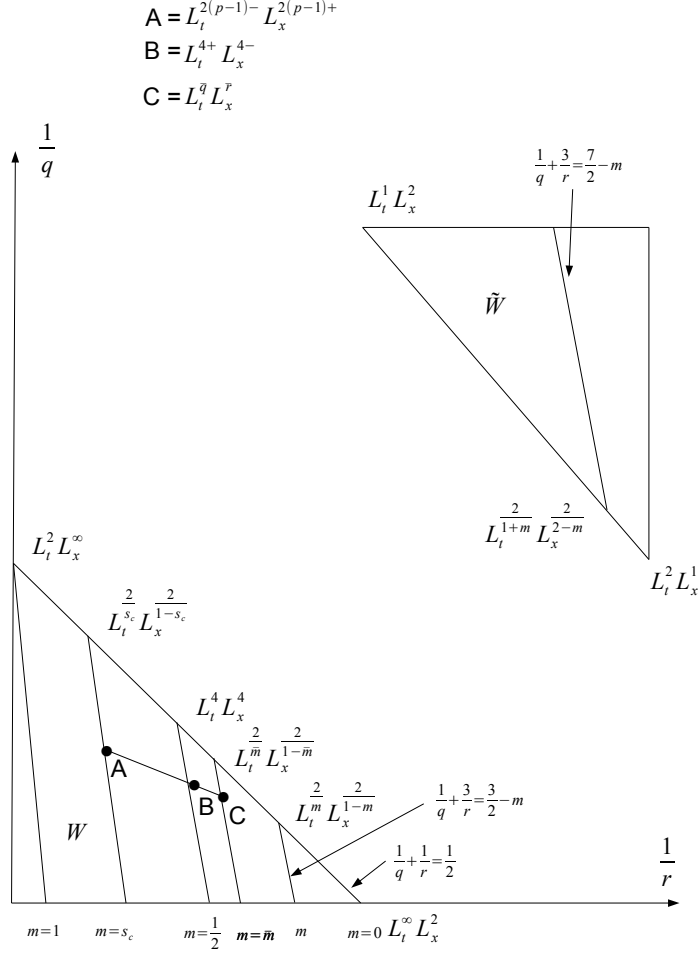


FIGURE 1. The Strichartz graph

Proposition 5.1. “Local boundedness” *There exists a constant c_2 such that if*

$$(5.4) \quad \|Iw\|_{L_t^{2(p-1)-} L_x^{2(p-1)+}(\bar{J}')} \leq c_2,$$

then

$$(5.5) \quad Z(\bar{J}', w) \lesssim X_w^{\frac{1}{2}}.$$

The second proposition shows that if a subinterval is too large in the sense of (5.6), then one can separate the mollified energy into two parts: one that is carried by a free Klein-Gordon solution and the other one that is carried by another solution w' of (1.1). The proof of Proposition 5.2 relies upon the combination of the I -method with a modification of arguments from Bourgain [1], or, more closely, Nakanishi [18, 19].

Proposition 5.2. “Separation of the localized mollified energy” Let $M \gg 1$. There exist $T \in J'$, and v , a solution of the free Klein-Gordon equation, and constants $c_3, C_1, C_2, \alpha_2, \dots, \alpha_5$ such that if

$$(5.6) \quad \|Iw\|_{L_t^{2(p-1)-} L_x^{2(p-1)+}(J')} \geq C_1^{(C_1^{A^{\alpha_2}} M)^{C_1 A^{\alpha_2}}},$$

then

$$(5.7) \quad E_c(Iv) \lesssim 1,$$

$$(5.8) \quad E(Iw'(T)) \leq \sup_{t \in J'} E(Iw(t)) - c_3 A^{-\alpha_3},$$

$$(5.9) \quad \text{and} \quad \|Iv\|_{L_t^{2(p-1)-} L_x^{2(p-1)+}(J'')} \leq C_2 \frac{A^{\alpha_4}}{M^{\alpha_5}}, \quad \text{for } J'' := (T, b) \quad \text{or} \quad \text{for } J'' := (a, T).$$

Here w' denotes the solution of (1.1) such that $w'(T) := w(T) - v(T)$.

The third proposition shows that if the target norm of v is small in the sense of (5.11), then the target norm of w can be estimated from that of w' :

Proposition 5.3. “Perturbation argument” Let T, v , and w' be defined in the previous proposition. Let $J'' := (T, b)$ (or $J'' := (a, T)$). Assume that w' satisfies (5.1) and (5.2) (with w substituted for w'). Then there exist $c_4, C_3, C_4, \alpha_6, \alpha_7, \alpha_8$, and k such that if

$$(5.10) \quad \frac{C_3^{(A\langle L_{w'} \rangle)^{\alpha_6}} (A\langle L_w \rangle)^{\alpha_6}}{N^{(1-s_c)-}} \leq c_4,$$

$$(5.11) \quad k \leq \frac{1}{C_4^{(A\langle L_{w'} \rangle)^{\alpha_7}}} \quad \text{and} \quad \|Iv\|_{L_t^{2(p-1)-} L_x^{2(p-1)+}(J'')} \leq k,$$

then

$$(5.12) \quad \|Iw\|_{L_t^{2(p-1)-} L_x^{2(p-1)+}(J'')} \lesssim (\langle L_{w'} \rangle A)^{\alpha_8}.$$

Next we show that if the mollified energy is small enough at one time, then we can have a very good control of the mollified energy and our target norm:

Proposition 5.4. “Small mollified energy theory”

Assume that there exists $\tilde{t} \in \mathbb{R}$ such that

$$(5.13) \quad E(Iw(\tilde{t})) \ll 1.$$

Then one can find constants α_9 and c_5 such that if

$$(5.14) \quad \frac{A^{\alpha_9}}{N^{(1-s_c)-}} \leq c_5,$$

then

$$(5.15) \quad \|Iw\|_{L_t^{2(p-1)-} L_x^{2(p-1)+}(\mathbb{R})} \lesssim 1,$$

and there exist c_6 and α_{10} such that

$$(5.16) \quad \sup_{t \in \mathbb{R}} E(Iw(t)) \leq E(Iw(\tilde{t}))(1 + c_6 A^{-\alpha_{10}}).$$

In fact, one can choose c_6 (resp. α_{10}) arbitrarily small (resp. arbitrarily large) in (5.16), by choosing c_5 (resp. α_9) small enough (resp. large enough) in (5.14).

5.2. The proof.

We are now in position to prove Theorem 1.1. We define the following statement of induction for $l \in \mathbb{N}$

$\mathcal{P}(l)$: let

$$\mathcal{C}_l := \left\{ w_l, \begin{array}{l} w_l \text{ solution of (1.1),} \\ \exists t_l \in \mathbb{R}^+ \text{ s.t. } E(Iw_l(t_l)) \leq C_{in,2}A^{p+1} - 0.9lc_3A^{-\alpha_3}; \end{array} \right\},$$

then there exists $\infty > L(l) := L_{N,s,A}(l)$ such that

$$(5.17) \quad \inf\{\bar{C}, w_l \in \mathcal{C}_l \text{ and } \|Iw_l\|_{L_t^{2(p-1)-}L_x^{2(p-1)+}(\mathbb{R}^+)} \leq \bar{C}\} = L(l)$$

and

$$(5.18) \quad \sup_{t \in \mathbb{R}^+} E(Iw_l(t)) \leq (C_{in,2}A^{p+1} - 0.9lc_3A^{-\alpha_3})(1 + c_6A^{-\alpha_{10}}).$$

We easily obtain that $\mathcal{P}(\bar{l})$ holds for some $\bar{l} \lesssim A^{p+1+\alpha_3}$ by applying Proposition 5.4, choosing N such that (5.14) holds.

Our goal is then to show that if $\mathcal{P}(l+1)$ holds, then $\mathcal{P}(l)$ also holds for N and s to be properly chosen. To this end let $w_l \in \mathcal{C}_l$. Let $T > 0$. Assume that (5.17) restricted to $[0, t_l + T]$ holds for some $L(l) < \infty$ to be chosen. Choose N such that (5.3), (5.10), and (5.14) hold with $L_w, L_{w'}$ substituted respectively for $L(l), L(l+1)$. Clearly (5.10) is the most constraining assumption to satisfy, choosing C_3 and α_6 (resp. c_4) large enough (resp. small enough). One may partition $[0, t_l + T]$ into subintervals J such that $\|Iw_l\|_{L_t^{2(p-1)-}L_x^{2(p-1)+}(J)} = c_2$ (except maybe the last one), one may apply Proposition 5.1 and Proposition 3.3 on each J , and then one may iterate to get for $t \in [0, t_l + T]$

$$\begin{aligned} |E(Iw_l(t)) - E(Iw_l(t_l))| &\lesssim \frac{(C_{in,2}A^{p+1} - 0.9lc_3A^{-\alpha_3})^{\frac{p+1}{2}} \langle L(l) \rangle^{2(p-1)-}}{N^{\frac{5-p}{2}-}} \\ &\leq c_6A^{-\alpha_{10}} (C_{in,2}A^{p+1} - 0.9lc_3A^{-\alpha_3}), \end{aligned}$$

using (5.3), and choosing α_1 (resp. c_1) large (resp. small) enough. Therefore (5.18) holds. It remains to prove (5.17). To this end we let M be such that

$$(5.19) \quad C_2 \frac{A^{\alpha_4}}{M^{\alpha_5}} = \frac{1}{C_4^{\langle A \langle L(l+1) \rangle \rangle^{\alpha_7}}},$$

Let $B > 0$. If $\|Iw_l\|_{L_t^{2(p-1)-}L_x^{2(p-1)+}([0, t_l+T])} \geq 3B$, then we can find $(\bar{t}_l, \bar{\bar{t}}_l) \in [0, t_l + T]^2$ such that

$$\|Iw_l\|_{L_t^{2(p-1)-}L_x^{2(p-1)+}([0, \bar{t}_l])} \geq B, \quad \|Iw_l\|_{L_t^{2(p-1)-}L_x^{2(p-1)+}([\bar{t}_l, \bar{\bar{t}}_l])} \geq B, \quad \text{and} \quad \|Iw_l\|_{L_t^{2(p-1)-}L_x^{2(p-1)+}([\bar{\bar{t}}_l, t_l+T])} \geq B.$$

Assume that $B \geq C_1^{(C_1^{A^{\alpha_2}}M)^{C_1A^{\alpha_2}}}$. Then, applying Proposition 5.2 to $J' := [\bar{t}_l, \bar{\bar{t}}_l]$, we see that there exists $T_l \in J'$ and w_l' solution of (1.1) such that

$$(5.20) \quad \begin{aligned} E(Iw_l', T_l) &\leq (C_{in,2}A^{p+1} - 0.9lc_3A^{-\alpha_3})(1 + c_6A^{-\alpha_{10}}) - c_3A^{-\alpha_3} \\ &\leq C_{in,2}A^{p+1} - 0.9(l+1)c_3A^{-\alpha_3}, \end{aligned}$$

choosing c_6 (resp. α_{10}) small (resp. large) enough. Therefore $w_l' \in \mathcal{C}_{l+1}$. We deal with the case where (5.9) holds with $J'' := [T_l, t_l + T]$ ⁶. Applying $\mathcal{P}(l+1)$ and Proposition 5.3, we see that

$$B \leq \|Iw_l\|_{L_t^{2(p-1)-}L_x^{2(p-1)+}([\bar{\bar{t}}_l, t_l+T])} \leq \|Iw_l\|_{L_t^{2(p-1)-}L_x^{2(p-1)+}([T_l, t_l+T])} \lesssim (\langle L(l+1) \rangle A)^{\alpha_8}.$$

⁶the other case $J'' := [t_l, T_l]$ is handled by a similar argument and therefore it is left to the reader.

Therefore we see that $\|Iw_l\|_{L_t^{2(p-1)-} L_x^{2(p-1)+}([0, t_l+T])} < \infty$, $\mathcal{P}(l)$ holds, and, moreover, there exist $\alpha \gtrsim 1$ and $C \gtrsim 1$ such that

$$(5.21) \quad L(l) \lesssim \max \left(C_1^{(C_1^{A^{\alpha_2}} C_2^{\frac{1}{\alpha_5}} A^{\frac{\alpha_4}{\alpha_5}} C_4^{\frac{(A\langle L(l+1)\rangle)^{\alpha_7}}{\alpha_5}})^{C_1 A^{\alpha_2}}}, (\langle L(l+1)\rangle A)^{\alpha_8} \right) \leq C^{C^{\max \alpha(A, \langle L(l+1)\rangle)} C^{\max \alpha(A, \langle L(l+1)\rangle)}}.$$

Iterating (5.21) \bar{l} times, we see that for $0 \leq l \leq \bar{l}$ (even if it means increasing the value of α and C)

$$L(l) \leq \|L\|_\infty := C^{C^{\dots C^\alpha}},$$

where the height of the tower is $\sim A^\alpha$. Such an iteration is possible if $\frac{C_3^{(A\langle \|L\|_\infty \rangle)^{\alpha_6}} (A\langle \|L\|_\infty \rangle)^{\alpha_6}}{N^{(1-s_c)-}} \leq c_4$.

Pick N such that $\frac{c_4}{2} \leq \frac{C_3^{(A\langle \|L\|_\infty \rangle)^{\alpha_6}} (A\langle \|L\|_\infty \rangle)^{\alpha_6}}{N^{(1-s_c)-}} \leq c_4$. From (2.1), we see that there exists \tilde{s} such that $\mathcal{P}(l)$ holds for $1 > s > \tilde{s}$. Moreover \tilde{s} can be chosen to be of the form (1.7) as $A \rightarrow \infty$.

Global existence

From $\mathcal{P}(0)$ and (3.3) we see that solutions of (1.1) with $1 > s > \tilde{s}$ and with data $(u_0, u_1) \in H^s \times H^{s-1}$ satisfying (1.6) satisfy

$$(5.22) \quad \sup_{t \in \mathbb{R}^+} E(Iu(t)) \lesssim A^{p+1}, \quad \text{and} \quad \|Iu\|_{L_t^{2(p-1)-} L_x^{2(p-1)+}(\mathbb{R}^+)} < \infty.$$

By time reversal symmetry, we may extend \mathbb{R}^+ to \mathbb{R} . By Plancherel, we have for all time $T \in \mathbb{R}$ $\|(u(T), \partial_t u(T))\|_{H^s \times H^{s-1}}^2 \lesssim A^{p+1}$. This proves global well-posedness.⁷

Global estimates

Now we claim that

$$(5.23) \quad Z_{m,s}(\mathbb{R}, u) \lesssim_A 1$$

for all $0 \leq m \leq s$. Indeed, by (5.22), we may divide $[0, \infty)$ in subintervals $J := [a, b]$ such that $\|Iu\|_{L_t^{2(p-1)-} L_x^{2(p-1)+}(J)} = c_2$ (except maybe the last one). Moreover, plugging $\langle D \rangle^{1-m} I$ into (3.2), and by (5.22), we have

$$(5.24) \quad \begin{aligned} Z_{m,s}(J, u) &\lesssim \|(\langle D \rangle Iu(a), \partial_t Iu(a))\|_{L^2 \times L^2} + \|\langle D \rangle^{1-m} I(|u|^{p-1} u)\|_{L_t^{\frac{2}{1+m}} L_x^{\frac{2}{2-m}}(J)} \\ &\lesssim A^{\frac{p+1}{2}} + \|\langle D \rangle^{1-m} Iu\|_{L_t^{\frac{2}{m}} L_x^{\frac{2}{1-m}}(J)} \| |u|^{p-1} \|_{L_t^{2-} L_x^{2+}(J)} \\ &\lesssim A^{\frac{p+1}{2}} + Z_{m,s}(J, u) \left(\|P_{\ll N} u\|_{L_t^{2(p-1)-} L_x^{2(p-1)+}(J)}^{p-1} + \|P_{\gtrsim N} u\|_{L_t^{2(p-1)-} L_x^{2(p-1)+}(J)}^{p-1} \right) \\ &\lesssim A^{\frac{p+1}{2}} + Z_{m,s}(J, u) \left(\|Iu\|_{L_t^{2(p-1)-} L_x^{2(p-1)+}(J)}^{p-1} + \frac{\|\langle D \rangle^{1-s_c} Iu\|_{L_t^{2(p-1)-} L_x^{2(p-1)+}(J)}^{p-1}}{N^{\frac{5-p}{2}-}} \right) \\ &\lesssim A^{\frac{p+1}{2}} + Z_{m,s}(J, u) \left(c_2^{p-1} + \frac{Z_{s_c, s}^{p-1}(u, J)}{N^{\frac{5-p}{2}-}} \right) \end{aligned}$$

⁷Notice that global well-posedness was already proved in [24] but, since it is a prerequisite to prove scattering, we reprove it.

By our choice of N , we have $\frac{A^{\frac{p(p+1)}{2}}}{N^{\frac{5-p}{2}}} \ll A^{\frac{p+1}{2}}$. Therefore a continuity argument (first for $m = s_c$, then for the other m), we see that $Z_{m,s}(J, u) \lesssim A^{\frac{p+1}{2}}$. Iterating over J , we get (5.23).

Scattering

Let $\mathbf{v}(t) := (u(t), \partial_t u(t))$, $\mathbf{v}_0 := (u_0, u_1)$ and

$$\mathbf{u}_{\mathbf{nl}}(t) := \begin{pmatrix} -\int_0^t \frac{\sin((t-t')\langle D \rangle)}{\langle D \rangle} (|u|^{p-1}(t')u(t')) dt' \\ -\int_0^t \cos((t-t')\langle D \rangle) (|u|^{p-1}(t')u(t')) dt' \end{pmatrix}.$$

Then we get from (1.2) $\mathbf{v}(t) = K(t)\mathbf{v}_0 + \mathbf{u}_{\mathbf{nl}}(t)$. Recall that the solution u scatters in $H^s \times H^{s-1}$ if there exists $\mathbf{v}_{+,0} := (u_{+,0}, u_{+,1})$ such that $\|\mathbf{v}(t) - K(t)\mathbf{v}_{+,0}\|_{H^s \times H^{s-1}} \rightarrow 0$ as $t \rightarrow \infty$. In other words, since K is bounded on $H^s \times H^{s-1}$, it suffices to prove that the quantity $\|K^{-1}(t)\mathbf{v}(t) - \mathbf{v}_{+,0}\|_{H^s \times H^{s-1}} \rightarrow 0$ as $t \rightarrow \infty$. A computation shows that

$$K^{-1}(t) = \begin{pmatrix} \cos(t\langle D \rangle) & -\frac{\sin(t\langle D \rangle)}{\langle D \rangle} \\ \langle D \rangle \sin(t\langle D \rangle) & \cos(t\langle D \rangle) \end{pmatrix}.$$

But $K^{-1}(t)\mathbf{v}(t) = \mathbf{v}_0 - K^{-1}(t)\mathbf{u}_{\mathbf{nl}}(t)$ and, by dualizing the Strichartz estimate $\|e^{it\langle D \rangle} f\|_{L_t^{\frac{2}{1-s}} L_x^{\frac{2}{1+s}}} \lesssim \|f\|_{H^{1-s}}$ (see Proposition 3.1), we have

$$\begin{aligned} \|K^{-1}(t_1)\mathbf{u}_{\mathbf{nl}}(t_1) - K^{-1}(t_2)\mathbf{u}_{\mathbf{nl}}(t_2)\|_{H^s \times H^{s-1}} &\lesssim \| |u|^{p-1}u \|_{L_t^{\frac{2}{1+s}} L_x^{\frac{2}{2-s}}([t_1, t_2])} \\ &\lesssim \|\langle D \rangle^{1-s} I(|u|^{p-1}u)\|_{L_t^{\frac{2}{1+s}} L_x^{\frac{2}{2-s}}([t_1, t_2])}. \end{aligned}$$

But, plugging $\langle D \rangle^{1-s} I$ into (3.2) and modifying slightly (5.24), we get

$$\|\langle D \rangle^{1-s} I(|u|^{p-1}u)\|_{L_t^{\frac{2}{1+s}} L_x^{\frac{2}{2-s}}([t_1, t_2])} \lesssim Z_{s,s}([t_1, t_2], u) \left(\|Iu\|_{L_t^{2(p-1)-} L_x^{2(p-1)+}([t_1, t_2])}^{p-1} + \frac{Z_{s_c,s}^{p-1}([t_1, t_2], u)}{N^{\frac{5-p}{2}-}} \right).$$

Therefore, from (5.22) and (5.23), we see that the Cauchy criterion is satisfied by $K^{-1}(t)v(t)$ and we conclude that $K^{-1}(t)v(t)$ has a limit in $H^s \times H^{s-1}$ as t goes to infinity. Moreover $\lim_{t \rightarrow \infty} \|\mathbf{v}(t) - K(t)\mathbf{v}_{+,0}\|_{H^s \times H^{s-1}} = 0$, with $\mathbf{v}_{+,0} := (u_{+,0}, u_{+,1})$ given explicitly by

$$\begin{aligned} u_{+,0} &:= u_0 + \int_0^\infty \frac{\sin(t'\langle D \rangle)}{\langle D \rangle} (|u|^{p-1}(t')u(t')) dt', \\ u_{+,1} &:= u_1 - \int_0^\infty \cos(t'\langle D \rangle) (|u|^{p-1}(t')u(t')) dt'. \end{aligned}$$

6. PROOF OF LOCAL BOUNDEDNESS

In this section we prove Proposition 5.1. Plugging $\langle D \rangle^{1-m} I$ into (3.2), we have (with $\bar{J}' := [\bar{a}', \bar{b}']$)

$$(6.1) \quad Z_{m,s}(J', w) \lesssim \|\langle D \rangle I w(\bar{a}')\|_{L^2} + \|\partial_t I w(\bar{a}')\|_{L^2} + \|\langle D \rangle^{1-m} I(|w|^{p-1}w)\|_{L_t^{\frac{2}{1+m}} L_x^{\frac{2}{2-m}}(\bar{J}')} ,$$

by (5.1). There are three cases:

- $m \leq s$. By slightly modifying (5.24)

$$\begin{aligned}
Z_{m,s}(\vec{J}', w) &\lesssim X_w^{\frac{1}{2}} + Z_{m,s}(\vec{J}', w) \left(\|Iw\|_{L_t^{2(p-1)-} L_x^{2(p-1)+}(\vec{J}')}^{p-1} + \frac{\|\langle D \rangle^{1-s_c} Iw\|_{L_t^{2(p-1)-} L_x^{2(p-1)+}(\vec{J}')}^{p-1}}{N^{\frac{5-p}{2}-}} \right) \\
&\lesssim X_w^{\frac{1}{2}} + Z_{m,s}(\vec{J}', w) \left(c_2^{p-1} + \frac{Z_{s_c,s}^{p-1}(\vec{J}', w)}{N^{\frac{5-p}{2}-}} \right)
\end{aligned}$$

Again, choosing α_1 (resp. c_1) large enough (resp. small enough) in (5.3), we have $\frac{A^{\frac{p(p+1)}{2}}}{N^{\frac{5-p}{2}-}} \ll A^{\frac{p+1}{2}}$. Therefore a continuity argument (first for $m = s_c$, then for the other m) shows that $Z_{m,s}(\vec{J}', w) \lesssim X_w^{\frac{1}{2}}$.

- $m = 1$. We estimate

$$\begin{aligned}
Z_{m,s}(\vec{J}', w) &\lesssim X_w^{\frac{1}{2}} + \|I(|w|^{p-1}w)\|_{L_t^1 L_x^2(\vec{J}')} \\
&\lesssim X_w^{\frac{1}{2}} + \| |w|^{p-1}w \|_{L_t^1 L_x^2(\vec{J}')} \\
&\lesssim X_w^{\frac{1}{2}} + \| |P_{\ll N} w|^{p-1} P_{\ll N} w \|_{L_t^1 L_x^2(\vec{J}')} + \| |P_{\ll N} w|^{p-1} P_{\gtrsim N} w \|_{L_t^1 L_x^2(\vec{J}')} \\
&\quad + \| |P_{\gtrsim N} w|^{p-1} P_{< N} w \|_{L_t^1 L_x^2(\vec{J}')} + \| |P_{\gtrsim N} w|^{p-1} P_{\gtrsim N} w \|_{L_t^1 L_x^2(\vec{J}')} \\
&\lesssim X_w^{\frac{1}{2}} + B_1 + B_2 + B_3 + B_4.
\end{aligned}$$

We estimate

$$\begin{aligned}
B_1 &\lesssim \|Iw\|_{L_t^{2(p-1)-} L_x^{2(p-1)+}(\vec{J}')}^{p-1} \|Iw\|_{L_t^{2+} L_x^{\infty-}(\vec{J}')} \\
&\lesssim c_2^{p-1} Z_{1,s}(w, \vec{J}'), \\
B_2 &\lesssim \|Iw\|_{L_t^{p-1} L_x^{\frac{6(p-1)}{p-3}}(\vec{J}')}^{p-1} \|P_{\gtrsim N} w\|_{L_t^{\infty} L_x^{\frac{6}{p-3}}(\vec{J}')} \\
&\lesssim Z_{1,s}^{p-1}(\vec{J}', w) \frac{\|\langle D \rangle Iw\|_{L_t^{\infty} L_x^2(\vec{J}')}}{N^{\frac{5-p}{2}-}} \\
&\lesssim \frac{Z_{1,s}^{p-1}(\vec{J}', w)}{N^{\frac{5-p}{2}-}} X_w^{\frac{1}{2}}, \\
B_3 &\lesssim \frac{\|\langle D \rangle^{1-s_c} Iw\|_{L_t^{2(p-1)-} L_x^{2(p-1)+}(\vec{J}')}^{p-1}}{N^{\frac{5-p}{2}-}} \|Iw\|_{L_t^{2+} L_x^{\infty-}(\vec{J}')} \\
&\lesssim \frac{Z_{s_c,s}^{p-1}(\vec{J}', w)}{N^{\frac{5-p}{2}-}} Z_{1,s}(\vec{J}', w) \lesssim \frac{X_w^{\frac{p-1}{2}}}{N^{\frac{5-p}{2}-}} Z_{1,s}(\vec{J}', w), \\
B_4 &\lesssim \|P_{\gtrsim N} w\|_{L_t^p L_x^{2p}(\vec{J}')}^p \lesssim \frac{\|\langle D \rangle^{1-(\frac{3p-5}{2p})} Iw\|_{L_t^p L_x^{2p}(\vec{J}')}^p}{N^{\frac{5-p}{2}-}} \\
&\lesssim \frac{Z_{\frac{3p-5}{2p},s}^p(\vec{J}', w)}{N^{\frac{5-p}{2}-}} \lesssim \frac{X_w^{\frac{p}{2}}}{N^{\frac{5-p}{2}-}}.
\end{aligned}$$

Therefore, since again N satisfies (5.3), we see by a continuity argument that $Z_{1,s}(\vec{J}', w) \lesssim X_w^{\frac{1}{2}}$.

- $s < m < 1$: $Z_{m,s}(\vec{J}', w) \lesssim X_w^{\frac{1}{2}}$ follows by interpolating between $m = s$ and $m = 1$.

7. PROOF OF SEPARATION OF THE LOCALIZED MOLLIFIED ENERGY

In this section we prove Proposition 5.2. The proof of Proposition 5.2 relies upon three lemmas that we show in the next subsections.

7.1. Lemmas 7.1, 7.2, and 7.3. The first lemma shows that if there is concentration of the target norm of the solution on a subinterval of J' in the sense of (7.1), then this also means that the potential term of the mollified energy and the size of this subinterval are substantial.

Lemma 7.1. *Assume that*

$$(7.1) \quad \|Iw\|_{L_t^{2(p-1)-} L_x^{2(p-1)+}(J')} = c_2.$$

Then there exist a subinterval $\bar{K}' \subset J'$, a number $R \geq 1$, a point $\bar{x}' \in \mathbb{R}^3$ and constants $C_5, c_7, c_8, \alpha_{11}, \dots, \alpha_{13}$ such that

$$(7.2) \quad R := C_5 A^{\alpha_{11}}, |\bar{K}'| = c_7 A^{-\alpha_{12}},$$

and for all $t \in \bar{K}'$

$$(7.3) \quad \int_{|x-\bar{x}'| \leq R} |Iw(t, x)|^{p+1} dx \geq c_8 A^{-\alpha_{13}}.$$

The second lemma shows that if we consider a partition of J' into subintervals where the target norm of the solution concentrates, then these subintervals must be large on average. In order to prove this lemma, we shall mostly use the previous lemma and the Almost Morawetz-Strauss estimate (3.5).

Lemma 7.2. *Let $(\bar{J}_j' = [\bar{a}_j', \bar{a}_{j+1}'])_{1 \leq j \leq \bar{j}}$ be a partition of J' such that $\|Iw\|_{L_t^{2(p-1)-} L_x^{2(p-1)+}(\bar{J}_j')} = c_2$, except maybe the last one. Then there exist $\bar{t}_j' \in \bar{J}_j'$ and a constant α_{14} such that*

$$(7.4) \quad \sum_{j=1}^{\bar{j}-1} \frac{1}{\bar{t}_j' + 1} \lesssim A^{\alpha_{14}}.$$

The third lemma shows that if the target norm of the solution is too large in the sense of (7.5) then we can find a large subinterval where some norms are small compare with the concentration of mollified energy in the sense of (7.6)

Lemma 7.3. *Let $M \geq 1$. Then, there exist C_6, c_9, α_{15} , and α_{16} for all $0 < \epsilon \leq 1$, there exist $R' \in (1, \infty)$, $\bar{x} \in \mathbb{R}^3$ and $J'' := [S, T]$ (or $J'' := [T, S]$) such that $J'' \subset J'$ and if*

$$(7.5) \quad \|Iw\|_{L_t^{2(p-1)-} L_x^{2(p-1)+}(J')} \geq C_6^{(C_6^{(A\epsilon^{-1})^{\alpha_{15}}} M)^{C_6 A^{\alpha_{15}}}},$$

then

$$(7.6) \quad \tilde{Z}(J'', w) \leq c_9 A^{-\alpha_{16}} \leq E(Iw(S), B(\bar{x}, R')),$$

$$(7.7) \quad |J''| \geq MR',$$

and

$$(7.8) \quad \left\| \frac{Iw(S)}{\langle x - \bar{x} \rangle} \right\|_{L^2} \leq \epsilon.$$

7.2. The proof. We may assume without loss of generality that $S < T$. We apply Lemma 7.3 with $\epsilon < \min \left(A^{-\frac{\alpha_{16}}{2}}, A^{-\alpha_{16} - \frac{p+1}{2}} \right)$. Notice that with this choice of ϵ , the condition (7.5) becomes (5.6), choosing α_2 large enough. The proof is made of several steps:

Step 1. *Construction of the free Klein-Gordon equation v and proof of (5.7).*

Let $P(y) := \{y \in \mathbb{R}^3, E(Iw(S), B(y, 1)) \leq c_9 A^{-\alpha_{16}}\}$. Then by (5.1) there exists $\bar{x} \in \mathbb{R}^3$ such that $|\bar{x} - \bar{x}| \lesssim A^{p+1+\alpha_{16}}$ and $P(\bar{x})$ is true. Hence, using also (7.6) we see that there exists a constant C and $\Gamma \in [\frac{1}{2}, R' + CA^{p+1+\alpha_{16}}]$ such that

$$(7.9) \quad E(Iw(S), B(\bar{x}, \Gamma)) = c_9 A^{-\alpha_{16}}.$$

Let v be the solution of the free Klein-Gordon equation with data

$$\begin{cases} v(S) := I^{-1} \left(\chi \left(\frac{x - \bar{x}}{\Gamma} \right) Iw(S) \right) \\ \partial_t v(S) := I^{-1} \left(\chi \left(\frac{x - \bar{x}}{\Gamma} \right) \partial_t Iw(S) \right), \end{cases}$$

where χ is a smooth function such that $\chi(x) = 1$ if $|x| \leq 1$ and $\chi(x) = 0$ if $|x| \geq 2$. By (5.1) and (7.8) we see that there exists a constant C such that

$$(7.10) \quad \begin{aligned} E_c(Iv(S)) &\leq E(Iw(S), B(\bar{x}, \Gamma)) + \frac{C}{\Gamma^2} \int_{\Gamma \leq |x - \bar{x}| \leq 2\Gamma} |Iw(S)|^2 dx + \\ &\quad \frac{C}{\Gamma} \int_{\Gamma \leq |x - \bar{x}| \leq 2\Gamma} |Iw(S)| |\nabla Iw(S)| dx \\ &\leq c_9 A^{-\alpha_{16}} + CA^{\frac{p+1}{2}} \left\| \frac{Iw(S)}{\langle x - \bar{x} \rangle} \right\|_{L^2} + C \left\| \frac{Iw(S)}{\langle x - \bar{x} \rangle} \right\|_{L^2}^2 \\ &\lesssim A^{-\alpha_{16}}, \end{aligned}$$

Hence, using also the conservation of $E_c(Iv)$, we see that (5.7) holds.

Step 2. *Proof of the decay (5.9).*

By interpolation we see that one can choose one can choose $m < s_c$ close to s_c and $\gamma > 0$ close to zero such that

$$(7.11) \quad \begin{aligned} \|Iv\|_{L_t^{2(p-1)-} L_x^{2(p-1)+}(T,b)} &\lesssim \|\langle D \rangle^{1-m} Iv\|_{L_t^{\frac{8}{3-2m}-} L_x^{\frac{8}{3-2m}+}(T,b)}^{1-\gamma} \|\langle D \rangle^{-(\frac{3}{2}++)} Iv\|_{L_t^\infty L_x^\infty(T,b)}^\gamma \\ &\lesssim A^{-(1-\gamma)\alpha_{16}} \|Iv\|_{L_t^\infty B_{\infty,2}^{-(\frac{3}{2}+)}(T,b)}^\gamma \\ &\lesssim A^{-(1-\gamma)\alpha_{16}} \frac{1}{|T-S|^{\frac{3\gamma}{2}}} \left(\|Iv(S)\|_{B_{1,2}^1}^\gamma + \|\partial_t Iv(S)\|_{B_{1,2}^{0-}}^\gamma \right) \\ &\lesssim A^{-(1-\gamma)\alpha_{16}} \frac{1}{|T-S|^{\frac{3\gamma}{2}}} \left(\Gamma^{\frac{3\gamma}{2}} \|Iw(S)\|_{L^2}^\gamma + \Gamma^{\frac{\gamma}{2}} \|\nabla Iw(S)\|_{L^2}^\gamma + \Gamma^{\frac{3\gamma}{2}} \|\partial_t Iw(S)\|_{L^2}^\gamma \right) \\ &\leq C_2 \frac{A^{\alpha_4}}{M^{\alpha_5}}, \end{aligned}$$

using also (7.7), (7.10) combined with (3.2), and the following dispersive estimate (see [9], Lemma 2.1)

$$(7.12) \quad \|e^{it\langle D \rangle} \phi\|_{B_{\infty,2}^{-\frac{5}{4}}} \lesssim \frac{1}{|t|^{\frac{3}{2}}} \|\phi\|_{B_{1,2}^{\frac{5}{4}}}.$$

Step 3. *Proof of the separation of the localized mollified energy (5.8).*

Let $\bar{w} := w - v$. Then

$$\begin{aligned}
E(I\bar{w}(S)) &\leq \frac{1}{2} \int_{\mathbb{R}^3} \left(1 - \chi^2 \left(\frac{x - \bar{x}}{\Gamma}\right)\right) |\nabla Iw(S)|^2 dx \\
&\quad + \frac{1}{p+1} \int_{\mathbb{R}^3} \left(1 - \chi^{p+1} \left(\frac{x - \bar{x}}{\Gamma}\right)\right) |Iw(S)|^{p+1} dx \\
&\quad + \frac{1}{2} \int_{\mathbb{R}^3} \left(1 - \chi^2 \left(\frac{x - \bar{x}}{\Gamma}\right)\right) |\partial_t Iw(S)|^2 dx + \frac{1}{2} \int_{\mathbb{R}^3} \left(1 - \chi^2 \left(\frac{x - \bar{x}}{\Gamma}\right)\right) |Iw(S)|^2 dx \\
&\quad + \frac{C}{\Gamma} \int_{\Gamma \leq |x - \bar{x}| \leq 2\Gamma} |Iw(S)| |\nabla Iw(S)| dx + \frac{C}{\Gamma^2} \int_{\Gamma \leq |x - \bar{x}| \leq 2\Gamma} |Iw(S)|^2 dx \\
&\leq \sup_{t \in J} E(Iw(t)) - c_9 A^{-\alpha_{16}} + C A^{\frac{p+1}{2}} \left\| \frac{Iw(S)}{\langle x - \bar{x} \rangle} \right\|_{L^2}^2 + C \left\| \frac{Iw(S)}{\langle x - \bar{x} \rangle} \right\|_{L^2}^2 \\
(7.13) \quad &\leq \sup_{t \in J} E(Iw(t)) - \frac{c_9 A^{-\alpha_{16}}}{2}.
\end{aligned}$$

Let

$$\bar{Z}([S, T], f) := \|If\|_{L_t^{2(p-1)-} L_x^{2(p-1)+}([S, T])} + \|\partial_t \langle D \rangle^{-\frac{1}{2}} If\|_{L_t^4 L_x^4([S, T])} + \|\langle D \rangle^{\frac{1}{2}} If\|_{L_t^{4+} L_x^{4-}([S, T])}$$

Plugging $\langle D \rangle^{1-\frac{1}{2}} I$ into (3.2), we see, by (7.6) and the Sobolev embedding

$$\|If\|_{L_t^{2(p-1)-} L_x^{2(p-1)+}([S, T])} \lesssim \|\langle D \rangle^{1-\frac{1}{2}} If\|_{L_t^{2(p-1)-} L_x^{\frac{6(p-1)}{2p-3}+}([S, T])},$$

that

(7.14)

$$\begin{aligned}
\bar{Z}([S, T], \bar{w}) &\lesssim \left\| \langle D \rangle^{1-\frac{1}{2}} I(|w|^{p-1} w) \right\|_{L_t^{\frac{4}{3}} L_x^{\frac{4}{3}}([S, T])} \\
&\lesssim \|\langle D \rangle^{1-\frac{1}{2}} Iw\|_{L_t^{4+} L_x^{4-}([S, T])} \left(\|P_{<N} w\|_{L_t^{2(p-1)-} L_x^{2(p-1)+}([S, T])}^{p-1} + \|P_{\gtrsim N} w\|_{L_t^{2(p-1)-} L_x^{2(p-1)+}([S, T])}^{p-1} \right) \\
&\lesssim \|\langle D \rangle^{1-\frac{1}{2}} Iw\|_{L_t^{4+} L_x^{4-}([S, T])} \left(\|Iw\|_{L_t^{2(p-1)-} L_x^{2(p-1)+}([S, T])}^{p-1} + \frac{\|\langle D \rangle^{1-\frac{1}{2}} Iw\|_{L_t^{2(p-1)-} L_x^{\frac{6(p-1)}{2p-3}+}([S, T])}^{p-1}}{N^{\frac{5-p}{2}}} \right) \\
&\lesssim \tilde{Z}^p([S, T], w) \\
&\leq \frac{c_9 A^{-\alpha_{16}}}{1000}.
\end{aligned}$$

We compute

$$\begin{aligned}
\partial_t E(I\bar{w}) &= \int_{\mathbb{R}^3} \Re \left(\overline{\partial_t I\bar{w}} (\partial_{tt} I\bar{w} - \triangle I\bar{w} + I\bar{w} + F(I\bar{w})) \right) dx \\
&= \int_{\mathbb{R}^3} \Re \left(\overline{\partial_t I\bar{w}} (F(I\bar{w}) - IF(w)) \right) dx.
\end{aligned}$$

Now, we decompose $E(I\bar{w}(T)) - E(I\bar{w}(S)) = X_{1,1} + X_{1,2} + X_2$ where

$$\begin{aligned}
 X_{1,1} &:= \int_S^T \int_{\mathbb{R}^3} \Re(\overline{\partial_t I\bar{w}} (F(w) - IF(w))) \, dx \, dt \\
 X_{1,2} &:= \int_S^T \int_{\mathbb{R}^3} \Re(\overline{\partial_t I\bar{w}} (F(Iw) - F(w))) \, dx \, dt \\
 X_2 &:= \int_S^T \int_{\mathbb{R}^3} \Re(\overline{\partial_t I\bar{w}} (F(I\bar{w}) - F(Iw))) \, dx \, dt.
 \end{aligned}
 \tag{7.15}$$

We estimate X_2 . From (7.6) and (7.14)

$$\begin{aligned}
 |X_2| &\lesssim \|\partial_t \langle D \rangle^{-\frac{1}{2}} I\bar{w}\|_{L_t^4 L_x^4([S,T])} \|\langle D \rangle^{\frac{1}{2}} (F(I\bar{w}) - F(Iw))\|_{L_t^{\frac{4}{3}} L_x^{\frac{4}{3}}([S,T])} \\
 &\lesssim \|\partial_t \langle D \rangle^{-\frac{1}{2}} I\bar{w}\|_{L_t^4 L_x^4([S,T])} \left(\|Iw\|_{L_t^{2(p-1)-} L_x^{2(p-1)+}([S,T])}^{p-1} + \|I\bar{w}\|_{L_t^{2(p-1)-} L_x^{2(p-1)+}([S,T])}^{p-1} \right) \\
 &\quad \left(\|\langle D \rangle^{1-\frac{1}{2}} Iw\|_{L_t^{4+} L_x^{4-}([S,T])} + \|\langle D \rangle^{\frac{1}{2}} I\bar{w}\|_{L_t^{4+} L_x^{4-}([S,T])} \right) \\
 &\leq \frac{c_9 A^{-\alpha_{16}}}{1000}.
 \end{aligned}$$

Hence, using also (7.13) and Result 10.1 (see Appendix A) we see that (5.8) holds, with w' solution of (1.1) such that $w'(T) := \bar{w}(T)$.

7.3. Proof of Lemma 7.1.

The proof is made of four steps:

Step 1. *Lower bound of the size of \bar{J}' .*

We see from Proposition 5.1 that if $p \geq 4$, then

$$\begin{aligned}
 c_2 &= \|Iw\|_{L_t^{2(p-1)-} L_x^{2(p-1)+}(\bar{J}')} \\
 &\lesssim |\bar{J}'|^{\frac{5-p}{2(p-1)+}} \|\langle D \rangle^{1-\frac{1}{2}} Iw\|_{L_t^{\frac{2(p-1)}{p-4}} L_x^{\frac{6(p-1)}{p+2}}(\bar{J}')} \\
 &\lesssim |\bar{J}'|^{\frac{5-p}{2(p-1)+}} A^{\frac{p+1}{2}},
 \end{aligned}$$

and if $p < 4$, then

$$\begin{aligned}
 c_2 &= \|Iw\|_{L_t^{2(p-1)-} L_x^{2(p-1)+}(\bar{J}')} \\
 &\lesssim |\bar{J}'|^{\frac{1}{2(p-1)+}} \|\langle D \rangle^{1-\frac{1}{2}} Iw\|_{L_t^\infty L_x^3(\bar{J}')} \\
 &\lesssim |\bar{J}'|^{\frac{1}{2(p-1)+}} A^{\frac{p+1}{2}}.
 \end{aligned}$$

Therefore, we conclude that there exists a constant c such that

$$|\bar{J}'| \geq c \times \begin{cases} A^{-\frac{(p+1)(p-1)}{5-p}+}, & p \geq 4 \\ A^{-(p+1)(p-1)+}, & p < 4 \end{cases}
 \tag{7.16}$$

Step 2. *Lower bound of $\|P_M w\|_{L_t^\infty L_x^\infty(\bar{J}')}$ for some $M \in 2^{\mathbb{N}}$*

From Proposition 5.1

$$\begin{aligned}
c_2 &= \|Iw\|_{L_t^{2(p-1)-} L_x^{2(p-1)+}(\bar{J}')} \\
&\lesssim \|\langle D \rangle^{1-(\frac{1}{2}+)} Iw\|_{L_t^{4-\frac{2}{p-1}} L_x^{4+}(\bar{J}')} \|\langle D \rangle^{(-\frac{1}{p-3})+} Iw\|_{L_t^\infty L_x^\infty(\bar{J}')}^{1-\frac{2}{p-1}} \\
&\lesssim A^{\frac{p+1}{p-1}} \|\langle D \rangle^{(-\frac{1}{p-3})+} Iw\|_{L_t^\infty L_x^\infty(\bar{J}')}^{1-\frac{2}{p-1}}.
\end{aligned}$$

Thus we have $\|\langle D \rangle^{(-\frac{1}{p-3})+} Iw\|_{L_t^\infty L_x^\infty(\bar{J}')} \gtrsim A^{-\frac{p+1}{p-3}}$ and, by the pigeonhole principle, we conclude that there exists $M \in 2^{\mathbb{N}}$ such that

$$(7.17) \quad \|P_M Iw\|_{L_t^\infty L_x^\infty(\bar{J}')} \gtrsim \langle M \rangle^{\frac{1}{p-3}} A^{-\frac{p+1}{p-3}}.$$

On the other hand,

$$(7.18) \quad \|P_M Iw\|_{L_t^\infty L_x^\infty(\bar{J}')} \lesssim \langle M \rangle^{\frac{1}{2}} \|\langle D \rangle Iw\|_{L_t^\infty L_x^2(\bar{J}')} \lesssim \langle M \rangle^{\frac{1}{2}} A^{\frac{p+1}{2}}.$$

Combining (7.17) and (7.18), we see that

$$(7.19) \quad \langle M \rangle \lesssim A^{\frac{(p+1)(p-1)}{5-p}}.$$

Step 3. *Control of $|P_M Iw(t, \bar{x}')|$ for some $\bar{x}' \in \mathbb{R}^3$ and for all $t \in \bar{K}'$, $\bar{K}' \subset \bar{J}'$ to be defined shortly.*

By (7.17), there exists (\bar{t}', \bar{x}') such that

$$(7.20) \quad |P_M Iw(\bar{t}', \bar{x}')| \gtrsim \langle M \rangle^{\frac{1}{p-3}} A^{-\frac{p+1}{p-1}}.$$

But, by (5.1) and (7.19), we see that

$$\begin{aligned}
|P_M Iw(t, \bar{x}') - P_M Iw(\bar{t}', \bar{x}')| &\lesssim \sup_{s \in (\bar{t}', t)} \|\partial_s Iw(s)\|_{L_x^2} \langle M \rangle^{\frac{3}{2}} |t - \bar{t}'| \\
(7.21) \quad &\lesssim A^{\frac{(p+1)^2}{5-p}} |t - \bar{t}'|,
\end{aligned}$$

Therefore, in view of (7.16), (7.20) and (7.21), choosing c_7 (resp. α_{12}) small enough (resp. large enough), we see that either $[\bar{t}', \bar{t}' + c_7 A^{-\alpha_{12}}] \subset \bar{J}'$ (in this case we let $\bar{K}' := [\bar{t}', \bar{t}' + c_7 A^{-\alpha_{12}}]$), or $[\bar{t}' - c_7 A^{-\alpha_{12}}, \bar{t}'] \subset \bar{J}'$ (in this case we let $\bar{K}' := [\bar{t}' - c_7 A^{-\alpha_{12}}, \bar{t}']$) and

$$(7.22) \quad |P_M Iw(t, \bar{x}')| \gtrsim A^{-\frac{p+1}{p-1}}, t \in \bar{K}'$$

Step 4. *Lower bound of potential mollified energy.*

Let $R > 0$ to be fixed shortly. Let $\Psi := \psi$ if $M > 1$ and $\Psi := \phi$ if $M = 1$. By (7.22) we have

$$M^3(B_1 + B_2) \gtrsim A^{-\frac{p+1}{p-1}}$$

where $B_1 := \int_{|y| \leq R} |\check{\Psi}(My)| |Iw(t, \bar{x}' - y)| dy$ and $B_2 := \int_{|y| \geq R} |\check{\Psi}(My)| |Iw(t, \bar{x}' - y)| dy$. We have

$$\begin{aligned}
B_1 &\lesssim \left(\int_{|y| \leq R} |\check{\Psi}(My)|^{\frac{p+1}{p}} dy \right)^{\frac{p}{p+1}} \left(\int_{|y| \leq R} |Iw(t, \bar{x}' - y)|^{p+1} dy \right)^{\frac{1}{p+1}} \\
&\lesssim \left(\int_{|y - \bar{x}'| \leq R} |Iw(t, y)|^{p+1} dy \right)^{\frac{1}{p+1}} M^{-\frac{3p}{p+1}}
\end{aligned}$$

and

$$\begin{aligned} B_2 &\lesssim M^{-\frac{3}{2}} \|\check{\Psi}\|_{L^2(|y|\geq MR)} \|Iw(t)\|_{L_x^2} \\ &\lesssim M^{-\frac{3}{2}} \|\check{\Psi}\|_{L^2(|y|\geq MR)} A^{\frac{p+1}{2}}. \end{aligned}$$

The fast decay of $\check{\Psi}$ implies

$$\|\check{\Psi}\|_{L^2(|y|\geq MR)} \lesssim \frac{1}{(MR)^{\frac{3}{2}}}.$$

Hence, if $R := C_5 A^{\alpha_{11}}$ (with C_5 and α_{11} large enough), then

$$\int_{|y-\bar{x}'|\leq R} |Iw(t, y)|^{p+1} dy \geq c_8 A^{-\alpha_{13}},$$

for all $t \in \bar{K}'$.

7.4. Proof of Lemma 7.2.

Let $j \in [1, \bar{j} - 1]$. Recall that, by Lemma 7.1, on each \bar{J}'_j , there exist $\bar{x}'_j \in \mathbb{R}^3$ and $\bar{K}'_j = [\bar{t}'_j, \bar{t}'_{j+1}] \subset \bar{J}'_j$ such that

$$(7.23) \quad \int_{|x-\bar{x}'_j|\leq R} |Iw(t, x)|^{p+1} dx \geq c_8 A^{-\alpha_{13}}$$

for all $t \in \bar{K}'_j$, with $R = C_5 A^{\alpha_{11}}$ and

$$(7.24) \quad |\bar{K}'_j| = c_7 A^{-\alpha_{12}}.$$

We construct (see [18]) a set $\mathcal{P} := \{j_1, \dots, j_l\} \subset \{1, \dots, \bar{j} - 1\}$. Initially $j_1 = 1$. Then let j_{k+1} be the minimal j such that

$$(7.25) \quad |\bar{x}'_j - \bar{x}'_{j_{k+1}}| \geq |\bar{t}'_j - \bar{t}'_{j_{k+1}}| + 100R$$

for $j := j_1, \dots, j_k$. Observe that $J' = \bigcup_{j_k \in \mathcal{P}} A'_{j_k}$ with

$$A'_{j_k} := \left\{ \bar{J}'_l, \bar{j} - 1 \geq l \geq j_k, \text{ and } |\bar{x}'_{j_k} - \bar{x}'_l| < |\bar{t}'_{j_k} - \bar{t}'_l| + 100R \right\}.$$

From Result 10.2 and the estimates above

$$\begin{aligned} A^{p+1} \text{card } \mathcal{P} &\gtrsim \sum_{j_k \in \mathcal{P}} \int_{J'} \int_{\mathbb{R}^3} \frac{|Iw(t, x)|^{p+1}}{|x - \bar{x}'_{j_k}|} dx dt \\ &\gtrsim A^{-\alpha_{11}} \sum_{j_k \in \mathcal{P}} \int_{J'} \int_{|x-\bar{x}'_{j_k}| \leq |t-\bar{t}'_{j_k}|+1000R} \frac{|Iw(t, x)|^{p+1}}{1 + |t - \bar{t}'_{j_k}|} dx dt \\ &\gtrsim A^{-\alpha_{11}-\alpha_{13}} \sum_{j_k \in \mathcal{P}} \sum_{l \in A'_{j_k}} \int_{K'_l} \frac{1}{1+t} dt \\ &\geq c A^{-\alpha_{11}-\alpha_{12}-\alpha_{13}} \sum_{j=1}^{\bar{j}-1} \frac{1}{1+\bar{t}'_j}, \end{aligned}$$

where we used at the second line the elementary inequality

$$\frac{1}{1000R + |t - \bar{t}'_{j_k}|} \gtrsim \frac{1}{R} \frac{1}{|t - \bar{t}'_{j_k}| + 1}.$$

Then it suffices to estimate $\text{card } \mathcal{P}$. Let $j_{k_{max}} := \max_{j_k \in \mathcal{P}} j_k$. By applying - $\text{card } \mathcal{P}$ - times Result 10.3, by the construction of \mathcal{P} and by (7.23)

$$\begin{aligned} A^{p+1} &\gtrsim E \left(Iw(\vec{t}_{j_{k_{max}}}), \bigcup_{j_k \in \mathcal{P}} B(\vec{x}'_{j_k}, R + |\vec{t}_{j_{k_{max}}} - \vec{t}_{j_k}|) \right) \\ &\geq \sum_{j_k \in \mathcal{P}} E \left(Iw(\vec{t}_{j_k}), B(\vec{x}'_{j_k}, R) \right) - \frac{c_8 A^{-\alpha_{13}} \text{card}(\mathcal{P})}{1000} \\ &\gtrsim \text{card}(\mathcal{P}) A^{-\alpha_{13}}. \end{aligned}$$

Hence, (7.4) follows.

7.5. Proof of Lemma 7.3.

Partitionning J' into the subintervals $(\vec{J}'_j)_{1 \leq j \leq \bar{j}}$ that were defined in Lemma 7.2, we see by (5.1) and Proposition 5.1 that

$$(7.26) \quad Z(\vec{J}'_j, w) \lesssim A^{\frac{p+1}{2}}.$$

By (7.3), there exists $(\vec{t}'_j, \vec{x}'_j) \in \vec{J}'_j \times \mathbb{R}^3$ such that

$$(7.27) \quad E \left(Iw(\vec{t}'_j), B(\vec{x}'_j, R) \right) \geq \frac{c_8 A^{-\alpha_{13}}}{p+1}$$

Hence, from Result 10.3 (see Appendix A), we see that there exist two constants c_9 and α_{16} such that

$$E \left(Iw(t), B(\vec{x}'_j, R + |t - \vec{t}'_j|) \right) \geq c_9 A^{-\alpha_{16}}, \quad t \in \vec{J}'_j.$$

We further chop out each subinterval \vec{J}'_j into smaller subintervals $(\vec{J}'_{j,k} = [\vec{t}'_{j,k}, \vec{t}'_{j,k+1}])_{k \in \mathbb{Z}}$ with $\vec{t}'_{j,0} := \vec{t}'_j$ and such that $\tilde{Z}(\vec{J}'_{j,k}, w) \leq c_9 A^{-\alpha_{16}}$ while $\tilde{Z}(\vec{J}'_{j,k}, w) \sim A^{-\alpha_{16}}$, except maybe the last interval. Notice that by (7.26), there exists α such that

$$(7.28) \quad \forall j : \text{card} \left((\vec{J}'_{j,k})_{k \in \mathbb{Z}} \right) \lesssim A^\alpha.$$

From Result 10.2 we see that given $\epsilon > 0$, there exist $C > 0$ and $\tilde{t}'_{j,k}$ such that

$$\begin{aligned} k \geq 0 : \quad & \tilde{t}'_{j,k} \in [\vec{t}'_{j,k}, \vec{t}'_{j,k} + C \langle \vec{t}'_{j,k} - \vec{t}'_j \rangle] \\ k < 0 : \quad & \tilde{t}'_{j,k} \in [\vec{t}'_{j,k+1} - C \langle \vec{t}'_{j,k+1} - \vec{t}'_j \rangle, \vec{t}'_{j,k+1}], \\ (7.29) \quad & \log(C) \lesssim A^{\frac{(p+1)^2}{2}} \epsilon^{-p-1}, \quad \left\| \frac{Iw(\tilde{t}'_{j,k})}{\langle x - \vec{x}'_j \rangle} \right\|_{L^2} \leq \epsilon. \end{aligned}$$

Next, we claim that there exists (k_0, j_0) and $M' := 1000 \langle C \rangle M$ such that

$$(7.30) \quad \begin{aligned} k_0 \geq 0 : \quad & |\vec{t}'_{j_0, k_0+1} - \vec{t}'_{j_0, k_0}| \geq M' \left(R + |\vec{t}'_{j_0, k_0} - \vec{t}'_{j_0}| \right), \\ k_0 < 0 : \quad & |\vec{t}'_{j_0, k_0+1} - \vec{t}'_{j_0, k_0}| \geq M' \left(R + |\vec{t}'_{j_0, k_0+1} - \vec{t}'_{j_0}| \right). \end{aligned}$$

If not, this implies, by simple induction on k , that (say) $|\vec{t}'_{j,k} - \vec{t}'_j| \lesssim (2M')^{|k|+1} R$ for all j and therefore, by (7.28), we see that there exist two positive constants α and C' such that $|\vec{J}'_j| \leq (M')^{C' A^\alpha}$ for all j ⁸. But, by Lemma 7.2, this imply that

⁸We allow the value of α and C' to increase in the sequel so that all the estimates hold down to the end of the section.

$$\frac{\log \left(1 + (\bar{j} - 1)(M')^{C'A^\alpha} \right)}{(M')^{C'A^\alpha}} \leq \sum_{j=1}^{\bar{j}-1} \frac{1}{1 + j(M')^{C'A^\alpha}} \lesssim A^{\alpha_{14}}.$$

Therefore $\log(\bar{j}) \leq (M')^{C'A^\alpha}$ and, combining this inequality with (7.29), this yields a contradiction with (7.5).

Assume that $k_0 \geq 0$. Then

$$\begin{aligned} \tilde{t}'_{j_0, k_0+1} - \tilde{t}'_{j_0, k_0} &\geq \tilde{t}'_{j_0, k_0+1} - \tilde{t}'_{j_0, k_0} - C \langle \tilde{t}'_{j_0, k_0} - \tilde{t}'_{j_0} \rangle \\ &\geq M' \left(R + |\tilde{t}'_{j_0, k_0} - \tilde{t}'_{j_0}| \right) - C \langle \tilde{t}'_{j_0, k_0} - \tilde{t}'_{j_0} \rangle \\ &\geq M \left(R + |\tilde{t}'_{j_0, k_0} - \tilde{t}'_{j_0}| \right) \end{aligned}$$

Hence, choosing $\bar{x} := \tilde{x}'_{j_0}$, $R' := R + |\tilde{t}'_{j_0, k_0} - \tilde{t}'_{j_0}|$, $S := \tilde{t}'_{j_0, k_0}$ and $T := \tilde{t}'_{j_0, k_0+1}$, we have (7.6), (7.7) and (7.8). The reader is invited to check that if $k_0 < 0$, then a similar argument shows that the same estimates hold if $R' := R + |\tilde{t}'_{j_0, k_0+1} - \tilde{t}'_{j_0}|$, $S := \tilde{t}'_{j_0, k_0+1}$ and $T := \tilde{t}'_{j_0, k_0}$.

8. PROOF OF PERTURBATION ARGUMENT

In this section we prove Proposition 5.3. We may assume without loss of generality that $J'' = (T, b)$. The proof is made of several steps:

Step 1 . *Bound of $Z([T, b], w)$ and $Z([T, b], w')$.*

We divide $[T, b]$ into subintervals K such that $\|Iw\|_{L_t^{2(p-1)-} L_x^{2(p-1)+}(K)} = c_2$, except maybe the last one. Proposition 5.1 yields $Z(K, w) \lesssim A^{\frac{p+1}{2}}$. Iterating over K and using (5.2)

$$(8.1) \quad Z([T, b], w) \lesssim \langle L_w \rangle^{2(p-1)-} A^{\frac{p+1}{2}}.$$

A bound of $Z(w', [T, b])$ is obtained similarly as above:

$$(8.2) \quad Z([T, b], w') \lesssim \langle L_{w'} \rangle^{2(p-1)-} A^{\frac{p+1}{2}}.$$

Step 2. *Decomposition.*

Let $\Gamma := w - w' - v$ and $K' := [t', t''] \subset [T, b]$. A simple computation shows that

$$\partial_{tt} I\Gamma - \triangle I\Gamma + I\Gamma = (IF(w') - F(Iw')) + (F(Iw) - IF(w)) + (F(Iw') - F(Iw)).$$

We decompose

$$I\Gamma(t) = I\Gamma_l^{K'}(t) + X_{F(Iw')-IF(w')}^{K'}(t) + X_{IF(w)-F(Iw)}^{K'}(t) + X_{F(Iw)-F(Iw')}^{K'}(t),$$

where the X numbers are defined in Section 2 and

$$\Gamma_l^{K'}(t) := \cos((t - t') \langle D \rangle) \Gamma(t') + \frac{\sin((t - t') \langle D \rangle)}{\langle D \rangle} \partial_t \Gamma(t').$$

We use the following notation: Given a function f , let

$$(8.3) \quad \bar{Z}(K', f) := \sup_{(q, r) - \frac{1}{2} \text{ wave adm}} \left\| \langle D \rangle^{s_c - \frac{1}{2}} I f \right\|_{L_t^q L_x^r(K')}.$$

Step 3. *Short-time perturbation argument.*

Lemma 8.1. *There exist $0 < \theta := \theta(p) < 1$ and c_{10} such that if $c \leq c_{10}$,*

$$(8.4) \quad \|Iw'\|_{L_t^{2(p-1)-} L_x^{2(p-1)+}(K')}, \|\langle D \rangle^{s_c - \frac{1}{2}} Iw'\|_{L_t^{4+} L_x^{4-}(K')} \leq c,$$

$$(8.5) \quad \bar{Z}(K', \Gamma_l^{K'}) \leq c, \text{ and}$$

$$(8.6) \quad \|Iv\|_{L_t^{2(p-1)-} L_x^{2(p-1)+}(K')} \leq c,$$

then

$$(8.7) \quad \begin{aligned} & \bar{Z}(K', I^{-1} X_{F(Iw')-IF(w')}^{K'}) + \bar{Z}(K', I^{-1} X_{IF(w)-F(Iw)}^{K'}) + \bar{Z}(K', I^{-1} X_{F(Iw)-F(Iw')}^{K'}) \\ & \lesssim \max \left(\frac{\max^{2p(p-1)-}(\langle L_w \rangle, \langle L_{w'} \rangle) A^{\frac{p(p+1)}{2}}}{N^{(1-s_c)-}}, c \right). \end{aligned}$$

Proof. We have

$$\bar{Z}(K', \Gamma) \leq \bar{Z}(K', \Gamma_l^{K'}) + \bar{Z}(K', I^{-1} X_{F(Iw')-IF(w')}^{K'}) + \bar{Z}(K', I^{-1} X_{IF(w)-F(Iw)}^{K'}) + \bar{Z}(K', I^{-1} X_{F(Iw)-F(Iw')}^{K'}).$$

We first estimate $\bar{Z}(K', I^{-1} X_{F(Iw)-F(Iw')}^{K'})$.

By interpolation (see points A , B , and C on Figure 1) there exist $\theta := \theta(p) > 0$, $m < \frac{1}{2}$, and (\bar{q}, \bar{r}) m -wave admissible such that

$$(8.8) \quad \|\langle D \rangle^{\frac{1}{2}} Iv\|_{L_t^{4+} L_x^{4-}(K')} \lesssim \|\langle D \rangle^{1-s_c} Iv\|_{L_t^{2(p-1)-} L_x^{2(p-1)+}(K')}^{\theta} \|\langle D \rangle^{1-m} Iv\|_{L_t^{\bar{q}} L_x^{\bar{r}}(K')}^{1-\theta}$$

Notice that, in view of (5.7) and (3.2), we have $\|\langle D \rangle^{1-m} Iv\|_{L_t^{\bar{q}} L_x^{\bar{r}}(K')} \lesssim \|\langle D \rangle Iv(t')\|_{L^2} \lesssim 1$. Hence

$$\|\langle D \rangle^{s_c - \frac{1}{2}} Iv\|_{L_t^{4+} L_x^{4-}(K')} \lesssim \|Iv\|_{L_t^{2(p-1)-} L_x^{2(p-1)+}(K')}^{\theta} \lesssim c^{\theta}.$$

Therefore, using again (3.2)

$$\begin{aligned} & \|\langle D \rangle^{s_c - \frac{1}{2}} X_{F(Iw)-F(Iw')}^{K'}\|_{L_t^q L_x^r(K')} \\ & \lesssim \|\langle D \rangle^{s_c - \frac{1}{2}} (F(Iw) - F(Iw'))\|_{L_t^{\frac{4}{3}} L_x^{\frac{4}{3}}(K')} \\ & \lesssim \left(\|Iw\|_{L_t^{2(p-1)-} L_x^{2(p-1)+}(K')}^{p-1} + \|Iw'\|_{L_t^{2(p-1)-} L_x^{2(p-1)+}(K')}^{p-1} \right) \|\langle D \rangle^{s_c - \frac{1}{2}} I(w - w')\|_{L_t^{4+} L_x^{4-}(K')} \\ & + \left(\|Iw\|_{L_t^{2(p-1)-} L_x^{2(p-1)+}(K')}^{p-2} + \|Iw'\|_{L_t^{2(p-1)-} L_x^{2(p-1)+}(K')}^{p-2} \right) \left(\|\langle D \rangle^{s_c - \frac{1}{2}} Iw\|_{L_t^{4+} L_x^{4-}(K')} + \|\langle D \rangle^{s_c - \frac{1}{2}} Iw'\|_{L_t^{4+} L_x^{4-}(K')} \right) \\ & \|I(w - w')\|_{L_t^{2(p-1)-} L_x^{2(p-1)+}(K')} \\ & \lesssim \left(\|Iw\|_{L_t^{2(p-1)-} L_x^{2(p-1)+}(K')}^{p-1} + \|Iw'\|_{L_t^{2(p-1)-} L_x^{2(p-1)+}(K')}^{p-1} \right) \times \left(\|\langle D \rangle^{s_c - \frac{1}{2}} Iv\|_{L_t^{4+} L_x^{4-}(K')} + \|\langle D \rangle^{s_c - \frac{1}{2}} I\Gamma\|_{L_t^{4+} L_x^{4-}(K')} \right) \\ & + \left(\|Iw\|_{L_t^{2(p-1)-} L_x^{2(p-1)+}(K')}^{p-2} + \|Iw'\|_{L_t^{2(p-1)-} L_x^{2(p-1)+}(K')}^{p-2} \right) \left(\|\langle D \rangle^{s_c - \frac{1}{2}} Iw\|_{L_t^{4+} L_x^{4-}(K')} + \|\langle D \rangle^{s_c - \frac{1}{2}} Iw'\|_{L_t^{4+} L_x^{4-}(K')} \right) \\ & \left(\|Iv\|_{L_t^{2(p-1)-} L_x^{2(p-1)+}(K')} + \|I\Gamma\|_{L_t^{2(p-1)-} L_x^{2(p-1)+}(K')} \right) \\ & \lesssim (\bar{Z}^{p-1}(K', \Gamma) + c^{p-1}) (c^{\theta} + \bar{Z}(K', \Gamma)) + (\bar{Z}^{p-2}(K', \Gamma) + c^{p-2}) (\bar{Z}(K', \Gamma) + c^{\theta}) (\bar{Z}(K', \Gamma) + c), \end{aligned}$$

substituting w for $w' + \Gamma + v$ and using the Sobolev embedding at the last line, i.e

$$(8.9) \quad \|I\Gamma\|_{L_t^{2(p-1)-} L_x^{2(p-1)+}(K')} \lesssim \|\langle D \rangle^{s_c - \frac{1}{2}} I\Gamma\|_{L_t^{2(p-1)-} L_x^{\frac{6(p-1)}{2p-3}+}(K')} \lesssim \bar{Z}(K', \Gamma).$$

Hence, collecting all these estimates, and using Result 10.4 (see Appendix A), we get (8.7) from a continuity argument. \square

Step 4. *Long-time perturbation argument.*

We divide $[T, b]$ into subintervals $(K'_q = [t'_q, t'_{q+1}])_{1 \leq q \leq Q}$ such that $\|Iw'\|_{L_t^{2(p-1)-} L_x^{2(p-1)+}(K'_q)} = c$ or $\|\langle D \rangle^{s_c - \frac{1}{2}} Iw'\|_{L_t^{4+} L_x^{4-}(K'_q)} = c$, while $\max(\|Iw'\|_{L_t^{2(p-1)-} L_x^{2(p-1)+}(K'_q)}, \|\langle D \rangle^{s_c - \frac{1}{2}} Iw'\|_{L_t^{4+} L_x^{4-}(K'_q)}) \leq c$, except maybe the last one. We can use the short-time perturbation argument on $K' := J'_q$ as long as (8.5) and (8.6) hold (with $c \leq c_{10}$). But since $\Gamma_l(T) = 0$ and

$$\bar{Z}(K'_q, \Gamma_l^{K'_q}) \lesssim \bar{Z}(K'_q, \Gamma_l^{[T, b]}) + \sum_{q=1}^{q'-1} \bar{Z}(K'_q, I^{-1} X_{F(Iw')-IF(w')}^{K'_q}) + \bar{Z}(K'_q, I^{-1} X_{IF(w)-F(Iw)}^{K'_q}) + \bar{Z}(K'_q, I^{-1} X_{F(Iw)-F(Iw')}^{K'_q})$$

we easily see, by iteration, that there exists a positive constant C such that

$$\bar{Z}(K'_q, \Gamma_l^{K'_q}) \leq C^{q'} \max \left(k, \frac{A^{\frac{p(p+1)}{2}} \max^{2p(p-1)-}(\langle L_w \rangle, \langle L_{w'} \rangle)}{N^{(1-s_c)-}} \right).$$

Hence we see from (8.2), (5.10) and (5.11), we see that (8.5) and (8.6) hold, by choosing $C_3, C_4, \alpha_6, \alpha_7$ (resp. c_4) large enough (resp. small enough).

By summation over q' and (8.9) (with K' substituted for $[T, b]$), we see that there exists a positive constant α such that $\|I\Gamma\|_{L_t^{2(p-1)-} L_x^{2(p-1)+}([T, b])} \lesssim (\langle L_{w'} \rangle A)^\alpha$. Hence (5.12) holds.

9. PROOF OF SMALL MOLLIFIED ENERGY THEORY

In this section we prove Proposition 5.4. The proof is made of two steps:

Control of $\|Iw\|_{L_t^{2(p-1)-} L_x^{2(p-1)+}(\mathbb{R})}$.

By (5.13) and (5.24) we realize that

$$\begin{aligned} Z_{s_c, s}(\mathbb{R}, w) &\lesssim E^{\frac{1}{2}}(Iw(\tilde{t})) + \|\langle D \rangle^{1-s_c} Iw\|_{L_t^{\frac{2}{s_c} + L_x^{\frac{2}{1-s_c}}(\mathbb{R})}} \times \left(\frac{\|Iw\|_{L_t^{2(p-1)-} L_x^{2(p-1)+}(\mathbb{R})}^{p-1}}{\|\langle D \rangle^{1-s_c} Iw\|_{L_t^{2(p-1)-} L_x^{2(p-1)+}(\mathbb{R})}^{p-1}} \right. \\ &\quad \left. + \frac{1}{N^{\frac{5-p}{2}}} \right) \\ &\lesssim E^{\frac{1}{2}}(Iw(\tilde{t})) + \left(Z_{s_c, s}^p(\mathbb{R}, w) + \frac{Z_{s_c, s}^p(\mathbb{R}, w)}{N^{\frac{5-p}{2}}} \right), \end{aligned}$$

where we used the Sobolev embedding, that is

$$\|Iw\|_{L_t^{2(p-1)-} L_x^{2(p-1)+}(\mathbb{R})} \lesssim \|\langle D \rangle^{1-s_c} Iw\|_{L_t^{2(p-1)-} L_x^{\frac{6(p-1)}{2p-3}+}(\mathbb{R})}.$$

Therefore we see by a continuity argument that $Z_{s_c, s}(\mathbb{R}, w) \lesssim E^{\frac{1}{2}}(Iw(\tilde{t}))$ and, consequently, (5.15) holds.

Control of $\sup_{t \in \mathbb{R}} E(Iw(t))$.

Let $T > 0$. From (5.15) one may divide $[\tilde{t}, \tilde{t} + T]$ and $[\tilde{t} - T, \tilde{t}]$ into subintervals J such that $\|Iw\|_{L_t^{2(p-1)-} L_x^{2(p-1)+}(J)} = c_2$, except maybe the last one. By choosing c_5 (resp. α_9) small enough

(resp. large enough) in (5.14), we see that we may apply Proposition 5.1 and Proposition 3.3 on each J and we get after iteration that

$$|E(Iw(\tilde{t} + T)) - E(Iw(\tilde{t}))|, |E(Iw(\tilde{t} - T)) - E(Iw(\tilde{t}))| \lesssim \frac{E^{\frac{p+1}{2}}(Iw(\tilde{t}))}{N^{\frac{5-p}{2}}},$$

assuming that $\sup_{t \in [\tilde{t}-T, \tilde{t}+T]} E(Iw(t)) \lesssim E(Iw(\tilde{t}))$. But, since again N satisfies (5.14) with c_5 (resp. α_9) small enough (resp. large enough), we see that not only this estimate holds but also (5.16).

10. APPENDIX A: ESTIMATES INVOLVING COMMUTATORS

10.1. Estimates involving commutators in Section 7. We prove all the estimates involving commutators that appear in Section 7.

10.1.1. Result 10.1.

Result 10.1. *Let $X_{1,1}$, $X_{1,2}$ be defined in (7.15). Then*

$$|X_{1,1}|, |X_{1,2}| \leq \frac{c_9 A^{-\alpha_{16}}}{1000}$$

Proof. In view of (5.1) and (7.10)

$$(10.1) \quad \|\partial_t I\bar{w}\|_{L_t^\infty L_x^2([S,T])} \lesssim A^{\frac{p+1}{2}}.$$

We first estimate $X_{1,2}$. From (5.3), (7.6), (10.1), and

$$|F(w) - F(Iw)| \lesssim \max(|Iw|^{p-1}, |w|^{p-1})|Iw - w|,$$

$$\begin{aligned} |X_{1,2}| &\lesssim \|\partial_t I\bar{w}\|_{L_t^\infty L_x^2([S,T])} \times \left(\|P_{<N} w\|_{L_t^{\frac{4(p-1)}{7-p}} L_x^{\frac{4(p-1)}{p-3}}([S,T])}^{p-1} \|P_{\gtrsim N} w\|_{L_t^{\frac{4}{p-3}} L_x^{\frac{4}{5-p}}([S,T])} \right. \\ &\quad \left. + \|P_{\gtrsim N} w\|_{L_t^p L_x^{2p}([S,T])}^{p-1} \|P_{\gtrsim N} w\|_{L_t^p L_x^{2p}([S,T])} \right) \\ &\lesssim \frac{1}{N^{\frac{5-p}{2}}} \|\partial_t I\bar{w}\|_{L_t^\infty L_x^2([S,T])} \times \left(\begin{aligned} &\|\langle D \rangle^{1-1} Iw\|_{L_t^{\frac{4(p-1)}{7-p}} L_x^{\frac{4(p-1)}{p-3}}([S,T])}^{p-1} \\ &\times \|\langle D \rangle^{1-\frac{p-3}{2}} Iw\|_{L_t^{\frac{4}{p-3}} L_x^{\frac{4}{5-p}}([S,T])} \\ &+ \|\langle D \rangle^{1-\frac{3p-5}{2p}} Iw\|_{L_t^p L_x^{2p}([S,T])}^p \end{aligned} \right) \\ &\leq \frac{c_9 A^{-\alpha_{16}}}{1000}, \end{aligned}$$

where at the last line we choose α_1 (resp. c_1) large enough (resp. small enough) in (5.3).

We turn to $X_{1,1}$. We use an argument in [25]: For low frequencies we use the smoothness of F (F is C^1) and for high frequencies, we use the regularity of w (in H^s). Indeed, we have

$$\begin{aligned} F(w) &:= F(P_{\ll N} w + P_{\gtrsim N} w) \\ &= F(P_{\ll N} w) + \left(\int_0^1 |P_{\ll N} w + y P_{\gtrsim N} w|^{p-1} dy \right) P_{\gtrsim N} w \\ (10.2) \quad &+ \left(\int_0^1 \frac{P_{\ll N} w + y P_{\gtrsim N} w}{\overline{P_{\ll N} w + y P_{\gtrsim N} w}} |P_{\ll N} w + y P_{\gtrsim N} w|^{p-1} dy \right) \overline{P_{\gtrsim N} w}. \end{aligned}$$

Therefore, we estimate

$$\begin{aligned} |X_{1,1}| &\lesssim \|\partial_t I\bar{w}\|_{L_t^\infty L_x^2([S,T])} (X_{1,1,1} + X_{1,1,2} + X_{1,1,3}) \\ &\lesssim A^{\frac{p+1}{2}} (X_{1,1,1} + X_{1,1,2} + X_{1,1,3}) \end{aligned}$$

with

$$\begin{aligned} X_{1,1,1} &:= \|P_{\gtrsim N} F(P_{\ll N} w)\|_{L_t^1 L_x^2([S,T])} \\ X_{1,1,2} &:= \| |P_{\ll N} w|^{p-1} P_{\gtrsim N} w \|_{L_t^1 L_x^2([S,T])} \\ X_{1,1,3} &:= \|P_{\gtrsim N} w\|_{L_t^p L_x^{2p}([S,T])}^p. \end{aligned}$$

We further estimate

$$\begin{aligned} X_{1,1,1} &\lesssim \frac{1}{N^{1-}} \|\nabla F(P_{\ll N} w)\|_{L_t^1 L_x^2([S,T])} \\ &\lesssim \frac{1}{N^{1-}} \|P_{\ll N} w\|_{L_t^{\frac{4(p-1)}{7-p}} L_x^{\frac{4(p-1)}{p-3}}([S,T])}^{p-1} \|\nabla P_{\ll N} w\|_{L_t^{\frac{4}{p-3}} L_x^{\frac{4}{5-p}}([S,T])} \\ &\lesssim \frac{1}{N^{\frac{5-p}{2}-}} \|\langle D \rangle^{1-1} Iw\|_{L_t^{\frac{4(p-1)}{7-p}} L_x^{\frac{4(p-1)}{p-3}}([S,T])}^{p-1} \|\langle D \rangle^{1-\frac{p-3}{2}} Iw\|_{L_t^{\frac{4}{p-3}} L_x^{\frac{4}{5-p}}([S,T])} \\ &\lesssim \frac{A^{-p\alpha_{16}}}{N^{\frac{5-p}{2}-}}, \\ X_{1,1,2} &\lesssim \| |P_{\ll N} w|^{p-1} P_{\gtrsim N} w \|_{L_t^1 L_x^2([S,T])} \\ &\lesssim \frac{1}{N^{\frac{5-p}{2}-}} \|\langle D \rangle^{1-1} Iw\|_{L_t^{\frac{4(p-1)}{7-p}} L_x^{\frac{4(p-1)}{p-3}}([S,T])}^{p-1} \|\langle D \rangle^{1-\frac{p-3}{2}} Iw\|_{L_t^{\frac{4}{p-3}} L_x^{\frac{4}{5-p}}([S,T])} \\ &\lesssim \frac{A^{-p\alpha_{16}}}{N^{\frac{5-p}{2}-}}, \end{aligned}$$

and

$$\begin{aligned} X_{1,1,3} &\lesssim \|P_{\gtrsim N} w\|_{L_t^p L_x^{2p}([S,T])}^p \\ &\lesssim \frac{\|\langle D \rangle^{1-\frac{3p-5}{2p}} Iw\|_{L_t^p L_x^{2p}([S,T])}^p}{N^{\frac{5-p}{2}-}} \\ &\lesssim \frac{A^{-p\alpha_{16}}}{N^{\frac{5-p}{2}-}}. \end{aligned}$$

Combining these estimates and using again (5.3) with α_1 (resp. c_1) large enough (resp. small enough), we obtain $|X_{1,1}| \leq \frac{c_9 A^{-\alpha_{16}}}{1000}$. □

10.1.2. Result 10.2.

Result 10.2. *Let $(\tilde{t}, \tilde{x}) \in \mathbb{R} \times \mathbb{R}^3$. Then*

$$(10.3) \quad \int_J \int_{\mathbb{R}^3} \frac{|Iw(t, x)|^{p+1}}{|x - \tilde{x}|} dx dt \lesssim A^{p+1} + \frac{\langle Lw \rangle^{2(p-1)-A^{\frac{(p+1)^2}{2}}}}{N^{\frac{5-p}{2}-}} \lesssim A^{p+1},$$

and

$$(10.4) \quad \int_J \left(\int_{\mathbb{R}^3} \frac{|Iw|^2}{\langle x - \tilde{x} \rangle^2} dx \right)^{\frac{p+1}{2}} \frac{1}{\langle t - \tilde{t} \rangle} dt \lesssim A^{\frac{(p+1)^2}{2}}.$$

Proof. From (5.2), one can chop J into subintervals K such that $\|Iw\|_{L_t^{2(p-1)-} L_x^{2(p-1)+}(K)} = c_2$, except maybe the last one. From (5.1), Proposition 5.1 and Proposition 3.4, we see, by iteration, that

$$\int_J \int_{\mathbb{R}^3} \frac{|Iw(t, x)|^{p+1}}{|x - \tilde{x}|} dx dt \lesssim A^{p+1} + \frac{\langle Lw \rangle^{2(p-1)-A^{\frac{(p+1)^2}{2}}}}{N^{\frac{5-p}{2}-}}.$$

Choosing α_1 (resp. c_1) large enough (resp. small enough) in (5.3), we get (10.3). Next we prove (10.4). Following Lemma 5.3, [18], we have

$$\int_J \left(\int_{\mathbb{R}^3} \frac{|Iw|^2}{\langle x - \tilde{x} \rangle^2} dx \right)^{\frac{p+1}{2}} \frac{1}{\langle t - \tilde{t} \rangle} dt \leq X_1 + X_2$$

where

$$X_1 := \int_J \left(\int_{|t-\tilde{t}| > |x-\tilde{x}|} \frac{|Iw|^2}{\langle x - \tilde{x} \rangle^2} dx \right)^{\frac{p+1}{2}} \frac{1}{\langle t - \tilde{t} \rangle} dt,$$

$$X_2 := \int_J \left(\int_{|t-\tilde{t}| \leq |x-\tilde{x}|} \frac{|Iw|^2}{\langle x - \tilde{x} \rangle^2} dx \right)^{\frac{p+1}{2}} \frac{1}{\langle t - \tilde{t} \rangle} dt.$$

By Hölder inequality and (10.3)

$$X_1 \leq \int_J \int_{|t-\tilde{t}| > |x-\tilde{x}|} \frac{|Iw|^{p+1}}{\langle t - \tilde{t} \rangle} dx dt \lesssim A^{p+1}$$

We also have

$$X_2 \leq \int_J \sup_{|x-\tilde{x}| \geq |t-\tilde{t}|} \left(\frac{1}{\langle x - \tilde{x} \rangle^2} \right)^{\frac{p+1}{2}} \left(\int_{|x-\tilde{x}| \geq |t-\tilde{t}|} |Iw|^2 \right)^{\frac{p+1}{2}} \frac{1}{\langle t - \tilde{t} \rangle} dt \lesssim A^{\frac{(p+1)^2}{2}}.$$

□

10.1.3. Result 10.3.

Result 10.3. *Let $x'_a \in \mathbb{R}^3$ and let $t'_b \geq t'_a$. Then we have*

$$(10.5) \quad E(Iw(t'_b), B(x'_a, R + t'_b - t'_a)) \geq E(Iw(t'_a), B(x'_a, R)) - \frac{c_8 A^{-\alpha_{13}}}{1000}.$$

Proof. Integrating the identity

$$\begin{aligned} & \partial_t \left(\frac{1}{2} |\partial_t Iw|^2 + \frac{1}{2} |\nabla Iw|^2 + \frac{|Iw|^{p+1}}{p+1} + \frac{|Iw|^2}{2} \right) - \partial_{x_i} (\Re(\partial_t Iw \partial_{x_i} Iw)) \\ & + \Re(\partial_t Iw (IF(w) - F(Iw))) = 0, \end{aligned}$$

inside the truncated cone $M := \{(t, x), t \in (t'_a, t'_b), t - t'_a - R \geq |x - x'_a|\}$, we obtain

$$(10.6) \quad \begin{aligned} & E(Iw(t'_b), B(x'_a, R + t'_b - t'_a)) - E(Iw(t'_a), B(x'_a, R)) \\ & = \frac{1}{\sqrt{2}} \int_{\partial M} \frac{|Iw|^2}{2} + \frac{|Iw|^{p+1}}{p+1} d\sigma \\ & + \frac{1}{\sqrt{2}} \int_{\partial M} \left| \frac{x - x'_a}{|x - x'_a|} \partial_t Iw + \nabla Iw \right|^2 d\sigma - \int_M \Re(\partial_t Iw (IF(w) - F(Iw))) dx dt. \end{aligned}$$

The boundary terms are nonnegative. In order to deal with the last integral, we chop J into subintervals K such that $\|Iw\|_{L_t^{2(p-1)-} L_x^{2(p-1)+}(K)} = c_2$, except maybe the last one; then, from (5.2), Proposition 3.3, Proposition 5.1, and iteration, we get

$$\begin{aligned} \left| \int_M \Re(\partial_t Iw (IF(w) - F(Iw))) dx dt \right| & \lesssim \frac{\langle L_w \rangle^{2(p-1)-} A^{\frac{(p+1)^2}{2}}}{N^{\frac{5-p}{2}-}} \\ & \leq \frac{c_8 A^{-\alpha_{13}}}{1000}, \end{aligned}$$

choosing α_1 (resp. c_1) large enough (resp. small enough) in (5.3).

□

10.2. Estimates involving commutators in Section 8. We prove all the estimates involving commutators that appear in Section 8.

10.2.1. *Result 10.4.*

Result 10.4.

$$\bar{Z}(K', I^{-1} X_{IF(w)-F(Iw)}^{K'}), \bar{Z}(K', I^{-1} X_{F(Iw')-IF(w')}^{K'}) \lesssim \frac{\max^{2p(p-1)-}(\langle L_w \rangle, \langle L_{w'} \rangle) A^{\frac{p(p+1)}{2}}}{N^{(1-s_c)-}}$$

Proof. Step 1 . *Bound of $\bar{Z}(K', I^{-1} X_{IF(w)-F(Iw)}^{K'})$ and $\bar{Z}(K', I^{-1} X_{F(Iw')-IF(w')}^{K'})$.*

We first estimate $\bar{Z}(K', I^{-1} X_{IF(w)-F(Iw)}^{K'})$.

We write $X_{IF(w)-F(Iw)}^{K'} = X_{IF(w)-F(w)}^{K'} + X_{F(w)-F(Iw)}^{K'}$. By (8.1) and (3.2) (with (q, r) defined in (8.3))

$$\begin{aligned} \|\langle D \rangle^{s_c - \frac{1}{2}} X_{F(w)-F(Iw)}^{K'}\|_{L_t^q L_x^r(K')} &\lesssim \|\langle D \rangle^{s_c - \frac{1}{2}} (F(w) - F(Iw))\|_{L_t^{\frac{4}{3}} L_x^{\frac{4}{3}}(K')} \\ &\lesssim \|w\|_{L_t^{2(p-1)} L_x^{2(p-1)}([T, b])}^{p-1} \|\langle D \rangle^{s_c - \frac{1}{2}} P_{\lesssim N} w\|_{L_t^4 L_x^4([T, b])} \\ &\quad + \|\langle D \rangle^{s_c - \frac{1}{2}} w\|_{L_t^4 L_x^4([T, b])} \|w\|_{L_t^{2(p-1)} L_x^{2(p-1)}([T, b])}^{p-2} \|P_{\lesssim N} w\|_{L_t^{2(p-1)} L_x^{2(p-1)}([T, b])} \\ &\lesssim \|\langle D \rangle^{1-s_c} Iw\|_{L_t^{2(p-1)} L_x^{2(p-1)}([T, b])}^{p-1} \frac{\|\langle D \rangle^{1-\frac{1}{2}} Iw\|_{L_t^4 L_x^4([T, b])}}{N^{(1-s_c)-}} \\ &\quad + \frac{\|\langle D \rangle^{1-\frac{1}{2}} Iw\|_{L_t^4 L_x^4([T, b])} \|\langle D \rangle^{1-s_c} Iw\|_{L_t^{2(p-1)} L_x^{2(p-1)}([T, b])}^{p-1}}{N^{(1-s_c)-}} \\ &\lesssim \frac{\langle L_w \rangle^{2p(p-1)-} A^{\frac{p(p+1)}{2}}}{N^{(1-s_c)-}}. \end{aligned}$$

By (10.2) we write $X_{IF(w)-F(w)}^{K'} = X_{Z_1}^{K'} + X_{Z_2}^{K'} + X_{Z_3}^{K'}$ where

$$\begin{aligned} Z_1 &:= (I-1)F(P_{\ll N} w) \\ Z_2 &:= (I-1) \int_0^1 |P_{\ll N} w + y P_{\lesssim N} w|^{p-1} P_{\lesssim N} w dy \\ Z_3 &:= (I-1) \int_0^1 \frac{P_{\ll N} w + y P_{\lesssim N} w}{P_{\ll N} w + y P_{\lesssim N} w} |P_{\ll N} w + y P_{\lesssim N} w|^{p-1} \overline{P_{\lesssim N} w} dy \end{aligned}$$

Again, we use the smoothness of F (F is C^1) to deal with $X_{Z_1}^{K'}$. We have (using again (8.1) and (3.2))

$$\begin{aligned} \|\langle D \rangle^{s_c - \frac{1}{2}} X_{Z_1}^{K'}\|_{L_t^q L_x^r(K')} &\lesssim \|\langle D \rangle^{s_c - \frac{1}{2}} Z_1\|_{L_t^{\frac{4}{3}} L_x^{\frac{4}{3}}(K')} \\ &\lesssim \|\nabla F(P_{\ll N} w)\|_{L_t^{\frac{4}{3}} L_x^{\frac{4}{3}}(K')} \\ &\lesssim \frac{1}{N^{\left(\frac{3}{2}-s_c\right)-}} \|P_{\ll N} w\|_{L_t^{2(p-1)} L_x^{2(p-1)}([T, b])}^{p-1} \|\nabla P_{\ll N} w\|_{L_t^4 L_x^4([T, b])} \\ &\lesssim \frac{1}{N^{(1-s_c)-}} \|\langle D \rangle^{1-s_c} Iw\|_{L_t^{2(p-1)} L_x^{2(p-1)}([T, b])}^{p-1} \|\langle D \rangle^{1-\frac{1}{2}} Iw\|_{L_t^4 L_x^4([T, b])} \\ &\lesssim \frac{\langle L_w \rangle^{2p(p-1)-} A^{\frac{p(p+1)}{2}}}{N^{(1-s_c)-}}. \end{aligned}$$

As for $X_{Z_2}^{K'}$, we have

$$\begin{aligned}
& \| \langle D \rangle^{s_c - \frac{1}{2}} X_{Z_2}^{K'} \|_{L_t^q L_x^r(K')} \lesssim \left\| \langle D \rangle^{s_c - \frac{1}{2}} Z_2 \right\|_{L_t^{\frac{4}{3}} L_x^{\frac{4}{3}}(K')} \\
& \lesssim \int_0^1 \left[\left\| \langle D \rangle^{s_c - \frac{1}{2}} (|P_{\ll N} w + y P_{\gtrsim N} w|^{p-1}) \right\|_{L_t^{\frac{4(p-1)}{3p-5}} L_x^{\frac{4(p-1)}{3p-5}}([T, b])} \|P_{\gtrsim N} w\|_{L_t^{2(p-1)} L_x^{2(p-1)}([T, b])} \right. \\
& \quad \left. + \left\| |P_{\ll N} w + y P_{\gtrsim N} w|^{p-1} \right\|_{L_t^2 L_x^2([T, b])} \left\| \langle D \rangle^{s_c - \frac{1}{2}} P_{\gtrsim N} w \right\|_{L_t^4 L_x^4([T, b])} \right] dy \\
& \lesssim \frac{\| \langle D \rangle^{\frac{1}{2}} I w \|_{L_t^4 L_x^4([T, b])} \| \langle D \rangle^{1-s_c} I w \|_{L_t^{2(p-1)} L_x^{2(p-1)}([T, b])}^{p-1}}{N^{(1-s_c)-}} \\
& \lesssim \frac{\langle L_w \rangle^{2p(p-1)-A \frac{p(p+1)}{2}}}{N^{(1-s_c)-}},
\end{aligned}$$

by using the product rule followed by a two-variable Leibnitz rule (see Appendix B) with $f := P_{\ll N} w$ and $g := P_{\gtrsim N} w$, $L_y(f, g) = |f + yg|^{p-1}$ and $\lambda = p - 2$.

$X_{Z_3}^{K'}$ is treated in a similar fashion. In fact, we get

$$\| \langle D \rangle^{s_c - \frac{1}{2}} X_{Z_3}^{K'} \|_{L_t^q L_x^r(K')} \lesssim \frac{\langle L_w \rangle^{2p(p-1)-A \frac{p(p+1)}{2}}}{N^{(1-s_c)-}}.$$

Combining together, we obtain

$$(10.7) \quad \bar{Z}(K', I^{-1} X_{IF(w)-F(w)}^{K'}) \lesssim \frac{\langle L_w \rangle^{2p(p-1)-A \frac{p(p+1)}{2}}}{N^{(1-s_c)-}}.$$

We can estimate $\bar{Z}(K', I^{-1} X_{IF(w')-F(w')}^{K'})$ by performing a similar decomposition as previously, using (8.2) instead of (8.1). We get the same bound that was found in (10.7), with w substituted for w' . \square

11. APPENDIX B: A TWO-VARIABLE LEIBNITZ RULE

In this section we provide the proof of a two-variable Leibnitz rule.

Lemma 11.1. *Let $L \in C^1(\mathbb{C}^2, \mathbb{C})$ such that $L(0, 0) = 0$ and such that for all $\mu \in [0, 1]$ and for all $(z_1, z_2, w_1, w_2) \in \mathbb{C}^4$ we have*

$$(11.1) \quad |L'(\mu z_1 + (1 - \mu)z_2, \mu w_1 + (1 - \mu)w_2)| \lesssim |z_1|^\lambda + |z_2|^\lambda + |w_1|^\lambda + |w_2|^\lambda$$

for some $\lambda > 0$. Then

$$(11.2) \quad \|L(f, g)\|_{H^{s,p}} \lesssim \left(\|f\|_{L^{p_1}}^\lambda \|f\|_{H^{s,p_2}} + \|g\|_{L^{\tilde{p}_1}}^\lambda \|f\|_{H^{s,\tilde{p}_2}} + \|f\|_{L^{r_1}}^\lambda \|g\|_{H^{s,r_2}} + \|g\|_{L^{\tilde{r}_1}}^\lambda \|g\|_{H^{s,\tilde{r}_2}} \right),$$

assuming that $(p, p_1, p_2, \tilde{p}_1, \tilde{p}_2, r_1, r_2, \tilde{r}_1, \tilde{r}_2) \in (1, \infty)^9$,

$$\begin{aligned}
\frac{1}{p} &= \frac{\lambda}{p_1} + \frac{1}{p_2} = \frac{\lambda}{\tilde{p}_1} + \frac{1}{\tilde{p}_2} \text{ and} \\
\frac{1}{p} &= \frac{\lambda}{r_1} + \frac{1}{r_2} = \frac{\lambda}{\tilde{r}_1} + \frac{1}{\tilde{r}_2}.
\end{aligned}$$

Proof. The proof relies upon a simple modification of the one-variable fractional Leibnitz rules (see e.g [5, 6, 26]). We recall the following inequalities (see e.g [26]): given $q : \mathbb{R}^3 \rightarrow \mathbb{C}$ a function, we have

$$\begin{aligned}
& \int_{\mathbb{R}^3} |P_{N_2} q(x) - P_{N_2} q(y)| |\psi_{N_1}(x - y)| dy \lesssim \min \left(\frac{N_2}{N_1}, 1 \right) M_h(\tilde{P}_{N_2} q)(x), \\
& \int_{\mathbb{R}^3} |P_{N_2} q(x) - P_{N_2} q(y)| |\psi_{N_1}(x - y)| L(y) dy \\
(11.3) \quad & \lesssim \min \left(\frac{N_2}{N_1}, 1 \right) \left(M_h(\tilde{P}_{N_2} q)(x) M_h L(x) + M_h(|\tilde{P}_{N_2} q| L)(x) \right)
\end{aligned}$$

where $(N_1, N_2) \in 2^{\mathbb{N}^*} \in \mathbb{N}$, H is a nonnegative function, $\psi_M(\xi) := \psi\left(\frac{\xi}{M}\right)$ (if $M \in 2^{\mathbb{N}}$), $\tilde{P}_M := P_{\frac{M}{2} \leq \cdot \leq 2M}$ (if $M \in 2^{\mathbb{N}^*}$), $\tilde{P}_1 := P_{\leq 2}$ and $(M_h(f))(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy$. Recall also the Paley-Littlewood inequalities (see [22])

$$\|L(f, g)\|_{H^{s,p}} \lesssim \|P_1(L(f, g))\|_{L^p} + \left\| \left(\sum_{N_1 \in 2^{\mathbb{N}^*}} N_1^{2s} |P_{N_1}(L(f, g))|^2 \right)^{\frac{1}{2}} \right\|_{L^p}$$

and

$$(11.4) \quad \left\| \left(\sum_{N_1 \in 2^{\mathbb{N}}} N_1^{2s} |\tilde{P}_{N_1} f|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \lesssim \|f\|_{H^{s,p}}.$$

We write

$$\begin{aligned} P_{N_1}(L(f, g))(x) &= \int_{\mathbb{R}^3} L(f(y), g(y)) \psi_{N_1}^{\check{}}(x - y) dy \\ &= \int_{\mathbb{R}^3} (L(f(y), g(y)) - L(f(x), g(x))) \psi_{N_1}^{\check{}}(x - y) dy \\ &= A_1 + A_2 + A_3 + A_4 \end{aligned}$$

where

$$\begin{aligned} A_1 &:= \int_0^1 \int_{\mathbb{R}^3} |\partial_z L(\mu f(y) + (1 - \mu)f(x), \mu g(y) + (1 - \mu)g(x))| |f(y) - f(x)| |\psi_{N_1}^{\check{}}(x - y)| d\mu dy \\ A_2 &:= \int_0^1 \int_{\mathbb{R}^3} |\partial_{\bar{z}} L(\mu f(y) + (1 - \mu)f(x), \mu g(y) + (1 - \mu)g(x))| |f(y) - f(x)| |\psi_{N_1}^{\check{}}(x - y)| d\mu dy \\ A_3 &:= \int_0^1 \int_{\mathbb{R}^3} |\partial_w L(\mu f(y) + (1 - \mu)f(x), \mu g(y) + (1 - \mu)g(x))| |g(y) - g(x)| |\psi_{N_1}^{\check{}}(x - y)| d\mu dy \\ A_4 &:= \int_0^1 \int_{\mathbb{R}^3} |\partial_{\bar{w}} L(\mu f(y) + (1 - \mu)f(x), \mu g(y) + (1 - \mu)g(x))| |g(y) - g(x)| |\psi_{N_1}^{\check{}}(x - y)| d\mu dy. \end{aligned}$$

Let us deal for example with A_1 .

$$\begin{aligned} \sum_{N_1 \in 2^{\mathbb{N}^*}} N_1^{2s} A_1^2 &\lesssim \left(\begin{aligned} &\sum_{N_1 \in 2^{\mathbb{N}^*}} N_1^{2s} \left(\int_{\mathbb{R}^3} |f(y)|^\lambda |f(y) - f(x)| |\psi_{N_1}^{\check{}}(x - y)| dy \right)^2 \\ &+ \sum_{N_1 \in 2^{\mathbb{N}^*}} N_1^{2s} |f(x)|^{2\lambda} \left(\int_{\mathbb{R}^3} |f(y) - f(x)| |\psi_{N_1}^{\check{}}(x - y)| dy \right)^2 \\ &+ \sum_{N_1 \in 2^{\mathbb{N}^*}} N_1^{2s} \left(\int_{\mathbb{R}^3} |g(y)|^\lambda |f(y) - f(x)| |\psi_{N_1}^{\check{}}(x - y)| dy \right)^2 \\ &+ \sum_{N_1 \in 2^{\mathbb{N}^*}} N_1^{2s} |g(x)|^{2\lambda} \left(\int_{\mathbb{R}^3} |f(y) - f(x)| |\psi_{N_1}^{\check{}}(x - y)| dy \right)^2 \end{aligned} \right) \\ &\lesssim A_{1,1}^2 + A_{1,2}^2 + A_{1,3}^2 + A_{1,4}^2. \end{aligned}$$

We have

$$\begin{aligned} A_{1,1}^2 &\lesssim \sum_{N_1 \in 2^{\mathbb{N}^*}} N_1^{2s} \left(\sum_{N_2 \in 2^{\mathbb{N}}} \int_{\mathbb{R}^3} |f(y)|^\lambda |P_{N_2}(f)(y) - P_{N_2}(f)(x)| |\psi_{N_1}^{\check{}}(x - y)| dy \right)^2 \\ &\lesssim \sum_{N_1 \in 2^{\mathbb{N}^*}} N_1^{2s} \left(\sum_{N_2 \leq N_1} \int_{\mathbb{R}^3} |f(y)|^\lambda |P_{N_2}(f)(y) - P_{N_2}(f)(x)| |\psi_{N_1}^{\check{}}(x - y)| dy \right)^2 \\ &\quad + \sum_{N_1 \in 2^{\mathbb{N}^*}} N_1^{2s} \left(\sum_{N_2 \geq N_1} \int_{\mathbb{R}^3} |f(y)|^\lambda |P_{N_2}(f)(y) - P_{N_2}(f)(x)| |\psi_{N_1}^{\check{}}(x - y)| dy \right)^2 \\ &\lesssim A_{1,1,1}^2 + A_{1,1,2}^2. \end{aligned}$$

But, by (11.3) we have

$$A_{1,1,1}^2 \lesssim \left(M_h(|f(x)|^\lambda)^2 \sum_{N_1 \in 2^{\mathbb{N}^*}} N_1^{2s} \left(\sum_{N_2 \leq N_1} \frac{N_2}{N_1} M_h((\tilde{P}_{N_2} f)(x)) \right)^2 \right. \\ \left. + \sum_{N_1 \in 2^{\mathbb{N}^*}} N_1^{2s} \left(\sum_{N_2 \leq N_1} \frac{N_2}{N_1} M_h((|\tilde{P}_{N_2} f| |f|^\lambda)(x)) \right)^2 \right).$$

Now, by Young's inequality we have

$$\sum_{N_1 \in 2^{\mathbb{N}^*}} N_1^{2s} \left(\sum_{N_2 \leq N_1} \frac{N_2}{N_1} |a_{N_2}| \right)^2 = \sum_{N_1 \in 2^{\mathbb{N}^*}} \left(\sum_{N_2 \leq N_1} \left(\frac{N_2}{N_1} \right)^{1-s} N_2^s |a_{N_2}| \right)^2 \\ \lesssim \sum_{N_1 \in 2^{\mathbb{N}}} N_1^{2s} |a_{N_1}|^2$$

which implies that

$$(11.5) \quad A_{1,1,1}^2 \lesssim \left(M_h(|f(x)|^\lambda)^2 \sum_{N_1 \in 2^{\mathbb{N}}} N_1^{2s} \left(M_h((\tilde{P}_{N_1} f)(x)) \right)^2 + \sum_{N_1 \in 2^{\mathbb{N}}} N_1^{2s} \left(M_h((|\tilde{P}_{N_1} f| |f|^\lambda)(x)) \right)^2 \right)$$

and therefore, by Fefferman-Stein maximal inequality [8], Hölder's inequality and (11.4) we have

$$\|A_{1,1,1}\|_{L^p} \lesssim \|f\|_{L^{p_1}}^\lambda \|f\|_{H^{s,p_2}}.$$

Also, by (11.3) we have

$$A_{1,1,2}^2 \lesssim \left(\left(M_h(|f(x)|^\lambda)^2 \sum_{N_1 \in 2^{\mathbb{N}^*}} N_1^{2s} \left(\sum_{N_2 > N_1} M_h((\tilde{P}_{N_2} f)(x)) \right)^2 \right) \right. \\ \left. + \sum_{N_1 \in 2^{\mathbb{N}^*}} N_1^{2s} \left(\sum_{N_2 > N_1} M_h((|\tilde{P}_{N_2} f| |f|^\lambda)(x)) \right)^2 \right) \\ \sum_{N_1 \in 2^{\mathbb{N}^*}} N_1^{2s} \left(\sum_{N_2 > N_1} |a_{N_2}| \right)^2 = \sum_{N_1 \in 2^{\mathbb{N}^*}} \left(\sum_{N_2 > N_1} \left(\frac{N_1}{N_2} \right)^s N_2^s |a_{N_2}| \right)^2 \\ \lesssim \sum_{N_1 \in 2^{\mathbb{N}}} N_1^{2s} |a_{N_1}|^2.$$

and therefore (11.5) also holds if $A_{1,1,1}$ is substituted for $A_{1,1,2}$.

The other terms ($A_{1,2}$, $A_{1,3}$, $A_{1,4}$ and then A_2 , A_3 , A_4) are treated in a similar fashion.

We also have $\|P_1(L(f, g))\|_{L^p} \lesssim \|L(f, g)\|_{L^p}$. Then writing $L(f, g) = L(f, g) - L(0, 0)$ and applying the fundamental theorem of calculus, we see that (11.2) holds if $s = 0$. \square

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