GLOBAL DYNAMICS ABOVE THE GROUND STATE FOR THE ENERGY-CRITICAL SCHRÖDINGER EQUATION WITH RADIAL DATA

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Abstract. Consider the focusing energy critical Schrödinger equation in three space dimensions with radial initial data in the energy space. We describe the global dynamics of all the solutions of which the energy is at most slightly larger than that of the ground states, according to whether it stays in a neighborhood of them, blows up in finite time or scatters. In analogy with [19], the proof uses an analysis of the hyperbolic dynamics near them and the variational structure far from them. The key step that allows to classify the solutions is the *one-pass* lemma. The main difference between [19] and this paper is that one has to introduce a scaling parameter in order to describe the dynamics near them. One has to take into account this parameter in the analysis around the ground states by introducing some orthogonality conditions. One also has to take it into account in the proof of the *one-pass* lemma by comparing the contribution in the variational region and in the hyperbolic region.

Contents

1. Introduction

In this paper, we consider the semilinear Schrödinger equation on $\mathbb{R}^3$ with the focusing energy-critical power for $u = u(t,x) : \mathbb{R}^{1+3} \to \mathbb{C}$:

$$i\partial_t u - \Delta u = |u|^4 u, \quad u(0,x) = u_0(x) \tag{1.1}$$

with radial initial data $u_0 \in \dot{H}^1$ (or H^1). Here $\dot{H}^1$ (resp. H^1) is the standard homogeneous (resp. inhomogeneous) Sobolev space in three dimensions, i.e., the completion of the Schwartz space with respect to the norm $\|f\|_{\dot{H}^1} := \|\nabla f\|_{L^2}$ (resp. $\|f\|_{H^1} := \|f\|_{L^2} + \|\nabla f\|_{L^2}$). Our consideration is restricted throughout this paper to the radial subspace:

$$\dot{H}^1_{\text{radial}} := \{\varphi \in \dot{H}^1 \mid \varphi(x) = \varphi(|x|)\}. \tag{1.2}$$

A strong solution of (1.1) is a solution that satisfies the Duhamel formula:

$$u(t) = e^{-it\Delta}u_0 - i\int_0^t e^{-i(t-t')\Delta}(|u|^4 u(t'))\,dt'. \tag{1.3}$$

It enjoys the following energy conservation law

$$E(u(t)) := \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u(t,x)|^2\,dx - \frac{1}{6}\int_{\mathbb{R}^3} |u(t,x)|^6\,dx = E(u(0)). \tag{1.4}$$

(1.1) can be written in the Hamiltonian form $\partial_t u = iE'(u)$, where $\langle E'(u), h\rangle = \partial_\lambda E(u+\lambda h)|_{\lambda=0} = -\langle \Delta u, h\rangle - \langle |u|^4 u, h\rangle$, and $\langle\cdot,\cdot\rangle$ denotes the real-valued inner product on $L^2(\mathbb{R}^3)$:

$$(f|g) := \int_{\mathbb{R}^3} f(x)\bar{g}(x)dx, \quad \langle f, g\rangle := \Re(f|g). \tag{1.5}$$

The symplectic form ω associated to this Hamiltonian system is

$$\omega(u,v) := \langle iu, v\rangle. \tag{1.6}$$

This equation admits a family of radial stationary solutions called the ground states, described by the rotation parameter $\theta \in \mathbb{R}$ and the scaling parameter $\sigma \in \mathbb{R}$:

$$W_{\theta,\sigma}(x) := e^{i\theta}e^{\sigma/2}W(e^\sigma x) \in \dot{H}^1_{\text{radial}}, \tag{1.7}$$

with

$$W(x) := \left(1 + \frac{|x|^2}{3}\right)^{-1/2}, \tag{1.8}$$

which satisfy

$$-\Delta W_{\theta,\sigma} = |W_{\theta,\sigma}|^4 W_{\theta,\sigma}. \tag{1.9}$$

The two dimensional manifold of those stationary solutions is denoted by

$$\mathcal{W} := \{W_{\theta,\sigma} \mid \theta,\sigma \in \mathbb{R}\} \subset \dot{H}^1_{\text{radial}}. \tag{1.10}$$

The distance from $\mathcal{W}$ and its δ-neighborhood are denoted by

$$d_{\mathcal{W}}(\varphi) := \inf_{\theta,\sigma\in\mathbb{R}} \|\varphi - W_{\theta,\sigma}\|_{\dot{H}^1}, \quad B_\delta(\mathcal{W}) := \{\varphi \in \dot{H}^1_{\text{radial}} \mid d_{\mathcal{W}}(\varphi) < \delta\}. \tag{1.11}$$

Note that this set is invariant for the complex rotation and $\dot{H}^1$ scaling. Recall (see [1, 23]) that W is an extremizer for the Sobolev inequality, i.e.,

$$\|W\|_{L^6}/\|W\|_{\dot{H}^1} = \sup\{\|f\|_{L^6}/\|f\|_{\dot{H}^1} \mid 0 \neq f \in \dot{H}^1\}. \tag{1.12}$$

The local well-posedness of (1.1) has been studied in [5, 4]. See [11] for a summary of these results. In particular, it is known that on an interval J such that $\|e^{it\triangle}u_0\|_{L^{10}_{t,x}(J\times\mathbb{R}^3)}$ is small enough, there exists a unique solution u of (1.3) in a subspace of $C(J, \dot{H}^1)$. This allows us to define the maximal time interval of existence $I(u) := (-T_-(u), T_+(u))$ with $T_+(u)$, $T_-(u)$ denoting respectively the forward, backward maximal time of existence (in this class): see again [11] for more detail. The next step is to understand the global behavior of (1.3). Classification of radial solutions of (1.1) was studied for $E(u_0) < E(W)$ in [11], and that for $E(u_0) = E(W)$ in [6]. These results are summarized as follows: For $u_0 \in \dot{H}^1_{\mathrm{radial}}$ with $E(u_0) \le E(W)$,

- If $\|\nabla u_0\|_{L^2} < \|\nabla W\|_{L^2}$, then the solution is either W^- up to symmetry, or scattering as $t \to \pm\infty$, i.e., $T_\pm(u) = \infty$ and there exist $u_\pm \in \dot{H}^1$ such that $\lim_{t\to\pm\infty} \|u(t) - e^{-it\Delta}u_\pm\|_{\dot{H}^1} = 0$.
- If $\|\nabla u_0\|_{L^2} = \|\nabla W\|_{L^2}$ then $u(t) = u(0) \in \mathcal{W}$.
- If $\|\nabla u_0\|_{L^2} > \|\nabla W\|_{L^2}$ and $u_0 \in L^2(\mathbb{R}^3)$ then the solution is either W^+ up to symmetry, or blowing up both in $t > 0$ and in $t < 0$ (i.e., $T_\pm(u) < \infty$),

where $W^\pm$ are the unique solutions which converge to W strongly in $\dot{H}^1$ as $t \to \infty$, satisfying $\pm(\|\nabla W^\pm\|_{L^2} - \|\nabla W\|_{L^2}) > 0$. W^- scatters as $t \to -\infty$, while W^+ blows up in $t < 0$.

The goal of this paper is to classify the global behavior of solutions with slightly more energy than the ground states. Our main result is the following. Let

$$\begin{aligned} \mathcal{H}^\epsilon &:= \{\varphi \in \dot{H}^1_{\mathrm{radial}} \mid E(\varphi) < E(W) + \epsilon^2\}, \\ \mathcal{S}_\pm &:= \{u_0 \in \dot{H}^1_{\mathrm{radial}} \mid \text{the solution } u \text{ scatters as } t \to \pm\infty\}, \\ \mathcal{B}_\pm &:= \{u_0 \in \dot{H}^1_{\mathrm{radial}} \mid \text{the solution } u \text{ blows up in } \pm t > 0\}. \end{aligned} \tag{1.13}$$

Theorem 1.1. *There is an absolute constant $\epsilon_\star \in (0,1)$ such that for each $\epsilon \in (0, \epsilon_\star]$, there exist a relatively closed set $\mathcal{X}_\epsilon \subset \mathcal{H}^\epsilon$, and a continuous function $\Theta : \mathcal{H}^\epsilon \setminus \mathcal{X}_\epsilon \to \{\pm 1\}$, with the following properties. $\mathcal{W} \subset \mathcal{X}_\epsilon \subset B_{C\epsilon}(\mathcal{W})$ for some absolute constant $C \in (0,\infty)$. The values of Θ are independent of ϵ. For each $u_0 \in \mathcal{H}^\epsilon$ and the solution u of (1.1),*

$$I_0(u) := \{t \in I(u) \mid u(t) \in \mathcal{X}_\epsilon\} \tag{1.14}$$

is either empty or an interval. Hence $I(u) \setminus I_0(u)$ consists of at most two open intervals. Let $\sigma \in \{\pm\}$. If $\Theta(u(t)) = +1$ for t close to $T_\sigma(u)$, then $u_0 \in \mathcal{S}_\sigma$. If $\Theta(u(t)) = -1$ for t close to $T_\sigma(u)$ and $u_0 \in L^2(\mathbb{R}^3)$, then $u_0 \in \mathcal{B}_\sigma$.

In other words, every solution with energy less than $E(W) + \epsilon^2$ can stay in $\mathcal{X}_\epsilon$ only for an interval of time, though it can be the entire existence time. Once the solution gets out of $\mathcal{X}_\epsilon$, it has to either scatter or blow-up, according to the sign function $\Theta(u)$, though we need an additional condition $u_0 \in L^2(\mathbb{R}^3)$ to ensure the blow-up.

The above properties hold in both the time directions. Concerning the relation between forward and backward dynamics, we have

Theorem 1.2. *For any $\epsilon > 0$, each of the 4 intersections*

$$\mathcal{S}_- \cap \mathcal{S}_+, \quad \mathcal{B}_- \cap \mathcal{S}_+, \quad \mathcal{S}_- \cap \mathcal{B}_+, \quad \mathcal{B}_- \cap \mathcal{B}_+ \tag{1.15}$$

has non-empty interior in $\mathcal{H}^\epsilon \cap L^2(\mathbb{R}^3)$.

In particular, there are infinitely many solutions which scatter on one side of time and blow up on the other. According to the previous theorem, such transition can occur only by changing $\Theta(u)$ from $+1$ to -1 or vice versa, going through $O(\epsilon)$ neighborhood of $\mathcal{W}$, but the change is allowed at most once for each solution. Note however that there may well exist blow-up inside the neighborhood of $\mathcal{W}$, as the equation is energy-critical. It is indeed the case for the energy-critical wave equation. More precise dynamics around $\mathcal{W}$ should be studied elsewhere.

In the proof of the above results, we will explicitly construct, in terms of the eigenfunctions of the linearized operator, the functionals $\tilde{d}_{\mathcal{W}}$ and Θ, as well as open initial data sets in the 4 intersections.

Now we explain the main ideas of this paper and how it is organized. The proof of Theorem 1.1 relies upon a strategy that was pioneered by the first author and Schlag in [17] in the study of the nonlinear Klein-Gordon equation with the focusing cubic nonlinearity. It relies upon two components: an *ejection lemma* and a *one-pass lemma*.

The *ejection lemma* aims at describing the dynamics of the solution when it is in the exit mode, i.e., when it is close to $\mathcal{W}$ and moving away from it. By analogy with the dynamics of solutions of linear differential equations, we would like its dynamics to be ruled by that of its unstable eigenmode of the linearized operator around $\mathcal{W}$. In order to verify this statement, one has to control the orthogonal component of the spectral decomposition of the remainder resulting from the linearization around $\mathcal{W}$. We would like to control this component by using the quadratic terms resulting from the Taylor expansion of the energy around $\mathcal{W}$. This can be done if and only if the remainder satisfies two orthogonality conditions: see Proposition 3.4. In order to satisfy these conditions, one has to give two degrees of freedom to the decomposition of the solution around $\mathcal{W}$: a rotation parameter (this was done in [19]) and a scaling parameter: see Propositions 3.1 and 3.3. Then, one also has to control the evolution of these two parameters. We prove in Proposition 3.7 that we can close the argument. More precisely the dynamic of the solution close to $\mathcal{W}$ and in the exit mode is dominated by the exponential growth of the unstable eigenmode; moreover, a relevant functional (denoted by K) grows exponentially and its sign eventually becomes opposite to that of the eigenmode.

The *one-pass lemma* (see Proposition 3.11 and Section 11 for more details) aims at classifying the fate of the solution. A direct consequence of this lemma is the dichotomy described in the statement of Theorem 1.1. It shows that the orbit cannot cross a neighborhood of $\mathcal{W}$ more than once. The proof is by contradiction. Assuming that the solution crosses this neighborhood more than once, then it means that the solution is at two different times t_a, t_b close to $\mathcal{W}$ and in the ejection mode (forward and backward respectively in time). So we can apply the ejection lemma as long as we are not so far from $\mathcal{W}$ and then variational estimates (see Proposition 3.9) far from $\mathcal{W}$. The contradiction appears when we integrate by part a localized virial identity (11.10). The left-hand side is much smaller than the right-hand side thanks to the exponential growth of a relevant functional (denoted by K) in the ejection mode and variational estimates far from $\mathcal{W}$. The process involves a parameter m, which is the cut-off radius for the localization. Notice that unlike the subcritical case, one has to take into account the scaling parameter defined by the ground states to which the solution is close to. This requires a much more complicated analysis since we have no control of this parameter. It is also harder than the energy-critical wave equation, for which it is easy to localize virial and energy estimates in space-time, thanks to the finite speed of propagation (see [12]). Indeed, this part of analysis is the main novelty of this paper. In the case where $\Theta(u(t)) = +1$ after ejection (see Section 11), one introduces a radius of the concentration of the kinetic part of the energy (see definition of m_V^+) and the hyperbolic parameter (see definition of m_H), estimates K along with some error terms (generated by the cut-off) in the hyperbolic region and the variational region, and compares these estimates. In the worst scenario, one proves a decay estimate (see (11.65)) in the variational region and uses this estimate to implement Bourgain's energy induction method [3]: this allows to construct a solution whose energy is smaller than the original one by a nontrivial amount (in particular it is smaller than that of the ground states), then the theory below the ground state energy (see [11]) implies that it is not close to the ground states, neither is the original solution by a perturbation argument, contradicting the assumption of returning orbit. In the case where $\Theta(u(t)) = -1$ after ejection, we introduce a threshold (see definition of m_V^-) that allows to compare K with the main part of the virial identity; then, by integrating the virial identity, we can prove that this threshold must be very large; then, by proving a decay estimate, one can show that this threshold is not so large, which leads to a contradiction.

The fate of the solution depends on $\Theta(u(t))$ when u is ejected. If $\Theta(u(t)) = -1$ and $u_0 \in L^2$ then we prove that it blows up in finite time; if $\Theta(u(t)) = +1$ then we prove that it is scattering: see Section 13. The scattering is proved by a modification of Kenig-Merle approach [11] and arguments from [19]. Unlike the subcritical case, one has to deal with possible blow-up in finite time, although the $\dot{H}^1$ norm is bounded. The proof is by contradiction. Assuming that scattering fails, then one can find a critical level of energy above which scattering does not hold for solutions that are far from the ground states and $\Theta(u) = +1$. But this means that there exists a sequence $(u_n)_{n\geq 1}$ that satisfies the properties that we have just mentioned (in fact, the distance can be upgraded from far to very far, by appealing to the ejection lemma), and, thanks to a concentration compactness procedure, the fact that the energy of u_n is just above that of the ground states, one can construct a critical element U_c that does not scatter, has energy equal to the critical level of energy, is far from the ground states, and satisfies $\Theta(U_c) = +1$. Moreover its orbit is precompact up to scaling. By using Kenig-Merle's arguments, one sees that U_c does not exist.

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2. Notation

In this section, we set up some notation that appear in this paper. If x is a complex number then $x = \Re(x) + i\Im(x) = x_1 + ix_2$. Here $\Re(x)$ and x_1 (resp. $\Im(x)$ and x_2) denote the real part (resp. the imaginary part) of x. Given x, y two real numbers, $x \lesssim y$ (resp. $\gtrsim$) means that there exists a universal constant $C > 0$ such that $x \leq Cy$ (resp. $x \geq Cy$). For any function f on $\mathbb{R}$ or $[0, \infty)$, and for any

$m \in (0,\infty)$, we denote by f_m the following rescaled function

$$f_m(r) := f(r/m). \tag{2.1}$$

$L^p_x = L^p$ denotes the standard L^p space on $\mathbb{R}^3$. Some estimates that we establish in this paper require the Littlewood-Paley technology, which we set up now. The Fourier transform of $\varphi \in \mathcal{S}'(\mathbb{R}^3)$ is denoted by $\widehat{\varphi}$. Let $\phi : \mathbb{R} \to [0,\infty)$ be a smooth even function satisfying $t\phi'(t) \le 0$ and

$$|t| \le 1 \implies \phi(t) = 1, \quad |t| \ge 2 \implies \phi(t) = 0. \tag{2.2}$$

The complement of this smooth cut-off is denoted by

$$\phi^C := 1 - \phi. \tag{2.3}$$

For any $m > 0$, Littlewood-Paley operators $P_{<m}$, $P_{\ge m}$ and P_m are defined by

$$\widehat{P_{<m} f}(\xi) := \phi_m(|\xi|)\widehat{f}(\xi), \quad P_{\ge m} := 1 - P_{<m}, \quad P_m := P_{<m} - P_{<m/2}. \tag{2.4}$$

The following functionals on $\dot{H}^1(\mathbb{R}^3)$ play crucial roles in variational arguments.

$$K(f) := \|\nabla f\|_{L^2}^2 - \|f\|_{L^6}^6, \tag{2.5}$$

$$I(f) := E(f) - K(f)/2 = \|f\|_{L^6}^6/3, \tag{2.6}$$

$$G(f) := E(f) - K(f)/6 = \|\nabla f\|_{L^2}^2/3. \tag{2.7}$$

It follows from (1.12) and similar arguments to [17] that

$$E(W) = \inf\{E(\varphi) \mid 0 \ne \varphi \in \dot{H}^1, K(\varphi) = 0\} \tag{2.8}$$

$$= \inf\{G(\varphi) \mid 0 \ne \varphi \in \dot{H}^1, K(\varphi) \le 0\} \tag{2.9}$$

$$= \inf\{I(\varphi) \mid 0 \ne \varphi \in \dot{H}^1, K(\varphi) \le 0\} \tag{2.10}$$

Let S_a^σ be the one-parameter group of dilation operators defined as follows

$$S_a^\sigma f(x) := e^{(3/2+a)\sigma} f(e^\sigma x), \tag{2.11}$$

and let $S_a' := \partial_\sigma S_a^\sigma|_{\sigma=0}$ be its generator. It is easy to see that the adjoint is given by $(S_a^\sigma)^* = S_{-a}^{-\sigma}$, hence by differentiating in σ,

$$(S_a')^* = -S_{-a}'. \tag{2.12}$$

We denote by S, W, $\bar{S}$ and N the following mixed L^p spaces on $\mathbb{R}^{1+3}$

$$\begin{aligned} S &:= L^{10}_{t,x}(\mathbb{R}^{1+3}), \quad W := L^{10}_t(\mathbb{R}; L^{30/13}_x(\mathbb{R}^3)), \\ \bar{S} &:= L^{10/3}_{t,x}(\mathbb{R}^{1+3}), \quad N := L^2_t(\mathbb{R}; L^{6/5}_x(\mathbb{R}^3)). \end{aligned} \tag{2.13}$$

We also use the homogeneous Sobolev spaces defined by completion of the Schwarz space with respect to the norm

$$\|v\|_{X^1} := \|\nabla v\|_X \quad (X = W, \bar{S}, N). \tag{2.14}$$

For any interval $J \subset \mathbb{R}$ and any function space X on $\mathbb{R}^{1+3}$, the restriction of X onto J is denoted by $X(J)$. The Sobolev embedding implies

$$\|v\|_{S(J)} \lesssim \|v\|_{W^1(J)}. \tag{2.15}$$

We recall the L^p decay and the Strichartz estimates (see e.g., [9]). For any $p \in [2,\infty]$

$$\|e^{it\Delta}\varphi\|_{L^p_x} \lesssim |t|^{-3(1/2-1/p)}\|\varphi\|_{L^{p'}_x}, \tag{2.16}$$

where $p' = p/(p-1)$, and for any interval $J \ni 0$,

$$\begin{aligned} &\|e^{-it\Delta}\varphi\|_{(L_t^\infty L_x^2 \cap \bar{S} \cap W)(\mathbb{R})} \lesssim \|\varphi\|_{L^2}, \\ &\left\| \int_0^t e^{-i(t-s)\Delta} F(s) ds \right\|_{(L_t^\infty L_x^2 \cap \bar{S} \cap W)(J)} \lesssim \|F\|_{N(J)}. \end{aligned} \tag{2.17}$$

In this paper, we constantly use the linearized operator $\mathcal{L}$ defined by

$$\mathcal{L}f = L_+ f_1 + iL_- f_2, \quad L_+ := -\Delta - 5W^4, \quad L_- := -\Delta - W^4. \tag{2.18}$$

We recall some spectral properties of $\mathcal{L}$ (see [6, 8, 22]):

- It has two resonance functions iW and W'.
- It has two simple eigenvalues $\pm\mu$ (with $\mu > 0$) and two smooth, exponentially decaying eigenfunctions $g_\pm = g_1 \mp ig_2$ that satisfy $i\mathcal{L}g_\pm = \pm\mu g_\pm$. In other words, $L_+ g_1 = -\mu g_2$ and $L_- g_2 = \mu g_1$.
- $L_- \geq 0$ and $\mathrm{Ker}(L_-) = \mathrm{span}\{W\}$ on $\dot{H}^1_{\mathrm{radial}}$.
- $\mathrm{Ker}(L_+) = \mathrm{span}\{W'\}$ on $\dot{H}^1_{\mathrm{radial}}$.

Fix a real-valued radial function $\chi \in \mathcal{S}_{\mathrm{radial}}(\mathbb{R}^3) \subset L^{6/5}(\mathbb{R}^3)$ such that

$$\langle W', \chi \rangle \neq 0, \quad \langle W, \chi \rangle \neq 0, \quad \langle \chi, g_1 \rangle = \langle \chi, g_2 \rangle = 0. \tag{2.19}$$

To see existence of such χ, suppose for contradiction that $\{g_1, g_2\}^\perp$ is included in $\{W'\}^\perp$ or $\{W\}^\perp$, where $A^\perp := \{\varphi \in \mathcal{S}(\mathbb{R}^3) \mid A \ni \forall\psi, \ \langle \varphi, \psi \rangle = 0\}$. Since $\mathcal{S}$ is reflexive, it implies that either W' or W is in $\mathrm{span}\{g_1, g_2\}$, contradicting the slow decay of W', W by the rapid decay of g_1, g_2. Therefore $\{g_1, g_2\}^\perp$ is not included in $\{W'\}^\perp$ nor $\{W\}^\perp$. The same conclusion holds under the radial restriction, since orthogonality against radial functions is determined by the spherical average. Hence there exists $\chi \in \mathcal{S}_{\mathrm{radial}}(\mathbb{R}^3)$ satisfying (2.19).

3. Proof of the main theorem

The proof of Theorem 1.1 relies upon some propositions stated below. The first proposition, proved in Section 4, gives a decomposition of a vector $\varphi \in \dot{H}^1$ close to the ground states $\mathcal{W}$, taking account of two parameters (the rotation parameter and the scaling parameter) and a constraint (the so-called orthogonality condition).

Proposition 3.1 (Orthogonal decomposition of φ)**.** *There exist an absolute constant $0 < \delta_E \ll 1$ and a C^1 function $(\tilde{\theta}, \tilde{\sigma}) : B_{\delta_E}(\mathcal{W}) \to (\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}$ with the following properties. For any $\varphi \in B_{\delta_E}(\mathcal{W})$, putting*

$$\varphi = e^{i\tilde{\theta}(\varphi)} S_{-1}^{\tilde{\sigma}(\varphi)}(W + v), \tag{3.1}$$

we have

$$(v|\chi) = 0, \quad d_{\mathcal{W}}(\varphi) \sim \|v\|_{\dot{H}^1}. \tag{3.2}$$

Moreover $(\tilde{\theta}(\varphi), \tilde{\sigma}(\varphi)) \in (\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}$ is unique for the above property. Furthermore, if $\|\varphi - W_{\theta,\sigma}\|_{\dot{H}^1} \ll 1$ for some $(\theta, \sigma) \in \mathbb{R}^2$, then

$$|(e^{i\tilde{\theta}(\varphi)} - e^{i\theta}, \tilde{\sigma}(\varphi) - \sigma)| \lesssim \|\varphi - W_{\theta,\sigma}\|_{\dot{H}^1}. \tag{3.3}$$

The second proposition, proved in Section 5, describes more precisely the decomposition in Proposition 3.1, taking into account the spectral properties of $\mathcal{L}$. This decomposition does not use the radial symmetry.

Proposition 3.2 (Spectral decomposition of v)**.** *For any $v \in \dot H^1$, there exists a unique decomposition*

$$v = \lambda_+ g_+ + \lambda_- g_- + \gamma, \quad \lambda_\pm \in \mathbb{R}, \quad \gamma \in \dot H^1, \tag{3.4}$$

such that $\omega(g_\pm, \gamma) = 0$. After normalizing $g_\pm$ (or g_1 and g_2) such that

$$\omega(g_+, g_-) = 2\langle g_1, g_2\rangle = 1, \quad \pm\omega(W, g_\mp) = \langle W, g_2\rangle > 0, \tag{3.5}$$

the above decomposition is given by

$$\lambda_\pm := \pm\omega(v, g_\mp). \tag{3.6}$$

Putting $\lambda_1 := (\lambda_+ + \lambda_-)/2$ and $\lambda_2 := (\lambda_+ - \lambda_-)/2$, it can also be written as

$$v = 2\lambda_1 g_1 - 2i\lambda_2 g_2 + \gamma, \quad \lambda_1 = \langle v_1|g_2\rangle, \quad \lambda_2 = -\langle v_2|g_1\rangle, \tag{3.7}$$

with $\langle \gamma_1|g_2\rangle = \langle \gamma_2|g_1\rangle = 0$.

The third proposition, also proved in Section 5, aims at describing the dynamics of the solution near the ground states, using the decomposition in Proposition 3.1. Again, this does not use the radial symmetry.

Proposition 3.3 (Linearization and parametrization around $\mathcal{W}$)**.** *Let u be a solution of* (1.1) *on an interval I in the form* (3.1)*, i.e., $(\theta, \sigma, v) : I \to (\mathbb{R}/2\pi\mathbb{Z})\times\mathbb{R}\times\dot H^1$ is defined by*

$$u(t) = e^{i\theta(t)} S_{-1}^{\sigma(t)}(W + v(t)), \quad \theta(t) := \tilde\theta(u(t)), \quad \sigma(t) := \tilde\sigma(u(t)). \tag{3.8}$$

Then, letting $\tau : I \to \mathbb{R}$ such that $\tau'(t) := e^{2\sigma(t)}$, we have

$$\partial_\tau v = i\mathcal{L}v - (i\theta_\tau + \sigma_\tau S'_{-1})(W + v) - iN(v), \tag{3.9}$$

where $\theta_\tau = \frac{\partial\theta}{\partial\tau}$ etc., and

$$N(f) := |W + f|^4(W + f) - W^5 - \partial_\lambda|_{\lambda=0}\left(|W + \lambda f|^4(W + \lambda f)\right). \tag{3.10}$$

Furthermore, $\partial_\tau(\theta, \sigma) = O(\|v\|_{\dot H^1})$ and, decomposing v by Proposition 3.2,

$$\partial_\tau \lambda_\pm = \pm\mu\lambda_\pm + O(\|v\|_{\dot H^1}^2), \tag{3.11}$$

or equivalently,

$$\partial_\tau\lambda_1 = \mu\lambda_2 + O(\|v\|_{\dot H^1}^2), \quad \partial_\tau\lambda_2 = \mu\lambda_1 + O(\|v\|_{\dot H^1}^2). \tag{3.12}$$

The next proposition, proved in Section 4, shows that the orthogonal direction γ of v in (3.4) can be controlled by the linearized energy:

Proposition 3.4 (Control of orthogonal direction)**.** *For any function $w \in \dot H^1_{\mathrm{radial}}$ satisfying $\langle w_1, g_2\rangle = 0$, we have*

$$\|\nabla w\|_{L^2}^2 \sim |(w|\chi)|^2 + \langle \mathcal{L}w, w\rangle. \tag{3.13}$$

Hence in the subspace $\{v \in \dot H^1 \mid (v|\chi) = 0\}$, we can define an equivalent norm E using the decomposition of Proposition 3.2

$$\|v\|_E^2 := \mu\left(\lambda_1^2 + \lambda_2^2\right) + \frac{1}{2}\langle \mathcal{L}\gamma, \gamma\rangle \sim \lambda_1^2 + \lambda_2^2 + \|\gamma\|_{\dot H^1}^2 \sim \|v\|_{\dot H^1}^2. \tag{3.14}$$

In particular, in the decomposition of Proposition 3.1, we have

$$d_{\mathcal{W}}(\varphi) \sim \|v\|_{\dot H^1} \sim \|v\|_E. \tag{3.15}$$

Henceforth, we assume that whenever a solution u of (1.1) is in $B_{\delta_E}(\mathcal{W})$, the coordinates σ, θ, v, $\lambda_\pm$, λ_1, λ_2 and γ are defined by (3.8), (3.4) and (3.7), while $\tau(t)$ is a solution of $\dot{\tau}(t) = e^{2\sigma}$. In short,

$$\begin{aligned} &e^{-i\theta} S_{-1}^{-\sigma} u - W = v = \lambda_+ g_+ + \lambda_- g_- + \gamma = 2\lambda_1 g_1 - 2i\lambda_2 g_2 + \gamma, \\ &0 = (v|\chi) = (\gamma|\chi) = \omega(g_\pm, \gamma) = \langle g_1, \gamma_2 \rangle = \langle g_2, \gamma_1 \rangle, \\ &\delta_E > d_{\mathcal{W}}(u) \sim \|v\|_{\dot{H}^1} \sim \|v\|_E, \quad (\theta, \sigma) = (\tilde{\theta}(u), \tilde{\sigma}(u)), \quad \dot{\tau} = e^{2\sigma}. \end{aligned} \tag{3.16}$$

The next proposition, proved in Section 7, ensures the existence of a solution u of (1.1) in a neighborhood of $\mathcal{W}$ as long as the scaling parameter σ is bounded from above.

Proposition 3.5 (Uniform local existence in τ)**.** *There exists an absolute constant $\delta_L \in (0, \delta_E/2)$ such that for any solution u of (1.1) with $d_{\mathcal{W}}(u(0)) =: \delta \in [0, 2\delta_L]$, we have $T_\pm(u) > 3e^{-2\sigma(0)} =: T_0$, $\pm(\tau(\pm T_0) - \tau(0)) > 2$, and for $|t| \le T_0$,*

$$\delta_E > d_{\mathcal{W}}(u(t)) \sim \delta, \quad \sigma(t) = \sigma(0) + O(\delta). \tag{3.17}$$

Now we are ready to define the nonlinear distance $\tilde{d}_{\mathcal{W}}$. Let $\varphi \in B_{\delta_E}(\mathcal{W})$. Consider the decomposition (3.1) of φ. Then we define a local distance $d_0 : B_{\delta_E}(\mathcal{W}) \to [0, \infty)$ by

$$d_0(\varphi)^2 := E(\varphi) - E(W) + 2\mu\lambda_1^2. \tag{3.18}$$

As observed in [19], this is close to be convex in τ when the solution is ejected out of a small neighborhood of $\mathcal{W}$, but it may have small oscillation around minima in τ. This is a difference for the Schrödinger equation from the Klein-Gordon equation, for which d_0^2 is strictly convex (see [17]). We could treat the possible oscillation as in [19] by waiting for a short time before the exponential instability dominates, which would however bring a certain amount of complication to the statements as well as the proof.

Here instead, we introduce a dynamical mollification of d_0^2, which yields a strictly convex function in τ. The same argument works in the subcritical setting as in [19]. Let u be the solution of (1.1) with initial data $u(0) := \varphi \in B_{2\delta_L}(\mathcal{W})$. Then Proposition 3.5 ensures that u exists at least for $|\tau - \tau(0)| \le 2$ in $B_{\delta_E}(\mathcal{W})$. Using the decomposition (3.16) with $\tau(0) := 0$, let

$$d_1(\varphi)^2 := \int_{\mathbb{R}} \phi(\tau) d_0(u)^2 d\tau, \tag{3.19}$$

where ϕ is the cut-off function in (2.2). This defines the function $d_1 : B_{2\delta_L}(\mathcal{W}) \to [0, \infty)$. Then, we define the nonlinear distance function $\tilde{d}_{\mathcal{W}} : \dot{H}^1_{\text{radial}} \to [0, \infty)$ by

$$\tilde{d}_{\mathcal{W}}(\varphi) := \phi_{\delta_L}(d_{\mathcal{W}}(\varphi)) d_1(\varphi) + \phi^C_{\delta_L}(d_{\mathcal{W}}(\varphi)) d_{\mathcal{W}}(\varphi). \tag{3.20}$$

The following proposition, proved in Section 8, gives the main static properties of the distance function.

Proposition 3.6 (Nonlinear distance function)**.** *The functional $\tilde{d}_{\mathcal{W}}$ on $\dot{H}^1_{\text{radial}}$ is invariant for the rotation and scaling, and equivalent to $d_{\mathcal{W}}$. Precisely, there exists an absolute constant $C \in (1, \infty)$ such that for all $\varphi \in \dot{H}^1_{\text{radial}}$ and $(\alpha, \beta) \in \mathbb{R}^2$,*

$$d_{\mathcal{W}}(\varphi)/C \le \tilde{d}_{\mathcal{W}}(\varphi) = \tilde{d}_{\mathcal{W}}(e^{i\alpha} S^\beta_{-1} \varphi) \le C d_{\mathcal{W}}(\varphi). \tag{3.21}$$

Moreover, there exists an absolute constant $c_D \in (0,1)$ *such that putting*

$$\check{\mathcal{H}} := \{\varphi \in \dot{H}^1_{\mathrm{radial}} \mid E(\varphi) < E(W) + (c_D \tilde{d}_{\mathcal{W}}(\varphi))^2\} \tag{3.22}$$

we have

$$\varphi \in B_{\delta_L}(\mathcal{W}) \cap \check{\mathcal{H}} \implies \tilde{d}_{\mathcal{W}}(\varphi) \sim |\lambda_1|. \tag{3.23}$$

Hence we can use $\tilde{d}_{\mathcal{W}}(\varphi)$ to measure the distance to $\mathcal{W}$, instead of the standard $d_{\mathcal{W}}(\varphi)$. The δ neighborhood with respect to this distance function is denoted by

$$\tilde{B}_\delta(\mathcal{W}) := \{\varphi \in \dot{H}^1_{\mathrm{radial}} \mid \tilde{d}_{\mathcal{W}}(\varphi) < \delta\}. \tag{3.24}$$

The next proposition, proved in Section 9, describes the dynamics close to the ground states in the ejection mode:

Proposition 3.7 (Dynamics in the ejection mode)**.** *There is an absolute constant* $\delta_X \in (0,1)$ *such that* $\tilde{B}_{\delta_X}(\mathcal{W}) \subset B_{\delta_L}(\mathcal{W})$*, and that for any solution* u *of* (1.1) *with*

$$u(t_0) \in \tilde{B}_{\delta_X}(\mathcal{W}) \cap \check{\mathcal{H}} \quad \text{and} \quad \partial_t \tilde{d}_{\mathcal{W}}(u(t_0)) \geq 0, \tag{3.25}$$

at some $t_0 \in I(u)$*, we have the following.* $\tilde{d}_{\mathcal{W}}(u(t))$ *is increasing until it reaches* δ_X *at some* $t_X \in (t_0, T_+(u))$*. For all* $t \in [t_0, t_X]$*, we have*

$$\tilde{d}_{\mathcal{W}}(u(t)) \sim |\lambda_1(t)| \sim e^{\mu(\tau(t)-\tau(t_0))}\tilde{d}_{\mathcal{W}}(u(t_0)), \tag{3.26}$$

$$\|\gamma(t)\|_{\dot{H}^1} \lesssim \tilde{d}_{\mathcal{W}}(u(t_0)) + \tilde{d}_{\mathcal{W}}(u(t))^2, \tag{3.27}$$

$$|(e^{i\theta(t)} - e^{i\theta(t_0)}, \sigma(t) - \sigma(t_0))| \lesssim \tilde{d}_{\mathcal{W}}(u(t)), \tag{3.28}$$

$\operatorname{sign}(\lambda_1(t))$ *is constant, and there exists an absolute constant* $C_K > 0$ *such that*

$$-\operatorname{sign}(\lambda_1(t))K(u(t)) \gtrsim (e^{\mu(\tau(t)-\tau(t_0))} - C_K)\tilde{d}_{\mathcal{W}}(u(t_0)). \tag{3.29}$$

Remark 3.8. *By time-reversal symmetry*[1]*, a similar result holds in the negative time direction, where the last condition of* (3.25) *is replaced with* $\partial_t \tilde{d}_{\mathcal{W}}(u(t)) \leq 0$*.*

The next proposition gives a variational estimate away from the ground states, and it is a consequence of (2.8), cf. [13, 20].

Proposition 3.9 (Variational estimates)**.** *There exist two increasing functions* ϵ_V *and* κ *from* $(0,\infty)$ *to* $(0,1)$*, and an absolute constant* $c_V > 0$*, such that for any* $\varphi \in \dot{H}^1_{\mathrm{radial}}$ *satisfying* $E(\varphi) < E(W) + \epsilon_V(\tilde{d}_{\mathcal{W}}(\varphi))^2$*, we have*

$$K(\varphi) \geq \min(\kappa(\tilde{d}_{\mathcal{W}}(\varphi)), c_V\|\nabla\varphi\|^2_{L^2}) \text{ or } K(u) \leq -\kappa(\tilde{d}_{\mathcal{W}}(\varphi)). \tag{3.30}$$

The next proposition, proved in Section 10, defines a functional Θ that decides the fate of the solution around $t = T_\pm(u)$, as well as at the exit time $t = t_X$ in the above proposition.

Proposition 3.10 (Sign functional)**.** *There exist an absolute constant* $\epsilon_S \in (0,1)$ *and a continuous function* $\Theta : \mathcal{H}^{\epsilon_S} \cap \check{\mathcal{H}} \to \{\pm 1\}$*, such that for some* $0 < \delta_1 < \delta_2 < \delta_X$ *and for any* $\varphi \in \mathcal{H}^{\epsilon_S} \cap \check{\mathcal{H}}$*, with the convention* $\operatorname{sign} 0 = +1$*,*

$$\begin{cases} \tilde{d}_{\mathcal{W}}(\varphi) \geq \delta_1 \implies \Theta(\varphi) = \operatorname{sign} K(\varphi), \\ \tilde{d}_{\mathcal{W}}(\varphi) \leq \delta_2 \implies \Theta(\varphi) = -\operatorname{sign}\lambda_1, \end{cases} \tag{3.31}$$

[1]If $u(t,x)$ is a solution of (1.1), then $\overline{u(-t,x)}$ is also a solution of (1.1).

and $\Theta(e^{i\alpha}S_{-1}^{\beta}\varphi)=\Theta(\varphi)$ *for all* $(\alpha,\beta)\in\mathbb{R}^2$. *Moreover, if* $E(\varphi)<E(W)$ *then* $\Theta(\varphi)=\operatorname{sign}K(\varphi)$.

Note that the region $\{\varphi\in\mathcal{H}^{\epsilon_S}\cap\check{\mathcal{H}}\mid\Theta(\varphi)=+1\}$ is bounded in $\dot{H}^1$, because

$$\begin{cases}K(\varphi)\ge 0\implies\|\nabla\varphi\|_2^2\le 3E(\varphi)\le 3(E(W)+\epsilon_S^2),\\ \tilde{d}_{\mathcal{W}}(\varphi)\le\delta_X\implies\|\varphi\|_{\dot{H}^1}\le\|W\|_{\dot{H}^1}+C\delta_X,\end{cases}\tag{3.32}$$

but it does not imply global existence for solutions staying in this region, because of the critical nature of (1.1).

The continuity of Θ implies that for any solution u in $\mathcal{H}^{\epsilon}\subset\mathcal{H}^{\epsilon_S}$, $\Theta(u)\in\{\pm1\}$ can change along $t\in I(u)$ only if u goes through the small neighborhood $\mathcal{H}^{\epsilon}\setminus\check{\mathcal{H}}\subset\tilde{B}_{\epsilon/c_D}(\mathcal{W})$. The next proposition, proved in Section 11, implies that such a transition can happen at most once for each solution.

Proposition 3.11 (One-pass)**.** *There exist an absolute constant* $\delta_B\in(0,\delta_X)$ *and an increasing function* $\epsilon_B:(0,\delta_B]\to(0,\epsilon_S]$ *satisfying* $\epsilon_B(\delta)<c_D\delta$ *for* $\delta\in(0,\delta_B]$, *and for any solution* u *of* (1.1) *with* $u(t_0)\in\mathcal{H}^{\epsilon_B(\delta)}\cap\tilde{B}_\delta(\mathcal{W})$ *at some* $t_0\in I(u)$,

$$\exists t_+(\delta)\in(t_0,T_+(u)],\ s.t.\ \begin{cases}t_0\le t<t_+(\delta)\implies\tilde{d}_{\mathcal{W}}(u(t))<\delta,\\ t_+(\delta)<t<T_+(u)\implies\tilde{d}_{\mathcal{W}}(u(t))>\delta.\end{cases}\tag{3.33}$$

Remark 3.12. *By time-reversal symmetry, there also exists* $t_-\in[-T_-(u),t_0)$ *such that* $\tilde{d}_{\mathcal{W}}(u(t))<\delta$ *for* $t_-<t<t_0$ *and* $\tilde{d}_{\mathcal{W}}(u(t))>\delta$ *for* $-T_-(u)<t<t_-$.

Remark 3.13. $\epsilon_B(\delta)\le\epsilon_S$ *and* $\epsilon_B(\delta)<c_D\delta$ *imply that* $\mathcal{H}^{\epsilon_B(\delta)}\setminus\tilde{B}_\delta(\mathcal{W})\subset\mathcal{H}^{\epsilon_S}\cap\check{\mathcal{H}}$, *where* Θ *is defined by Proposition 3.10.*

The above proposition tells that if a solution gets out of $\tilde{B}_\delta(\mathcal{W})$, then it can never return there. Moreover, it applies to all $\delta\in(0,\delta_B]$ satisfying $E(u)<E(W)+\epsilon_B(\delta)^2$. The solution u stays around $\mathcal{W}$ iff $t_+(\delta)=T_+(u)$. The following proposition, proved in Section 12, gives more precise description of such solutions.

Proposition 3.14. *Under the assumption of Proposition 3.11, suppose that* $t_+(\delta)=T_+(u)$. *Then there exists* $t_1\in[t_0,T_+(u)]$ *such that* $\tilde{d}_{\mathcal{W}}(u(t))$ *is decreasing on* $[t_0,t_1)$, *and* $u(t)\notin\check{\mathcal{H}}$ *for all* $t\in[t_1,T_+(u))$. *If* $t_1=T_+(u)$, *then* $\tilde{d}_{\mathcal{W}}(u(t))\searrow 0$ *as* $t\nearrow T_+(u)$, *which implies* $E(u)=E(W)$. *We have similar statements in the case* $t_-(\delta)=T_-(u)$ *by the time-reversal symmetry.*

The next proposition, proved in Section 13, describes the asymptotic behavior of solutions which are away from the ground states.

Proposition 3.15 (Asymptotic behavior)**.** *There exists an increasing function* $\epsilon_*:(0,\delta_B]\to(0,\epsilon_S]$ *with* $\epsilon_*(\delta)\le\epsilon_B(\delta)$ *for* $\delta\in(0,\delta_B)$ *and the following properties. Suppose that* u *is a solution of* (1.1) *satisfying* $u([t_0,T_+(u)))\subset\mathcal{H}^{\epsilon_*(\delta)}\setminus\tilde{B}_\delta(\mathcal{W})$ *for some* $t_0\in I(u)$. *If* $\Theta(u(t))=+1$ *at some* $t\in[t_0,T_+(u))$, *then* $T_+(u)=\infty$, *and* u *scatters as* $t\to\infty$. *If* $\Theta(u(t))=-1$ *and* $u(t)\in L_x^2$ *at some* $t\in[t_0,T_+(u))$, *then* $T_+(u)<\infty$. *By time-reversal symmetry, the same statements hold for the negative time direction* $(-T_-(u),t_0]$.

Note that by Remark 3.13 and $\epsilon_*\le\epsilon_B$, $\Theta(u(t))$ is well defined for all $t\in[t_0,T_+(u))$ in the above statement.

Armed with the above propositions, we are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Using $c_D, \delta_B, \epsilon_*$ in Propositions 3.6, 3.11 and 3.15, define

$$\epsilon_\star := \epsilon_*(\delta_B) > 0, \quad \mathcal{X}_\epsilon := \{\varphi \in \mathcal{H}^\epsilon \mid c_D \tilde{d}_{\mathcal{W}}(\varphi) \le \epsilon\}. \tag{3.34}$$

Then $\mathcal{X}_\epsilon$ is relatively closed in $\mathcal{H}^\epsilon$, and $\mathcal{W} \subset \mathcal{X}_\epsilon \subset \tilde{B}_{2\epsilon/c_D}(\mathcal{W})$. Since $\epsilon_\star \le \epsilon_S$ and

$$\varphi \in \mathcal{H}^\epsilon \setminus \mathcal{X}_\epsilon \implies E(\varphi) - E(W) < \epsilon^2 < (c_D \tilde{d}_W(\varphi))^2 \implies \varphi \in \check{\mathcal{H}}, \tag{3.35}$$

the functional Θ is defined by Proposition 3.10 on $\mathcal{H}^\epsilon \setminus \mathcal{X}_\epsilon$ for all $\epsilon \in (0, \epsilon_\star]$.

Next we consider the dynamics. Let $\epsilon \in (0, \epsilon_\star]$ and let $u \in \mathcal{H}^\epsilon$ be a solution of (1.1). Let $I_0(u)$ be as in the theorem and let $I_C(u) := I(u) \setminus I_0(u)$.

Take any $\delta \in (0, \delta_B]$ satisfying $\epsilon \le \epsilon_*(\delta)$. First suppose that there exists $t_0 \in I_C(u)$ such that

$$\tilde{d}_{\mathcal{W}}(u(t_0)) < \delta, \quad \partial_t \tilde{d}_{\mathcal{W}}(u(t_0)) \ge 0. \tag{3.36}$$

Then by Proposition 3.7, $\tilde{d}_{\mathcal{W}}(u(t))$ is increasing until it reaches δ_X at some $t_X \in (t_0, T_+(u))$. Since $\tilde{d}_{\mathcal{W}}(u(t_0)) < \delta < \delta_X$, there exists $t' \in (t_0, t_X)$ such that $\tilde{d}_{\mathcal{W}}(u(t')) = \delta$. Then Proposition 3.11 implies that $\tilde{d}_{\mathcal{W}}(u(t)) > \delta$ for all $t \in (t', T_+(u))$. Hence $\tilde{d}_{\mathcal{W}}(u(t)) \ge \tilde{d}_{\mathcal{W}}(u(t_0))$ for all $t \in [t_0, T_+(u))$, which implies $[t_0, T_+(u)) \subset I_C(u)$. Then by Proposition 3.15, $\Theta(u(t))$ on $[t_0, T_+(u))$ decides the behavior of u towards $T_+(u)$.

If the last condition of (3.36) is replaced with $\partial_t \tilde{d}_{\mathcal{W}}(u(t_0)) \le 0$, then the time reversed version of the above argument implies that $(-T_-(u), t_0) \subset I_C(u)$ and the behavior of u towards $-T_-(u)$ is determined by $\Theta(u(t))$ there.

Next consider the case where there exist $t_1 \in I_0(u)$ and $t_2 \in I_C(u)$. Suppose that $t_1 < t_2$. Since $\tilde{d}_{\mathcal{W}}(u) \le \epsilon/c_D < \delta$ on $I_0(u)$, we may assume $\tilde{d}_{\mathcal{W}}(u(t_2)) < \delta$ by decreasing t_2 if necessary. Then the above argument works with $t_0 := t_2$, either forward or backward in time, but the latter case leads to a contradiction with the existence of $t_1 \in I_0(u)$ smaller than t_2. Hence we have $\partial_t \tilde{d}_{\mathcal{W}}(u(t_2)) \ge 0$ and $[t_2, T_+(u)) \subset I_C(u)$. If $t_1 > t_2$, then in the same way, we deduce that $(-T_-(u), t_2] \subset I_C(u)$. Therefore, $I_0(u)$ is either empty or an interval.

Concerning the behavior of u towards $T_+(u)$, it only remains to consider the following case: $I(u) = I_C(u)$ but (3.36) is never satisfied by any $t_0 \in I(u)$. In this case, there are only two possibilities: either $\tilde{d}_{\mathcal{W}}(u(t)) \ge \delta$ all over $I(u)$, or $\tilde{d}_{\mathcal{W}}(u(t))$ goes below δ and then stays there. In the former case, we can apply Proposition 3.15 to decide the behavior around $T_\pm(u)$. In the latter case, we can apply Proposition 3.14 on some interval $[t_0, T_+(u))$ where $\tilde{d}_{\mathcal{W}}(u) < \delta$. Then

$$\limsup_{t \nearrow T_+(u)} c_D \tilde{d}_{\mathcal{W}}(u(t)) \le \sqrt{E(u) - E(W)} < \epsilon, \tag{3.37}$$

contradicting $I(u) = I_C(u)$. This completes the investigation around $T_+(u)$, and the behavior towards $-T_-(u)$ is treated in the same way. Theorem 1.1 is proved. □

Remark 3.16. *The same argument as above works if we replace X_ϵ with*

$$\tilde{X}_\epsilon := \mathcal{H}^\epsilon \setminus \check{\mathcal{H}} = \{\varphi \in \mathcal{H}^\epsilon \mid E(u) \ge E(W) + (c_D \tilde{d}_{\mathcal{W}}(u))^2\}, \tag{3.38}$$

which is smaller and essentially independent of ϵ. In that case, however, we need to modify our conclusion for the special solutions $W^\pm$ constructed by Duyckaerts and Merle [6] *on the threshold $E(u) = E(W)$, namely those two solutions (unique modulo the invariance) which are exponentially convergent to W as $t \to \infty$, and scattering or blowing up in $t < 0$. These solutions are in $\mathcal{H}^\epsilon \cap \check{\mathcal{H}}$ for all $t \in I(u)$ and*

$\epsilon > 0$, where $\Theta = \pm 1$ according to its behavior in $t < 0$. Thus Θ fails to give the correct prediction for $t > 0$ in this case. This is exactly the case $t_0 = t_1 = T_+(u)$ in Proposition 3.14, namely $\tilde{d}_{\mathcal{W}}(u) \searrow 0$ as $t \nearrow T_+(u)$. The classification in [6] also implies that it happens only for those special solutions. In other words, X_ϵ has been enlarged from $\tilde{X}_\epsilon$ in order to eliminate those solutions.

4. Orthogonal decomposition

In this section, we prove Proposition 3.1. Define a C^1 function $F : \mathbb{R} \times \mathbb{R} \times \dot{H}^1_{\text{radial}} \to \mathbb{C}$ by

$$F(\theta, \sigma, \psi) := (e^{-i\theta} S_{-1}^{-\sigma}(W + \psi) - W | \chi). \tag{4.1}$$

Since $F(0,0,0) = 0$ and (writing $F = F_1 + iF_2 = (F_1, F_2) \in \mathbb{R}^2$)

$$\partial_{\theta,\sigma} F(0,0,0) = \begin{pmatrix} \partial_\theta F_1(0,0,0) & \partial_\sigma F_1(0,0,0) \\ \partial_\theta F_2(0,0,0) & \partial_\sigma F_2(0,0,0) \end{pmatrix} = \begin{pmatrix} 0 & -\langle W', \chi \rangle \\ -\langle W, \chi \rangle & 0 \end{pmatrix}, \tag{4.2}$$

the implicit function theorem yields $\delta > 0$ and a C^1 function $(\theta, \sigma) : B_\delta(0) \to \mathbb{R}^2$, where $B_\delta(0)$ denotes the δ neighborhood of 0 in $\dot{H}^1_{\text{radial}}$, such that $F(\theta(\psi), \sigma(\psi), \psi) = 0$ and $\theta(0) = \sigma(0) = 0$, which is unique in $B_\delta(0)$ and a neighborhood of $0 \in \mathbb{R}^2$. For any $\varphi \in B_\delta(\mathcal{W})$, there exists $(\alpha, \beta, \psi) \in (\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R} \times B_\delta(0)$ such that $\varphi = e^{i\alpha} S_{-1}^{\beta}(W + \psi)$. Then we put

$$\tilde{\theta}(\varphi) := \theta(\psi) + \alpha \in \mathbb{R}/2\pi\mathbb{Z}, \quad \tilde{\sigma}(\varphi) := \sigma(\psi) + \beta \in \mathbb{R}. \tag{4.3}$$

Then defining v by (3.1), we have $v = e^{-i\theta(\psi)} S_{-1}^{-\sigma(\psi)}(W + \psi) - W$ and so

$$\begin{aligned} &(v|\chi) = F(\theta(\psi), \sigma(\psi), \psi) = 0, \\ &\|v\|_{\dot{H}^1} \lesssim |\theta(\psi)| + |\sigma(\psi)| + \|\psi\|_{\dot{H}^1} \lesssim \|\psi\|_{\dot{H}^1}. \end{aligned} \tag{4.4}$$

This implies (3.3), as well as $d_{\mathcal{W}}(\varphi) \sim \|v\|_{\dot{H}^1}$, choosing (α, β) such that $d_{\mathcal{W}}(\varphi) \sim \|\varphi - e^{i\alpha} S_{-1}^{\beta} W\|_{\dot{H}^1}$.

To see the uniqueness of $(\tilde{\theta}, \tilde{\sigma})$ for each $\varphi \in B_\delta(\mathcal{W})$, suppose that we have two ways of decomposition

$$\varphi = e^{i\alpha_1} S_{-1}^{\beta_1}(W + v) = e^{i\alpha_2} S_{-1}^{\beta_2}(W + v') \tag{4.5}$$

with (3.2) for both v and v', then putting $(\alpha, \beta) := (\alpha_2, \beta_2) - (\alpha_1, \beta_1)$,

$$0 = (v - v'|\chi) = ((e^{i\alpha} S_{-1}^{\beta} - 1)(W + v')|\chi). \tag{4.6}$$

Since $W \in \dot{H}^1$ and $\chi \in \dot{H}^{-1}$, we have

$$\|(e^{i\alpha} S_{-1}^{\beta} - 1) W\|_{\dot{H}^1} \sim \min(|e^{i\alpha} - 1| + |\beta|, 1) \gtrsim \|(e^{i\alpha} S_1^{\beta} - 1)\chi\|_{\dot{H}^{-1}}. \tag{4.7}$$

Then by (4.6), we obtain

$$0 = (1 + O(\|v'\|_{\dot{H}^1}))(|e^{i\alpha} - 1| + |\beta|), \tag{4.8}$$

hence the uniqueness of $(\tilde{\theta}, \tilde{\sigma})$. $\square$

5. Evolution around the ground states

In this section, we prove Proposition 3.2 and Proposition 3.3.

First we show that we can normalize g_+ and g_- such that (3.5) holds. Since $L_- \geq 0$ with $\mathrm{Ker}(L_-) = \mathrm{span}\{W\}$, we have

$$0 < c := \langle L_- g_2, g_2 \rangle = -\langle L_+ g_1, g_1 \rangle = \mu\omega(g_+, g_-)/2. \tag{5.1}$$

Hence it is enough to show $\langle W, g_2 \rangle \neq 0$ (see [19] for a similar argument). Suppose for contradiction that $\langle W, g_2 \rangle = 0$. Then $4\langle W^5, g_1 \rangle = -\langle L_+ W, g_1 \rangle = \langle W, \mu g_2 \rangle = 0$. Let $0 < \delta \ll 1$ and $v = \alpha W + \delta g_1$ with $\alpha = O(\delta^2)$ to be chosen shortly. By expansion of the energy

$$E(W+v) = E(W) + \frac{1}{2}\langle L_+ v, v \rangle + O(\|v\|_{\dot H^1}^3) < E(W) - \frac{c}{2}\delta^2 + O(\delta^3) \tag{5.2}$$

and by expansion of K

$$K(W+v) = -4\langle W^5, v \rangle + O(\|v\|_{\dot H^1}^2) = -4\alpha\|W\|_{L^6}^6 + O(\delta^2), \tag{5.3}$$

so that one can find $\alpha = O(\delta^2)$ such that $K(W+v) = 0$, which contradicts (2.8). Hence $\langle W, g_2 \rangle \neq 0$ and we can normalize $g_\pm$ such that (3.5) holds. Then (3.6) and (3.7) are immediate consequences. Thus we obtain Proposition 3.2.

Next, injecting the decomposition (3.8) into the equation (1.1), we obtain, after straightforward computations using that $W_{\theta,\sigma}$ is a real-valued stationary solution,

$$v_t = ie^{2\sigma}\mathcal{L}v - (i\theta_t + \sigma_t S'_{-1})(W+v) - ie^{2\sigma}N(v). \tag{5.4}$$

Applying the change of variable $t \mapsto \tau$ with $\dot\tau = e^{2\sigma}$ to the above yields (3.9).

Next we consider the equations for the parameters. Using (3.9), we have

$$\begin{aligned} \partial_\tau \lambda_+ = \omega(\partial_\tau v, g_-) &= \omega(i\mathcal{L}v, g_-) - \omega\left((i\theta_\tau + \sigma_\tau S'_{-1})W, g_-\right) \\ &\quad + \langle N(v), g_- \rangle - \omega\left((i\theta_\tau + \sigma_\tau S'_{-1})v, g_-\right). \end{aligned} \tag{5.5}$$

Using (2.12) and $\mathcal{L}^* = \mathcal{L}$, we see that

$$\begin{aligned} &\omega(i\mathcal{L}v, g_-) = \omega(i\mathcal{L}g_-, v) = \mu\lambda_+, \quad \omega\left((i\theta_\tau + \sigma_\tau S'_{-1})W, g_-\right) = 0, \\ &\omega\left((i\theta_\tau + \sigma_\tau S'_{-1})v, g_-\right) = -\theta_\tau\langle v, g_- \rangle - \sigma_\tau\langle iv, S'_1 g_- \rangle. \end{aligned} \tag{5.6}$$

Hence

$$\partial_\tau \lambda_+ = \mu\lambda_+ + \theta_\tau\langle v, g_- \rangle + \sigma_\tau\langle iv, S'_1 g_- \rangle + \langle N(v), g_- \rangle. \tag{5.7}$$

Similarly one finds that

$$\partial_\tau \lambda_- = -\mu\lambda_- - \theta_\tau\langle v, g_+ \rangle - \sigma_\tau\langle iv, S'_1 g_+ \rangle - \langle N(v), g_+ \rangle. \tag{5.8}$$

Next we differentiate with respect to τ the orthogonality condition $(v|\chi) = 0$. Then plugging (3.9) into v_τ, and using $(g_\pm, \chi) = 0$, we get

$$0 = (v_\tau, \chi) = (i\mathcal{L}\gamma, \chi) - i\theta_\tau\langle W, \chi \rangle - \sigma_\tau[\langle W', \chi \rangle - (v, S'_1\chi)] - (iN(v), \chi). \tag{5.9}$$

Hence, using that $\langle W, \chi \rangle \langle W', \chi \rangle \neq 0$, we obtain

$$\partial_\tau(\theta, \sigma) = O\left(\|\gamma\|_{\dot H^1} + \|v\|_{\dot H^1}^2\right). \tag{5.10}$$

Now plugging (5.10) into (5.7) and (5.8), we see that (3.11) holds. Thus we obtain Proposition 3.3. □

6. Control by the linearized energy

In this section, we prove Proposition 3.4. First we prove that, for any $f \in \dot{H}^1_{\text{radial}}$,

$$\langle f, g_2 \rangle = 0 \implies \langle L_+ f, f \rangle \geq 0. \tag{6.1}$$

If the above fails, then for all $(a, b) \in \mathbb{R}^2$,

$$\langle L_+(af + bg_1), af + bg_1 \rangle = a^2 \langle L_+ f, f \rangle - b^2 \langle L_- g_2, g_2 \rangle < 0. \tag{6.2}$$

So L_+ is negative on a two dimensional subspace, which contradicts the fact that L_+ has only one negative eigenvalue. Hence (6.1) holds.

We are now in position to prove (3.13). (3.13) with the $\gtrsim$ sign is obvious, so we only prove (3.13) with the $\lesssim$ sign. Assume for contradiction that this is false. Then there exists a sequence $\{w_n\}_{n\geq 1} \subset \dot{H}^1_{\text{radial}}$ such that $\|w_n\|_{\dot{H}^1} = 1$, $(w_n|\chi) \to 0$, $\langle w_{n,1}, g_2 \rangle = 0$ and

$$\langle L_+ w_{n,1}, w_{n,1} \rangle \to 0, \quad \langle L_- w_{n,2}, w_{n,2} \rangle \to 0, \tag{6.3}$$

as $n \to \infty$, where $w_n = w_{n,1} + i w_{n,2}$. Since w_n is bounded in $\dot{H}^1$, passing to a subsequence if necessary, we may assume that it is weakly converging to some $w \in \dot{H}^1_{\text{radial}}$. Then

$$\begin{aligned} &(w|\chi) = 0, \quad \langle w_1, g_2 \rangle = 0, \\ &\langle L_+ w_1, w_1 \rangle \leq 0, \quad \langle L_- w_2, w_2 \rangle \leq 0. \end{aligned} \tag{6.4}$$

Now (6.1) and $L_- \geq 0$ imply that $\langle L_+ w_1, w_1 \rangle = \langle L_- w_2, w_2 \rangle = 0$, and so

$$\|\nabla w_n\|^2_{L^2} \to \|\nabla w\|^2_{L^2}. \tag{6.5}$$

Moreover, w_1 and w_2 are minimizers for the quadratic forms under the constraint $\langle w_1, g_2 \rangle = 0$. Hence $L_+ w_1 = L_- w_2 = 0$, which implies

$$w_1 \in \operatorname{span}\{W'\}, \quad w_2 \in \operatorname{span}\{W\}. \tag{6.6}$$

Then by (2.19), we see that $w = 0$, which contradicts (6.5) and $\|w_n\|_{\dot{H}^1} = 1$. □

7. Uniform local existence

In this section, we prove Proposition 3.5. Let $u(0) \in B_\delta(\mathcal{W})$ for some constant $\delta \in (0, \delta_E)$, whose smallness will be required in the following. By rotation and scaling, we may reduce to the case $\theta(0) = \sigma(0) = 0$. Since W is a global solution, it is a consequence of the local wellposedness in $\dot{H}^1$ that for $\|u(0) - W\|_{\dot{H}^1} = d_{\mathcal{W}}(u(0)) =: \delta$ small enough, the solution u exists and remains in $O(\delta)$ neighborhood of W for $|t| \leq 3$. Since we can solve the equation back to $t = 0$ as well, it implies that $d_{\mathcal{W}}(u(t)) \sim d_{\mathcal{W}}(u(0)) = \delta$ for $|t| \leq 3$. Then by Proposition 3.3, we have

$$\dot{\tau} = e^{2\sigma}, \quad \dot{\sigma} = \dot{\tau}\sigma_\tau = e^{2\sigma} O(\delta), \tag{7.1}$$

and $\sigma(0) = 0$. Hence $e^{-2\sigma} = 1 + O(\delta)$ and $\tau = \tau(0) + (1 + O(\delta))t$ for $|t| \leq 3$. In particular, $\tau(t)$ reaches $\tau(0) \pm 2$ within $I(u)$, if $\delta > 0$ is small enough. □

8. Nonlinear distance function

In this section, we prove Proposition 3.6.

First we prove $\tilde{d}_{\mathcal{W}} \sim d_{\mathcal{W}}$. Since $\tilde{d}_{\mathcal{W}} = d_{\mathcal{W}}$ for $d_{\mathcal{W}}(\varphi) \ge 2\delta_L$, it suffices to consider the case $d_{\mathcal{W}}(\varphi) \le 2\delta_L$. Decompose φ by Propositions 3.1 and 3.2. Then we have

$$E(\varphi) - E(W) = \frac{1}{2}\langle \mathcal{L}v, v\rangle - C(v) = -\mu\lambda_+\lambda_- + \frac{1}{2}\langle \mathcal{L}\gamma, \gamma\rangle - C(v), \tag{8.1}$$

where we used (3.5). Here $C(\cdot)$ denotes the following functional on $\dot{H}^1$:

$$C(v) := \frac{1}{6}\|W+v\|_{L^6}^6 - \sum_{k=0}^{2} \frac{1}{k!}\partial_\lambda^k \frac{1}{6}\|W+\lambda v\|_{L^6}^6\Big|_{\lambda=0} = O(\|v\|_{\dot{H}^1}^3), \tag{8.2}$$

whose Fréchet derivative is $N(v)$. Hence, using Proposition 3.4,

$$d_0(\varphi)^2 = \|v\|_E^2 - C(v) = \|v\|_E^2 + O(\|v\|_{\dot{H}^1}^3) \sim d_{\mathcal{W}}(\varphi)^2. \tag{8.3}$$

Then by Proposition 3.5, we have $d_1(\varphi) \sim d_{\mathcal{W}}(\varphi)$, and so $\tilde{d}_{\mathcal{W}}(\varphi) \sim d_{\mathcal{W}}(\varphi)$.

Next we prove (3.23). If $d_{\mathcal{W}}(\varphi) < \delta_L$ and $E(\varphi) - E(W) < (c_D d_{\mathcal{W}}(\varphi))^2$, then

$$\tilde{d}_{\mathcal{W}}(\varphi)^2 \sim d_0(\varphi)^2 = E(\varphi) - E(W) + 2\mu\lambda_1^2, \tag{8.4}$$

and so $\tilde{d}_{\mathcal{W}}(\varphi)^2 \lesssim \lambda_1^2$. From (3.14) we see that $\tilde{d}_{\mathcal{W}}(\varphi)^2 \sim \|v\|_E^2 \gtrsim \lambda_1^2$.

Finally, we check the invariance for the rotation and scaling. Let $(\alpha, \beta) \in \mathbb{R}^2$, $\varphi \in B_{\delta_E}(\mathcal{W})$ and let u and u' be the solutions of (1.1) with the initial data

$$u(0) = \varphi, \quad u'(0) = e^{i\alpha} S_{-1}^{\beta}\varphi, \tag{8.5}$$

with the decompositions by Proposition 3.1 and the rescaled time functions

$$\begin{aligned} &u = e^{i\theta} S_{-1}^{\sigma}(W+v), \quad u' = e^{i\theta'} S_{-1}^{\sigma'}(W+v'), \\ &\dot{\tau} = e^{2\sigma}, \quad \dot{\tau}' = e^{2\sigma'}, \quad \tau(0) = 0 = \tau'(0). \end{aligned} \tag{8.6}$$

Then the uniqueness of $(\tilde{\theta}, \tilde{\sigma})$ in Proposition 3.1 implies

$$(\theta', \sigma') = (\theta, \sigma) + (\alpha, \beta), \quad \tau' = \tau e^{2\beta}, \tag{8.7}$$

while the invariance of the equation (1.1) implies $u'(t) = e^{i\alpha} S_{-1}^{\beta} u(e^{2\beta}t)$. Hence v is invariant in the rescaled time, namely

$$\tau(t) = \tau'(t') \implies v(t) = v'(t'), \tag{8.8}$$

which is inherited by λ_* and γ. Therefore d_0 and d_1 are invariant, so is $\tilde{d}_{\mathcal{W}}$.

9. Dynamics in the ejection mode

In this section, we prove Proposition 3.7. Let u be a solution in the ejection mode (3.25) at $t = t_0 \in I(u)$. Since $\delta_X < \delta_L$, Proposition 3.5 implies that either there is a minimal $\tau_X > \tau_0$ such that $\tilde{d}_{\mathcal{W}}(u) = \delta_X$ at $\tau = \tau_X$, or $\tilde{d}_{\mathcal{W}}(u) < \delta_X$ for all $\tau \in (\tau_0, \infty)$. Let $\tau_X := \infty$ in the latter case. Then in both the cases, we have $\tilde{d}_{\mathcal{W}}(u) < \delta_X$ for $\tau \in (\tau_0, \tau_X)$. Choosing δ_X small enough ensures that $\tilde{B}_{\delta_X}(\mathcal{W}) \subset B_{\delta_L}(\mathcal{W})$. Then $\tilde{d}_{\mathcal{W}}(u) = d_1(u) \sim |\lambda_1|$ on $\tau \in (\tau_0, \tau_X)$. Hence, using the definition (3.18)-(3.19) of d_1 and the equations (3.12) of λ_j,

$$\begin{aligned} &\partial_\tau \tilde{d}_{\mathcal{W}}(u)^2 = \phi * \partial_\tau 2\mu\lambda_1^2 = \phi * [4\mu^2\lambda_1\lambda_2 + O(\lambda_1^3)], \\ &\partial_\tau^2 \tilde{d}_{\mathcal{W}}(u)^2 = \phi * [4\mu^3(\lambda_1^2 + \lambda_2^2) + O(\lambda_1^3)] + \phi' * O(\lambda_1^3) \sim \lambda_1^2 \sim \tilde{d}_{\mathcal{W}}(u)^2, \end{aligned} \tag{9.1}$$

where we also used Proposition 3.5 to remove the convolution in the last step. Since $\partial_t\tilde{d}_{\mathcal{W}}(u(t_0)) \geq 0$, the last estimate implies that $\tilde{d}_{\mathcal{W}}(u)$ is strictly increasing for $\tau \in (\tau_0, \tau_X)$. It also implies exponential growth in τ of $\tilde{d}_{\mathcal{W}}$, so it is impossible to have the case $\tau_X = \infty$ above. In other words, there exists $T_X < T_+(u)$ such that $\tilde{d}_{\mathcal{W}}(u)$ reaches δ_X at $\tau = \tau_X = \tau(T_X)$. Since $\tilde{d}_{\mathcal{W}} \sim |\lambda_1|$ is positive continuous on (τ_0, τ_X), $\lambda_1(\tau)$ cannot change the sign. Let $\mathfrak{s} := \operatorname{sign}\lambda_1(\tau) \in \{\pm\}$ be its sign.

Next we show the more precise exponential behavior. Since $\partial_\tau\tilde{d}_{\mathcal{W}}(u)^2 \geq 0$ at $\tau = \tau_0$, there exists $\tau \in (\tau_0 - 2, \tau_0 + 2)$ where $\partial_\tau\lambda_1^2 \geq 0$, and so $\lambda_1\lambda_2 \gtrsim -|\lambda_1|^3$. Since $\partial_\tau(\lambda_1\lambda_2) \sim \lambda_1^2 \sim \lambda_1(\tau_0)^2$ for $|\tau - \tau_0| < 2$, there exists $\tau_1 \in (\tau_0, \tau_0 + 2)$ such that $\lambda_1(\tau_1)\lambda_2(\tau_1) \gtrsim -|\lambda_1(\tau_1)|^3$, or equivalently $\mathfrak{s}\lambda_2(\tau_1) \gtrsim -\lambda_1(\tau_1)^2$. Then $\mathfrak{s}\lambda_+(\tau_1) \geq |\lambda_1(\tau_1)|/2$ and $\mathfrak{s}\lambda_-(\tau_1) \geq 0$. Let $\tilde{R} := |\lambda_1(\tau_1)|$, and suppose that for some $\tau_2 \in (\tau_1, \tau_X)$

$$\tau_1 < \tau < \tau_2 \implies |\lambda_1(\tau)| \leq 2\tilde{R}e^{\mu(\tau-\tau_1)} \lesssim \delta_X. \tag{9.2}$$

Then the equations (3.11) of $\lambda_\pm$ together with $\|v\|_{\dot{H}^1} \sim |\lambda_1|$ imply for $\tau \in (\tau_1, \tau_2)$,

$$|\lambda_\pm(\tau) - e^{\pm\mu(\tau-\tau_1)}\lambda_\pm(\tau_1)| \lesssim \tilde{R}^2 e^{2\mu(\tau-\tau_1)} \lesssim \delta_X\tilde{R}e^{\mu(\tau-\tau_1)}, \tag{9.3}$$

and so

$$|\lambda_1| = \frac{\mathfrak{s}}{2}(\lambda_+ + \lambda_-)\begin{cases} \leq (1 + C\delta_X)\tilde{R}e^{\mu(\tau-\tau_1)} < 2\tilde{R}e^{\mu(\tau-\tau_1)}, \\ \geq (1/4 - C\delta_X)\tilde{R}e^{\mu(\tau-\tau_1)} > \tilde{R}e^{\mu(\tau-\tau_1)}/5. \end{cases} \tag{9.4}$$

Hence the continuity in τ allows us to take $\tau_2 = \tau_X$. Moreover the above estimates together with $|\lambda_1| \sim \tilde{R}$ on (τ_0, τ_1) implies that, with $R := \tilde{d}_{\mathcal{W}}(u(\tau_0))$,

$$\tau_0 \leq \tau \leq \tau_X \implies \tilde{d}_{\mathcal{W}} \sim \mathfrak{s}\lambda_1 \sim Re^{\mu(\tau-\tau_0)}. \tag{9.5}$$

In order to estimate γ, consider the expansion of the energy (8.1) without the γ terms. We denote this expansion by $E_{\gamma^\perp}$:

$$E_{\gamma^\perp}(u) := -\mu\lambda_+\lambda_- - C(\lambda_+ g_+ + \lambda_- g_-) \tag{9.6}$$

Notice that $C'(f) = N(f)$. By this observation, (5.7) and (5.8) we see that

$$\begin{aligned} \partial_\tau E_{\gamma^\perp}(u) = {} & \langle N(v) - N(\lambda_+ g_+ + \lambda_- g_-), g_+\rangle\partial_\tau\lambda_+ \\ & + \langle N(v) - N(\lambda_+ g_+ + \lambda_- g_-), g_-\rangle\partial_\tau\lambda_- \\ & + \theta_\tau\left(\langle v, g_-\rangle\partial_\tau\lambda_- + \langle v, g_+\rangle\partial_\tau\lambda_+\right) \\ & + \sigma_\tau\left(\langle iv, S_1' g_-\rangle\partial_\tau\lambda_- + \langle iv, S_1' g_+\rangle\partial_\tau\lambda_+\right). \end{aligned} \tag{9.7}$$

This together with (5.10) implies that, for $\tau \in [\tau_0, \tau_X]$,

$$|\partial_\tau(E(u) - E(W) - E_{\gamma^\perp}(u))| \lesssim \lambda_1^2\|\gamma\|_{\dot{H}^1} + \lambda_1^4. \tag{9.8}$$

Moreover, using the elementary inequality $|N(f)| \lesssim \max(W^3|f|^2, |f|^5)$, Sobolev and Hölder, we see that

$$|C(v) - C(\lambda_+ g_+ + \lambda_- g_-)| \lesssim |\lambda_1|^2\|\gamma\|_{\dot{H}^1}. \tag{9.9}$$

Now, by Proposition 3.4

$$\begin{aligned} E(u) - E(W) - E_{\gamma^\perp}(u) &= \frac{1}{2}\langle\mathcal{L}\gamma, \gamma\rangle + C(\lambda_+ g_+ + \lambda_- g_-) - C(v) \\ &\sim \|\gamma\|_{\dot{H}^1}^2 + O(|\lambda_1|^2\|\gamma\|_{\dot{H}^1}), \end{aligned} \tag{9.10}$$

which implies, by (9.5) and (9.8), that (3.27) holds. Plugging (3.27) into (5.10) and then integrating in τ, we obtain (3.28) as well.

Next we show (3.29). Expanding K, and using $L_+W = -4W^5$, we have

$$\begin{aligned} K(W+v) &= -4\langle W^5, v\rangle + O(\|v\|_{\dot H^1}^2) \\ &= -2\mu\lambda_1\langle W, g_2\rangle - 4\langle W^5,\gamma\rangle + O(\|v\|_{H^1}^2). \end{aligned} \tag{9.11}$$

This, combined with $\langle W, g_2\rangle > 0$ (see (3.5)) as well as the above estimates on λ_1, γ, v, proves (3.29). $\square$

10. Sign functional

In this section, we prove Proposition 3.10. On one hand, Proposition 3.9 implies that $\operatorname{sign} K$ is constant on each connected component of $\mathcal{H}^\epsilon \setminus \tilde B_\delta(\mathcal{W})$, provided that $\epsilon \le \epsilon_V(\delta)$. On the other hand, Proposition 3.6 implies that $\operatorname{sign}\lambda_1$ is constant on each connected component of $B_{\delta_L}(\mathcal{W}) \cap \check{\mathcal{H}}$. Hence, after fixing $\delta_1, \delta_2, \epsilon_S$ such that $0 < \delta_1 < \delta_2 \ll \delta_X$ and $\epsilon_S \le \epsilon_V(\delta_1)$, the functional Θ is well-defined and continuous on $\check{\mathcal{H}}^{\epsilon_S} := \mathcal{H}^{\epsilon_S} \cap \check{\mathcal{H}}$ by (3.31), once we prove that $-\operatorname{sign}\lambda_1 = \operatorname{sign} K$ on

$$Y := \{\varphi \in \check{\mathcal{H}}^{\epsilon_S} \mid \delta_1 \le \tilde d_{\mathcal{W}}(\varphi) \le \delta_2\}. \tag{10.1}$$

To this end, take any solution u of (1.1) with initial data $u(0) \in Y$. By applying Proposition 3.7 either forward or backward in time, there exists $t_X \in I(u)$ such that $\tilde d_{\mathcal{W}}(u(t_X)) = \delta_X$ and $\tilde d_{\mathcal{W}}(u(t)) \in [\tilde d_{\mathcal{W}}(u(0)), \delta_X] \subset [\delta_1, \delta_X]$ between $t = 0$ and $t = t_X$, so $u(t)$ remains in $\check{\mathcal{H}}^{\epsilon_S} \cap B_{\delta_L}(\mathcal{W}) \setminus \tilde B_{\delta_1}(\mathcal{W})$. Hence $\operatorname{sign}\lambda_1(u(t))$ and $\operatorname{sign} K(u(t))$ are unchanged between $t = 0$ and $t = t_X$, whereas $-\operatorname{sign}\lambda_1(u(t_X)) = \operatorname{sign} K(u(t_X))$ by (3.29) with $\delta_2 \ll \delta_X$. Therefore $-\operatorname{sign}\lambda_1 = \operatorname{sign} K$ on Y.

Finally we prove that $\Theta = \operatorname{sign} K$ on $\mathcal{H}^0$. Since $0 \in \check{\mathcal{H}}$ is away from $\mathcal{W}$, we have $\Theta(0) = \operatorname{sign} K(0) = +1$. If $E(u) < E(W)$ and $u \ne 0$, then $K(u) \ne 0$ by (2.8). By

$$\lambda\partial_\lambda E(\lambda u) = K(\lambda u) = \lambda^2\|\nabla u\|_{L^2}^2 - \lambda^6\|u\|_{L^6}^6, \tag{10.2}$$

there is a unique $\lambda_0 > 0$ such that for $0 < \lambda_- < \lambda_0 < \lambda_+ < \infty$

$$K(\lambda_- u) > 0 = K(\lambda_0 u) > K(\lambda_+ u). \tag{10.3}$$

If $K(u) > 0$, then $\{\lambda u\}_{0\le\lambda\le1}$ is a C^0 curve in $\mathcal{H}^0 \subset \check{\mathcal{H}}$ connecting u and 0. Hence by continuity $\Theta(u) = +1$. If $K(u) < 0$, then $\{\lambda u\}_{1\le\lambda<\infty}$ is a C^0 curve in $\mathcal{H}^0$ connecting u with the region $E(u) < 0$, where $\mathcal{W}$ is far and so $\Theta = \operatorname{sign} K = -1$. Hence by continuity $\Theta(u) = -1$. The invariance of Θ for the rotation and scaling follows from that of K and λ_1, the latter being proved in Section 8. $\square$

11. One-pass lemma

In this section we prove Proposition 3.11.

11.1. **Setting.** Let $\delta, \epsilon > 0$ and let u be a solution satisfying $u(t_0) \in \mathcal{H}^\epsilon \cap \tilde B_\delta(\mathcal{W})$ at some $t_0 \in I(u)$. The solution u is fixed for the rest of proof, so we denote for brevity,

$$\tilde d(t) := \tilde d_{\mathcal{W}}(u(t)). \tag{11.1}$$

We will define shortly the hyperbolic and the variational regions in $\mathcal{H}^\epsilon$. In order to distinguish them, we use small parameters $\delta_V, \delta_M \in (0, \delta_X]$, which will be fixed as absolute constants in the end. First we impose the following upper bounds on δ and ϵ

$$0 < \delta \ll \delta_V \ll \delta_M \le \delta_X, \quad 0 < \epsilon \le \min(\epsilon_S, \epsilon_V(\delta_V)), \quad \epsilon < c_D\delta. \tag{11.2}$$

Since $\epsilon \le \epsilon_S$ and $\epsilon < c_D\delta$, we have

$$\mathcal{H}^\epsilon \setminus \tilde{B}_\delta(\mathcal{W}) \subset \mathcal{H}^{\epsilon_S} \cap \check{\mathcal{H}}. \tag{11.3}$$

Put $t_a := \sup\{t_1 \in (t_0, T_+(u)) \mid t_0 < t \le t_1 \implies \tilde{d}(t) < \delta\}$. Since $\tilde{d}(t_0) < \delta$, we have $t_a \in (t_0, T_+(u)]$. If $t_a = T_+(u)$ then (3.33) holds with $t_+ = T_+(u)$. Hence we may assume without loss of generality that $t_a < T_+(u)$ and so $\tilde{d}(t_a) = \delta$. If $\{t \in (t_a, T_+(u)) \mid \tilde{d}(t) \le \delta\}$ is empty, then (3.33) holds with $t_+ = t_a$. If not, let $t_b := \inf\{t \in (t_a, T_+(u)) \mid \tilde{d}(t) \le \delta\}$. Applying Proposition 3.7 at $t = t_a$ implies $t_a < t_b$. Thus in the remaining case, we have

$$t_0 < \exists t_a < \exists t_b < T_+(u), \quad \tilde{d}(t_a) = \delta = \tilde{d}(t_b) = \min_{t \in [t_a, t_b]} \tilde{d}(t), \tag{11.4}$$

from which we will derive a contradiction for small $\delta > 0$ and for small $\epsilon > 0$ with $\delta-$dependent smallness.

The rest of proof concentrates on the interval $[t_a, t_b]$, where $u(t)$ stays in $\check{\mathcal{H}}$, and so $\Theta(u(t)) \in \{\pm 1\}$ is a constant, abbreviated by Θ in the following. $\tilde{d}(t_a) = \delta$ implies $E(u) = (1 + O(\delta^2))E(W) \gtrsim 1$.

11.2. **Hyperbolic and variational regions.** Let s be any local minimizer of the function $\tilde{d}(t)$ on $[t_a, t_b]$ such that $\tilde{d}(s) < \delta_V$. Then Proposition 3.7 from $t = s$, forward and backward in time if $s \notin \{t_a, t_b\}$, forward in time if $s = t_a$, and backward in time if $s = t_b$, yields a unique subinterval $I[s] \subset [t_a, t_b]$ such that

(1) $\tilde{d}(t) \sim -\Theta\lambda_1(t) \sim \tilde{d}(s)e^{\mu|\tau(t)-\tau(s)|}$ on $I[s]$,
(2) $\tilde{d}(t)^2$ is strictly convex as a function of τ on $I[s]$ with a minimum $< \delta_V$ at $t = s$,
(3) $\tilde{d}(t) = \delta_M$ on $\partial I[s] \setminus \{t_a, t_b\}$.

Let $\mathscr{L}$ be the set of those local minimum points. Since $\tilde{d}(t)$ ranges over $[\delta_V, \delta_M]$ between any pair of points in $\mathscr{L}$, its uniform continuity on $[t_a, t_b]$ implies that $\mathscr{L}$ is a finite set. Decompose the interval $[t_a, t_b]$ into the hyperbolic time I_H and the variational time I_V defined by the following

$$I_H := \bigcup_{s \in \mathscr{L}} I[s], \quad I_V := [t_a, t_b] \setminus I_H. \tag{11.5}$$

By the definition of $\mathscr{L}$ and $I[s]$, we have

$$t \in I_H \implies \delta \le \tilde{d}(t) \le \delta_M, \quad t \in I_V \implies \tilde{d}(t) \ge \delta_V. \tag{11.6}$$

In particular, the coordinates $\sigma, \theta, v, \lambda_+, \lambda_-, \lambda_1, \lambda_2, \gamma$ and τ are defined on I_H. Since u is fixed, we regard those as functions of $t \in I_H$ in the rest of proof.

The soliton size on I_H is measured by

$$m_H := \sup_{t \in I_H} e^{-\sigma(t)} \sim \max_{s \in \mathscr{L}} e^{-\sigma(s)}, \tag{11.7}$$

where the equivalence follows from (3.28) on each $I[s]$. The hyperbolic dynamics in τ on $I[s]$ together with the time scaling $\dot{\tau} = e^{2\sigma}$ implies that

$$e^{2\sigma(s)}|I[s]| \sim \log(\delta_M/\tilde{d}(s)) \in [\log(\delta_M/\delta_V), \log(\delta_M/\delta)]. \tag{11.8}$$

11.3. **Virial identity.** Now we consider a localized virial identity. For $m > 0$, put

$$\mathscr{V}_m(t) := \langle \phi_m u, ir\partial_r u\rangle. \tag{11.9}$$

Then from the equation (1.1), we obtain

$$\begin{aligned}\dot{\mathscr{V}}_m &= 2\langle |u_r|^2, \partial_r r\phi_m\rangle - \frac{1}{2}\langle |u|^2, \Delta(r\partial_r + 3)\phi_m\rangle - \frac{2}{3}\langle |u|^6, (r\partial_r + 3)\phi_m\rangle \\ &= 2K(\phi_m u) + 2\int_{m\le|x|\le 2m} \left(|u_r|^2\partial_r r\phi_m - |\partial_r(\phi_m u)|^2\right) dx \\ &\qquad - \frac{1}{2}\int_{m\le|x|\le 2m} |u|^2\Delta(r\partial_r + 3)\phi_m dx \\ &\qquad - \frac{2}{3}\int_{m\le|x|\le 2m} (|u|^6(r\partial_r + 3)\phi_m + 3|\phi_m u|^6)\, dx \\ &= 2K(\phi_m u) + O(E_m),\end{aligned} \tag{11.10}$$

where

$$E_m(t) := \int_{m\le|x|\le 2m} |\nabla u|^2 + |u/r|^2 + |u|^6\, dx. \tag{11.11}$$

Using the decomposition (3.8), (3.14), and Proposition 3.5, placing W_σ, $S^\sigma_{-1}v$ in L^6_x, and placing $\partial_r W_\sigma$, $\partial_r(S^\sigma_{-1}v)$ in L^2_x, we see that

$$t \in \{t_a, t_b\} \implies |\mathscr{V}_m(t)| \lesssim m^2\delta. \tag{11.12}$$

We need to estimate $K(\phi_m u)$ and E_m on I_H for m to be chosen properly. Using the scale invariance of K, (9.11), and Sobolev's inequality as well, we obtain

$$\begin{aligned}&K(\phi_m u) = K(\phi_{\tilde m}(W + v)) = K(W + (v - \phi^C_{\tilde m}(W + v)) \\ &= -4\langle W^5, v - \phi^C_{\tilde m}(W + v)\rangle + O(\|\phi_{\tilde m} v - \phi^C_{\tilde m} W\|^2_{\dot H^1}) \\ &= -2\mu\lambda_1\langle W, g_2\rangle + O(\|\gamma\|_{\dot H^1} + \|W^5\|_{L^{6/5}(|x|>\tilde m)} + \|\phi^C_{\tilde m} W\|^2_{\dot H^1} + \|v\|^2_{\dot H^1}),\end{aligned} \tag{11.13}$$

where $\tilde m(t) := me^{\sigma(t)}$. The decay of W implies $\|W^5\|_{L^{6/5}(|x|>\tilde m)} \lesssim \langle\tilde m\rangle^{-5/2}$ and $\|\phi^C_{\tilde m} W\|^2_{\dot H^1} \lesssim \langle\tilde m\rangle^{-1}$. Plugging these into the above yields, for $s \in \mathscr{L}$ and $t \in I[s]$,

$$K(\phi_m u) = -2\mu\langle W, g_2\rangle\lambda_1 + O(\tilde d(s) + \lambda_1^2 + \langle\tilde m\rangle^{-1}). \tag{11.14}$$

Similarly, the decomposition (3.5) and Hardy inequality yield for $t \in I_H$

$$\int_{|x|>m} (|\nabla u|^2 + |u/r|^2 + |u|^6)dx \lesssim \lambda_1^2 + \langle\tilde m\rangle^{-1}. \tag{11.15}$$

Putting these estimates into (11.10) yields, on each $I[s]$,

$$\dot{\mathscr{V}}_m = -4\mu\langle W, g_2\rangle\lambda_1 + O(\tilde d(s) + \lambda_1^2 + \langle\tilde m\rangle^{-1}). \tag{11.16}$$

Then using the hyperbolic dynamics of λ_1 and $\dot\tau = e^{2\sigma} \sim e^{2\sigma(s)}$, we obtain, with some absolute constant $C_H \ge 1$,

$$\begin{aligned}[\Theta\mathscr{V}_m]_{\partial I[s]} &\gtrsim e^{-2\sigma(s)}[\delta_M - C_H\langle\tilde m(s)\rangle^{-1}\log(\delta_M/\delta)] \\ &\ge e^{-2\sigma(s)}[\delta_M - C_H(m_H/m)\log(\delta_M/\delta)].\end{aligned} \tag{11.17}$$

Thus we obtain

$$m > m_X := 2C_H m_H\delta_M^{-1}\log(\delta_M/\delta) \implies \int_{I_H} \Theta\dot{\mathscr{V}}_m dt \gtrsim m_H^2\delta_M. \tag{11.18}$$

In order to control the cut-off error in I_V, we introduce

$$\mathcal{I}_V := \int_{I_V}\int_{\mathbb{R}^3} |\nabla u|^2 + |u/r|^2 + |u|^6 \, dx \, dt \sim \int_{I_V}\int_{\mathbb{R}^3} |\nabla u|^2 \, dx \, dt, \tag{11.19}$$

where the equivalence follows from Hardy's inequality and $E(u) \sim E(W) > 0$. Since

$$\int_0^\infty \int_{I_V}\int_{|x|>m} \frac{m}{r}(|\nabla u|^2 + |u/r|^2 + |u|^6) dx dt \frac{dm}{m} = \mathcal{I}_V, \tag{11.20}$$

there exists $m \in (m_0, m_1)$ for any $m_1 > m_0 > 0$ such that

$$\int_{I_V}\int_{|x|>m} \frac{m}{r}(|\nabla u|^2 + |u/r|^2 + |u|^6) dx \, dt \le \frac{\mathcal{I}_V}{\log(m_1/m_0)}. \tag{11.21}$$

11.4. **Blow-up region.** We start with the simpler case $\Theta(u) = -1$, where the solution will blow up. First consider $\dot{\mathscr{V}}_m$ on I_V. Since $\epsilon < \epsilon_V(\delta_V)$ and $\tilde{d} > \delta_V$ on I_V, Proposition 3.9 implies

$$t \in I_V \implies -K(u) \ge \kappa(\delta_V) \ge c_0(\delta_V)\|\nabla u\|_{L^2}^2, \tag{11.22}$$

for some constant $c_0(\delta_V) > 0$, since for $\|\nabla u\|_{L^2}^2 \ge 4E(W)$ we have

$$-K(u) = 2\|\nabla u\|_{L^2}^2 - 6E(u) > 2\|\nabla u\|_{L^2}^2 - 6(E(W) + O(\delta^2)) \ge \frac{\|\nabla u\|_{L^2}^2}{4}. \tag{11.23}$$

In order to estimate $\dot{\mathscr{V}}_m$ with $K(u)$ on I_V, the optimal cut-off radius is given by

$$m_V^- := \sup\Big\{R > 0 \Bigm| \int_{I_V}\int_{|x|>R} (|\nabla u|^2 - |u|^6) dx dt \le 0\Big\}. \tag{11.24}$$

$K(u) < 0$ on I_V implies $m_V^- > 0$, while the Sobolev inequality on $|x| > R$ implies that $m_V^- < \infty$. For any $m \ge m_V^-$, we have from (11.10) and (11.22),

$$\begin{aligned} t \in I_V \implies -\dot{\mathscr{V}}_m &= -2\int_{|x|\le m} (|\nabla u|^2 - |u|^6) dx + O(E_m) \\ &\ge -2K(u) + O(E_m) \ge c_0(\delta_V)\|\nabla u\|_{L^2}^2 + O(E_m). \end{aligned} \tag{11.25}$$

The last term can be absorbed by the other, using (11.21). Hence for any $m_0 \ge m_V^-$, there exists $m \in (m_0, C_1(\delta_V)m_0)$ for some constant $C_1(\delta_V) > 1$ such that

$$\int_{I_V} -\dot{\mathscr{V}}_m dt \gtrsim c_0(\delta_V)\mathcal{I}_V. \tag{11.26}$$

The minimal $m > 0$ satisfying this and (11.18) satisfies

$$m \le C_1(\delta_V)\max(m_V^-, m_X), \tag{11.27}$$

for which (11.12) with $[t_a, t_b] = I_H \cup I_V$ implies

$$\begin{aligned} m_H^2\delta_M + c_0(\delta_V)\mathcal{I}_V &\lesssim [-\mathscr{V}_m]_{t_a}^{t_b} \lesssim m^2\delta \\ &\lesssim (C_1(\delta_V)m_V^-)^2\delta + (C_1(\delta_V)m_X)^2\delta. \end{aligned} \tag{11.28}$$

Now we impose an upper bound on δ by the condition

$$(C_1(\delta_V)m_X)^2\delta \ll m_H^2\delta_M, \tag{11.29}$$

which is equivalent to

$$(C_1(\delta_V)C_H)^2\delta\log^2(\delta_M/\delta) \ll \delta_M^3. \tag{11.30}$$

Then the last term in (11.28) is absorbed by the first one, hence

$$m_H^2\delta_M + c_0(\delta_V)\mathcal{I}_V \lesssim (m_V^- C_1(\delta_V))^2\delta. \tag{11.31}$$

In order to bound m_V^-, we use the equation for $|u|^2$:

$$\partial_t|u|^2 = 2\Im\left(\nabla\cdot(\nabla u\bar{u})\right). \tag{11.32}$$

Multiplying it with $\phi^C_{m/2}/r^2$ and integrating on any interval $J\subset I_V$, we obtain

$$\begin{aligned}[\langle|u/r|^2,\phi^C_{m/2}\rangle]_{\partial J} &\lesssim \int_J\int_{|x|>m/2}\frac{|uu_r|}{r^3}dxdt\\ &\lesssim \frac{1}{|\mathcal{I}_V|}\int_{I_V}\int_{|x|>m/2}\frac{m^2}{r^2}(|u_r|^2+|u/r|^2)dxdt,\end{aligned}\tag{11.33}$$

for $m\ge\mathcal{I}_V^{1/2}$. On the other hand, we have from (11.15),

$$t\in\partial I_V \implies \int_{|x|>m/2}|u/r|^2dx \lesssim m_H/m+\delta_M^2. \tag{11.34}$$

Hence by (11.21), there exists $m\sim\max(m_H,\mathcal{I}_V^{1/2})$ such that

$$\sup_{t\in I_V}\|u/r\|_{L^2(|x|>m)} \ll 1. \tag{11.35}$$

Using the radial Gagliardo-Nirenberg inequality

$$\|r^{1/2}\varphi\|_{L^\infty(|x|>m)} \lesssim \|\partial_r\varphi\|^{1/2}_{L^2(|x|>m)}\|\varphi/r\|^{1/2}_{L^2(|x|>m)}, \tag{11.36}$$

we have, for any $m\ge 0$ and $\varphi\in\dot{H}^1_{\text{radial}}$,

$$\begin{aligned}\int_{|x|>m}|\varphi|^6dx &\le \|r^{1/2}\varphi\|^4_{L^\infty(|x|>m)}\|\varphi/r\|^2_{L^2(|x|>m)}\\ &\lesssim \|\nabla\varphi\|^2_{L^2(|x|>m)}\|\varphi/r\|^4_{L^2(|x|>m)}.\end{aligned}\tag{11.37}$$

Plugging (11.35), we obtain for $t\in I_V$,

$$\|u\|^6_{L^6(|x|>m)} \lesssim \|\nabla u\|^2_{L^2(|x|>m)}\|u/r\|^4_{L^2(|x|>m)} \ll \|\nabla u\|^2_{L^2(|x|>m)}, \tag{11.38}$$

which implies $\int_{|x|>m}(|\nabla u|^2-|u|^6)dx\ge 0$, and, by the definition of m_V^-,

$$m_V^- \le m \sim \max(m_H,\mathcal{I}_V^{1/2}). \tag{11.39}$$

Hence, imposing another upper bound on δ by the condition:

$$C_1^2(\delta_V)\delta \ll c_0(\delta_V), \tag{11.40}$$

we see that (11.39) contradicts (11.31).

In conclusion, the smallness conditions on δ,ϵ in the case $\Theta=-1$ are (11.30), (11.40), and (11.2), which determine δ_B and ϵ_B.

11.5. **Scattering region.** Now we consider the case $\Theta(u) = +1$, where the solution will scatter. The argument is similar to that in the previous case, but more involved. In particular, we need several smallness conditions on δ_M.

First observe that there exists an absolute constant $C_E \sim 1$ such that

$$1/C_E \le \|u(t)\|_{\dot H^1}^2 \le C_E \tag{11.41}$$

for all $t \in I(u)$. Indeed, since $E(u) \sim E(W) \sim 1$, the upper bound follows from (2.7), while the lower bound follows from $E(\varphi) \sim \|\varphi\|_{\dot H^1}^2$ for small $\varphi \in \dot H^1$.

Next we estimate $K(\phi_m u)$ on I_V. Using (2.6) and Proposition 3.9, we see that

$$I(\phi_m u) \le I(u) < E(W) + \epsilon^2 - \tilde\kappa(\delta_V)/2 \le I(W) - \tilde\kappa(\delta_V)/3, \tag{11.42}$$

provided that

$$6\epsilon^2 \le \tilde\kappa(\delta_V) := \min(\kappa(\delta_V), c_V/C_E). \tag{11.43}$$

Then using (1.12) and (1.9), we obtain

$$\begin{aligned} K(\phi_m u) &\ge \|\nabla(\phi_m u)\|_{L^2}^2 \left(1 - \|W\|_{L^6}^2 \|\phi_m u\|_{L^6}^4 / \|W\|_{\dot H^1}^2\right) \\ &\ge \|\nabla(\phi_m u)\|_{L^2}^2 \left[1 - \|W\|_{L^6}^2 (\|W\|_{L^6}^6 - \tilde\kappa)^{4/6} / \|W\|_{\dot H^1}^2\right] \\ &\gtrsim \tilde\kappa(\delta_V) \|\nabla(\phi_m u)\|_{L^2}^2 \ge \tilde\kappa(\delta_V) \|\nabla u\|_{L^2(|x|<m)}^2. \end{aligned} \tag{11.44}$$

In order to decide the cut-off for I_V, we put

$$m_V^+(\delta_M) := \inf\Big\{R > 0 \Bigm| \int_{I_V} \int_{|x|<R} |\nabla u|^2 \, dx \, dt \ge \delta_M^3 |I_V|\Big\} \in (0, \infty). \tag{11.45}$$

By (11.10), (11.21), (11.41), and (11.44), there exists $C_2(\delta_V) \in (1, \infty)$ such that for any $m_0 \ge m_V^+(\delta_M)$, there exists $m \in (m_0, C_2(\delta_V) m_0)$ such that

$$\int_{I_V} \dot{\mathscr V}_m dt \gtrsim \tilde\kappa(\delta_V) \delta_M^3 |I_V|. \tag{11.46}$$

Note that δ_V depends on δ_M through the condition $\delta_V \ll \delta_M$ in (11.2), which allows us to determine C_2 in terms of δ_V only. The minimal $m > 0$ satisfying both this and (11.18) must satisfy

$$m \le C_2(\delta_V) \max(m_V^+(\delta_M), m_X), \tag{11.47}$$

for which (11.12) implies

$$\begin{aligned} m_H^2 \delta_M + \tilde\kappa(\delta_V) \delta_M^3 |I_V| \sim [\mathscr V_m]_{t_a}^{t_b} &\lesssim m^2 \delta \\ &\le (C_2(\delta_V) m_V^+(\delta))^2 \delta + (C_2(\delta_V) m_X)^2 \delta. \end{aligned} \tag{11.48}$$

Now we impose an upper bound on δ by the condition

$$(C_2(\delta_V) m_X)^2 \delta \ll m_H^2 \delta_M, \tag{11.49}$$

which is equivalent to

$$(C_2(\delta_V) C_H)^2 \delta \log^2(\delta_M/\delta) \ll \delta_M^3. \tag{11.50}$$

Then the last term in (11.48) is absorbed by the first one, and

$$\begin{aligned} m_H^2 \delta_M + \tilde\kappa(\delta_V) \delta_M^3 |I_V| &\lesssim (C_2(\delta_V) m_V^+(\delta))^2 \delta \\ &\ll (m_V^+(\delta_M))^2 \delta_M^3 \log^{-2}(\delta_M/\delta). \end{aligned} \tag{11.51}$$

Imposing another upper bound on δ by

$$\tilde{\kappa}(\delta_V)^{1/2}\log(\delta_M/\delta) \gg e^{1/\delta_M^3}, \tag{11.52}$$

yields

$$m_H + |I_V|^{1/2} \ll e^{-1/\delta_M^3} m_V^+(\delta_M). \tag{11.53}$$

To compare m_H with $|I_V|^{1/2}$, use (11.21) with $m_0 := m_H + |I_V|^{1/2}$ and $m_1 := m_V^+(\delta_M)/2$. Then there exists $m \in (m_0, m_1)$ such that

$$\int_{I_V}\int_{|x|>m} \frac{m}{r}(|\nabla u|^2 + |u|^6 + |u/r|^2)dxdt \lesssim \delta_M^3 |I_V|, \tag{11.54}$$

because of (11.53). Using Hardy, we have

$$\begin{aligned}\|u/r\|_{L^2(|x|<2m)} &\lesssim \|\phi_m u\|_{\dot{H}^1} + \|u/r\|_{L^2(m<|x|<2m)} \\ &\lesssim \|u/r\|_{L^2(m<|x|<2m)} + \|\nabla u\|_{L^2(|x|<2m)}.\end{aligned} \tag{11.55}$$

Integrating its square over I_V, and using the definition of $m_V^+ > 2m$ for the $\|\nabla u\|^2_{L^2(|x|<2m)}$ term, and (11.54) for the the $\|u/r\|^2_{L^2(m<|x|<2m)}$ term, we obtain

$$\int_{I_V}\int_{|x|<2m} [|u/r|^2 + |\nabla u|^2]dxdt \lesssim \delta_M^3 |I_V|. \tag{11.56}$$

Using the radial Sobolev inequality, we have for any $\varphi \in \dot{H}^1_{\text{radial}}$ and $m > 0$,

$$\|\varphi\|^6_{L^6(|x|<m)} \lesssim \|\varphi/r\|^2_{L^2(|x|<m)} \|r^{1/2}\varphi\|^4_{L^\infty} \lesssim \|\varphi/r\|^2_{L^2(|x|<m)} \|\varphi_r\|^4_{L^2}. \tag{11.57}$$

Inserting it to the above estimate yields

$$\int_{I_V}\int_{|x|<2m} [|\nabla u|^2 + |u/r|^2 + |u|^6]dxdt \lesssim \delta_M^3 |I_V|. \tag{11.58}$$

Noting that $m_X \ll m_H < m$ by (11.52) and $m_H < m_0$, we see from the above estimate that if $m_H^2 \gg \delta_M^2 |I_V|$, then by (11.10) and (11.18)

$$\int_{I_V} |\dot{\mathscr{V}}_{m_H}|dt \ll m_H^2 \delta_M \lesssim [\mathscr{V}_{m_H}]_{t_a}^{t_b} \lesssim m_H^2 \delta, \tag{11.59}$$

which contradicts $\delta \ll \delta_M$. Therefore

$$m_H^2 \lesssim \delta_M^2 |I_V|. \tag{11.60}$$

Next we estimate $\|u/r\|_{L^2(I_V \times \mathbb{R}^3)}$. By the same argument as for (11.33), we have for any interval $J \subset I_V$,

$$\begin{aligned}\big[\langle |u/r|^2, \phi_m^C\rangle\big]_{\partial J} &\lesssim \int_J\int_{|x|>m} \frac{|uu_r|}{r^3}\, dx\, dt \\ &\lesssim \frac{1}{|I_V|}\int_{I_V}\int_{|x|>m} \frac{m^2}{r^2}(|u_r|^2 + |u/r|^2)dxdt \lesssim \delta_M^3,\end{aligned} \tag{11.61}$$

where we used $m > m_0 > |I_V|^{1/2}$ and (11.54). On the other hand, we have from (11.15) and then (11.60),

$$t \in \partial I_V \implies \int_{|x|>m} |u/r|^2 dx \lesssim \frac{m_H}{m} + \delta_M^2 < \frac{m_H}{|I_V|^{1/2}} + \delta_M^2 \lesssim \delta_M. \tag{11.62}$$

Combining the above two estimates yields

$$\sup_{t\in I_V} \|u/r\|_{L^2(|x|>2m)}^2 \lesssim \delta_M. \tag{11.63}$$

Then using (11.58) for the integral over $|x| < 2m$, we obtain

$$\int_{I_V}\int_{\mathbb{R}^3} |u/r|^2 dxdt \lesssim \delta_M |I_V|. \tag{11.64}$$

Decomposing I_V into its connected components, we obtain an interval $I \subset I_V$ such that $\partial I \subset \partial I_V$ and

$$\int_{I}\int_{\mathbb{R}^3} |u/r|^2 dxdt \lesssim \delta_M |I|. \tag{11.65}$$

Now we resort to an argument by Bourgain [3], in order to reduce the problem to energy below the ground state $E(W)$. Although Bourgain in [3] treated the defocusing case, the perturbative argument works as well for the focusing equation (1.1) under the uniform bound (11.41) in $\dot H^1$, while the non-perturbative argument with the Morawetz estimate can be replaced with (11.65), as is shown below.

In order to apply the argument to the interval I, the first observation is

$$\|u\|_{S(I)} \gtrsim 1. \tag{11.66}$$

Proof. Let $t_0 := \inf I \in \partial I \subset \partial I_V$. Then by the definition of I_V, we have $\tilde d(t_0) = \delta_M$ and Proposition 3.7 from $t = t_0$ yields some $t_1 \in I_V$ such that $\tilde d(t_1) = \delta_X$ and $\partial_t \tilde d(t) > 0$ on $(t_0, t_1) \subset I$. It suffices to show $\|u\|_{S(t_0,t_1)} \gtrsim 1$. The scaling invariance reduces it to the case $\sigma(t_0) = 0$. Then $\delta_M \ll \delta_X$ and $\dot\tau = e^{2\sigma}$ with (3.28) and (3.26) imply that $t_1 > t_0 + 1$. Put $v := u - W$. Then from the equation

$$i\dot v - \Delta v = 5W^4 v_1 + iW^4 v_2 + N(v), \tag{11.67}$$

the embedding $W^1 \subset S$, and the Strichartz estimate (2.17), we obtain for any interval $J = [a, b] \subset (t_0, t_0 + 1)$,

$$\begin{aligned} \|v\|_{(W^1\cap L_t^\infty \dot H^1)(J)} &\lesssim \|v(a)\|_{\dot H^1} + [\|W\|_{W^1(J)} + \|v\|_{W^1(J)}]^4 \|v\|_{W^1(J)} \\ &\lesssim \|v(a)\|_{\dot H^1} + |J|^{2/5}\|v\|_{W^1(J)} + \|v\|_{W^1(J)}^5. \end{aligned} \tag{11.68}$$

Hence if $|J| \ll 1$ then

$$\|v\|_{(W^1\cap L_t^\infty \dot H^1)(J)} \lesssim \|v(a)\|_{\dot H^1}. \tag{11.69}$$

Repeating this estimate from $t = t_0$ on consecutive small intervals, we obtain

$$\|v\|_{S(t_0,t_0+1)} \lesssim \|v\|_{W^1(t_0,t_0+1)} \lesssim \|v(t_0)\|_{\dot H^1} \lesssim \delta_M, \tag{11.70}$$

so, $\|u\|_{S(t_0,t_0+1)} \ge \|W\|_{L_x^{10}} - O(\delta_M) \gtrsim 1$. □

Hence as in [3], we can decompose the interval I such that

$$I = [t_0, t_N], \quad t_0 < t_1 < \cdots < t_N, \quad I_j := [t_j, t_{j+1}], \quad \|u\|_{S(I_j)} \in [\eta, 2\eta) \tag{11.71}$$

for a small fixed constant $\eta > 0$. In the following, c denotes a small positive constant, and $C(\eta)$ denotes a large positive constant which may depend on η, both allowed to change from line to line.

By the perturbation argument from Section 3 to (4.11) in [3], where the sign of nonlinearity is irrelevant, we have for each j,

$$\|u\|_{\bar{S}^1(I_j)} \lesssim 1, \tag{11.72}$$

and there exist a subinterval $I_j' \subset I_j$ and $R_j \lesssim |I_j'|^{1/2}$ such that

$$\inf_{t\in I_j'} \min(\|\nabla u(t)\|_{L^2(|x|<C(\eta)R_j)}, \|u(t)\|_{L^6(|x|<C(\eta)R_j)}) \gtrsim \eta^{3/2}. \tag{11.73}$$

Combining it with (11.57) and (11.65) yields

$$\sum_{j=1}^N |I_j'| \le C(\eta)\delta_M \sum_{j=1}^N |I_j|. \tag{11.74}$$

Hence there exists $j \in \{1, \dots, N\}$ such that

$$R_j^2 \lesssim |I_j'| \le C(\eta)\delta_M|I_j|. \tag{11.75}$$

Fix $s \in I_j'$. By the time reversal symmetry, we may assume without loss of generality

$$t_{j+1} - s > s - t_j. \tag{11.76}$$

By [3, Lemma 5.12], there exists $R \le C(\eta)R_j$ such that

$$\|\phi_R^C u(s)\|_{\dot{H}^1}^2 < \|u(s)\|_{\dot{H}^1}^2 - c\eta^3. \tag{11.77}$$

Let v be the solution of the free Schrödinger equation with initial data

$$v(s) := \phi_R u(s), \tag{11.78}$$

and $w := u - v$. By the L^p decay estimate (2.16), Hölder's inequality (placing u in L^6 and ∇u in L^2), and (11.41), we see that

$$\begin{aligned} &\|v(t)\|_{L^6_x} \lesssim |t-s|^{-1}\|v(s)\|_{L^{6/5}_x} \lesssim R^2|t-s|^{-1}, \\ &\|\nabla v(t)\|_{L^{30/13}_x} \lesssim |t-s|^{-1/5}\|\nabla v(s)\|_{L^{30/17}_x} \lesssim R^{1/5}|t-s|^{-1/5}. \end{aligned} \tag{11.79}$$

Hence using (11.75) and (11.76), we obtain

$$\begin{aligned} &\|v(t_{j+1})\|_{L^6_x} \lesssim R^2/|t_{j+1}-s| \le C(\eta)\delta_M \\ &\|v\|_{W^1(t_{j+1},\infty)} \lesssim (R^2/|t_{j+1}-s|)^{1/10} \lesssim C(\eta)\delta_M^{1/10}. \end{aligned} \tag{11.80}$$

By the equation, integration by part, and Hölder's inequality, we have

$$\begin{aligned} \left[\|u(t)\|_{L^6}^6\right]_s^{t_{j+1}} &= 6\int_s^{t_{j+1}} \Re\langle |u|^4 u, \dot{u}\rangle dt \\ &= 6\int_s^{t_{j+1}} \Im\langle \nabla(|u|^4 u), \nabla u\rangle dt \lesssim \|u\|_{\bar{S}^1(I_j)}^2 \|u\|_{S(I_j)}^4 \lesssim \eta^4. \end{aligned} \tag{11.81}$$

By the Strichartz estimate (2.17), we have

$$\|w(t_{j+1})\|_{\dot{H}^1} - \|w(s)\|_{\dot{H}^1} \lesssim \|u\|_{W^1(I_j)}\|u\|_{S(I_j)}^4 \lesssim \eta^4. \tag{11.82}$$

Let $\tilde{w}$ be the solution of (1.1) with initial data $\tilde{w}(t_{j+1}) := w(t_{j+1})$. Then by the above estimates together with (11.77), (11.80), and (11.81), we get

$$
\begin{aligned}
E(\tilde{w}) &\leq \frac{1}{2}\|\nabla w(t_{j+1})\|_{L^2}^2 - \frac{1}{6}\|u(t_{j+1})\|_{L^6}^6 + O(\eta^4) \\
&\leq \frac{1}{2}\|\nabla w(s)\|_{L^2}^2 - \frac{1}{6}\|u(s)\|_{L^6}^6 + O(\eta^4) \\
&\leq E(u) - c\eta^3 \leq E(W) + \epsilon^2 - c\eta^3 < E(W) - c\eta^3/2,
\end{aligned} \tag{11.83}
$$

where in the first and last steps, we imposed upper bounds on δ_M and ϵ respectively:

$$
C(\eta)\delta_M \ll \eta^4, \quad \epsilon \ll \eta^{3/2}. \tag{11.84}
$$

Similarly, plugging (11.77) into (11.82) yields

$$
\|\nabla \tilde{w}(t_{j+1})\|_{L^2}^2 \leq \|\nabla u(s)\|_{L^2}^2 - c\eta^3, \tag{11.85}
$$

while $E(u) < E(W) + \epsilon^2$ and (2.7) together with $K(u(s)) > 0$ implies

$$
\|\nabla u(s)\|_{L^2}^2 < \|\nabla W\|_{L^2}^2 + 3\epsilon^2 \tag{11.86}
$$

Hence $\|\nabla \tilde{w}(t_{j+1})\|_{L^2}^2 < \|\nabla W\|_{L^2}^2$ and therefore by (2.9),

$$
K(\tilde{w}(t_{j+1})) > 0. \tag{11.87}
$$

Hence by the result of Kenig and Merle [11] below the ground state energy, (11.41) and (11.85), $\tilde{w}$ scatters in both time directions with a uniform Strichartz bound:

$$
\|\tilde{w}\|_{W^1(\mathbb{R})} < C(\eta). \tag{11.88}
$$

In order to control u by this, we use the long-time perturbation [11, Theorem 2.14]:

Lemma 11.1 ([11])**.** *Let u be a solution of* (1.1)*. Let I be an interval with some $t_0 \in I \cap I(u)$. Let $e \in N^1(I)$ and let $\tilde{u} \in C(I; \dot{H}^1)$ be a solution of*

$$
i\partial_t \tilde{u} - \triangle \tilde{u} = |\tilde{u}|^4 \tilde{u} + e. \tag{11.89}
$$

Assume that for some $B_1, B_2, B_3 > 0$

$$
\|\tilde{u}\|_{L_t^\infty \dot{H}^1(I)} \leq B_1, \quad \|\tilde{u}\|_{S(I)} \leq B_2, \quad \|\tilde{u}(t_0) - u(t_0)\|_{\dot{H}^1} \leq B_3. \tag{11.90}
$$

Then there exists $\nu_P = \nu_P(B_1, B_2, B_3) > 0$ such that if

$$
\|e^{-i(t-t_0)\Delta}(\tilde{u}(t_0) - u(t_0))\|_{S(I)} + \|e\|_{N^1(I)} =: \nu \leq \nu_P, \tag{11.91}
$$

then $I \subset I(u)$ and

$$
\|\tilde{u}\|_{S(I)} \lesssim 1, \quad \|\tilde{u} - u\|_{L_t^\infty \dot{H}^1(I)} \lesssim \nu + B_3, \tag{11.92}
$$

where the implicit constants depend on B_1, B_2, B_3.

Apply the above lemma to u and $\tilde{u} := \tilde{w} + v$ with $I = [t_{j+1}, \infty)$ and initial data at $t = t_{j+1}$. From the bounds on $\tilde{w}$ and v, we have

$$
\|\tilde{u}\|_{L_t^\infty \dot{H}^1} \lesssim 1, \quad \|\tilde{u}\|_S \leq C(\eta), \quad \tilde{u}(t_{j+1}) - u(t_{j+1}) = 0, \tag{11.93}
$$

and, using (11.80), there exists a large positive constant $C_*(\eta)$ such that

$$
\begin{aligned}
\|e\|_{N^1(I)} &= \||\tilde{w}|^4\tilde{w} - |\tilde{u}|^4\tilde{u}\|_{N^1(I)} \\
&\lesssim (\|\tilde{w}\|_{W^1(I)} + \|v\|_{W^1(I)})^4 \|v\|_{W^1(I)} \leq C_*(\eta)\delta_M^{1/10}.
\end{aligned} \tag{11.94}
$$

So by imposing another smallness condition on δ_M:

$$C_*(\eta)\delta_M^{1/10} \ll \nu_P(C(\eta), C(\eta), 0), \tag{11.95}$$

we can apply the above lemma. Hence there exists another large positive constant $C_{**}(\eta)$ such that

$$\|\tilde{u} - u\|_{L_t^\infty \dot{H}_x^1(t_{j+1},\infty)} \le C_{**}(\eta)\delta_M^{1/10}. \tag{11.96}$$

Since $K(\tilde{w}) > 0$, which is preserved in time because of $E(\tilde{w}) < E(W)$, we have, using (2.6) and (11.83),

$$\|\tilde{w}(t)\|_{L_x^6}^6 \le \|W\|_{L_x^6}^6 - \frac{3c}{2}\eta^3. \tag{11.97}$$

Taking δ_M smaller if necessary we have

$$C_{**}(\eta)\delta_M^{1/10} \ll \eta^3. \tag{11.98}$$

Hence, combining the above estimates with (11.79) and (11.80), and taking δ_M smaller if necessary, we obtain

$$\|u(t)\|_{L_x^6}^6 \le \|W\|_{L_x^6}^6 - c\eta^3, \tag{11.99}$$

which contradicts $\tilde{d}_{\mathcal{W}}(u(t_b)) = \delta \ll \delta_M$, since (11.98) implies $\delta_M \ll \eta^3$. In conclusion, after fixing the constant $\delta_M > 0$ such that (11.95), (11.84) and (11.98) hold, the smallness conditions on δ, ϵ in the case $\Theta = +1$ are (11.2), (11.50), (11.52), and (11.84), which determine δ_B and ϵ_B. □

12. Solutions staying around the ground states

In this section, we prove Proposition 3.14. Let u be a solution of (1.1) satisfying $u(t_0) \in \mathcal{H}^{\epsilon_B(\delta)} \cap \tilde{B}_\delta(\mathcal{W})$ for some $\delta \in (0, \delta_B]$ and $t_0 \in I(u)$, and $t_+ = T_+(u)$, namely $\tilde{d}_{\mathcal{W}}(u(t)) < \delta$ for $t \in [t_0, T_+(u))$.

If $u(t) \in \check{\mathcal{H}}$ and $\partial_t \tilde{d}_{\mathcal{W}}(u(t)) \ge 0$ at some $t \in [t_0, T_+(u))$, then Proposition 3.7 implies that $\tilde{d}_{\mathcal{W}}(u(t))$ increases up to $\delta_X > \delta_B > \delta$, contradicting $t_+ = T_+(u)$. Hence for all $t \in [t_0, T_+(u))$,

$$c_D \tilde{d}_{\mathcal{W}}(u(t)) \le \sqrt{E(u) - E(W)} \quad \text{or} \quad \partial_t \tilde{d}_{\mathcal{W}}(u(t)) < 0, \tag{12.1}$$

so by the mean value theorem, there are only two possibilities:

1. There exists $t_1 \in [t_0, T_+(u))$ such that $u(t) \notin \check{\mathcal{H}}$ for all $t \in [t_1, T_+(u))$.
2. $u(t) \in \check{\mathcal{H}}$ and $\partial_t \tilde{d}_{\mathcal{W}}(u(t)) < 0$ for all $t \in [t_0, T_+(u))$.

In the first case, if we choose the minimal t_1, then for $t_0 \le t < t_1$, we have $\partial_t \tilde{d}_{\mathcal{W}}(u(t)) < 0$. Hence it suffices to treat the latter case, for which $t_1 = T_+(u)$. Since $\tilde{B}_\delta(\mathcal{W}) \subset \tilde{B}_{\delta_X}(\mathcal{W}) \subset B_{\delta_L}(\mathcal{W})$, Proposition 3.5 implies that $\tau \to \infty$ as $t \nearrow T_+(u)$. Apply Proposition 3.7 backward in time from any $t \in (t_0, T_+(u))$, corresponding to $\tau \in (\tau(t_0), \infty)$. Then

$$\tilde{d}_{\mathcal{W}}(u(t_0)) \sim e^{\mu(\tau(t)) - \tau(t_0))} \tilde{d}_{\mathcal{W}}(u(t)). \tag{12.2}$$

Sending $t \nearrow T_+(u)$ yields $\tilde{d}_{\mathcal{W}}(u(t)) \to 0$. □

13. Long-time behavior away from the ground states

In this section, we prove Proposition 3.15. Let u be a solution of (1.1) satisfying $u([t_0, T_+(u)) \subset \mathcal{H}^\epsilon \setminus \tilde{B}_\delta(\mathcal{W})$ for some $\epsilon \in (0, \epsilon_B(\delta))$. By Remark 3.13, u stays in $\check{\mathcal{H}}$, so $\Theta(u) \in \{\pm 1\}$ is a constant. Moreover, Proposition 3.14 implies that $t_+(\delta') < T_+(u)$ for all $\delta' \in [\delta, \delta_B]$, so Proposition 3.11 yields $t_1 \in I(u)$ such that

$$u([t_1, T_+(u))) \subset \mathcal{H}^\epsilon \setminus \tilde{B}_{\delta_B}(\mathcal{W}). \tag{13.1}$$

Without losing generality, we may assume $t_1 = 0$ by time translation.

13.1. **Blow-up after ejection.** In the case of $\Theta(u) = -1$ and $u_0 \in H^1_{\text{radial}}$, we prove that $T_+(u) < \infty$. Let $m \gg 1$. We rewrite (11.10) in the following way

$$\dot{\mathscr{V}}_m = 2K(u) - 2\langle |u_r|^2, f_{0,m}\rangle + \frac{1}{2}\langle |u/m|^2, f_{1,m}\rangle + 2\langle |u|^6, f_{2,m}\rangle \tag{13.2}$$

with

$$f_0 := 1 - \phi - r\partial_r\phi, \quad f_1 := -\Delta(r\partial_r + 3)\phi, \quad f_2 := 1 - \phi - r\partial_r\phi/3. \tag{13.3}$$

By the property of ϕ, we have $\operatorname{supp} f_{j,m} \subset \{m \le |x| \le 2m\}$ and $0 \le f_{2,m} \le f_{0,m}$. Hence using (11.37) and the L^2 conservation, we obtain

$$\begin{aligned} \langle |u|^6, f_{2,m}\rangle &\lesssim \int_m^\infty f_{0,m}(r)|u|^6 r^2 dr \\ &= \int_m^\infty \int_m^r f'_{0,m}(s)ds|u|^6 r^2 dr \sim \int_m^\infty f'_{0,m}(s)\|u\|^6_{L^6(|x|>s)}ds \\ &\lesssim \int_m^\infty f'_{0,m}(s)\frac{1}{s^4}\|\nabla u\|^2_{L^2(|x|>s)}\|u\|^4_{L^2(|x|>s)}ds \\ &\lesssim \frac{\|u_0\|^4_{L^2}}{m^4}\int_m^\infty f_{0,m}|u_r|^2 dx \sim \frac{\|u_0\|^4_{L^2}}{m^4}\langle |u_r|^2, f_{0,m}\rangle. \end{aligned} \tag{13.4}$$

We also have $\int_{\mathbb{R}^3} |f_{1,m}(r)||u/m|^2 dx \lesssim m^{-2}\|u_0\|^2_{L^2}$. Hence for $m \gg \|u_0\|_{L^2}/\kappa(\delta_B)$ and $0 < t < T_+(u)$ we have

$$-\dot{\mathscr{V}}_m(t) \ge -K(u(t)) \ge \kappa(\delta_B) > 0. \tag{13.5}$$

Now assume for contradiction that u exists for all time $t > 0$, namely $T_+(u) = \infty$. Then choosing $m \gg \|u_0\|_{L^2}/\kappa(\delta_B)$, we have from (13.5)

$$m\|u_r(t)\|_{L^2}\|u_0\|_{L^2} \gtrsim -\mathscr{V}_m(t) \to \infty, \tag{13.6}$$

as $t \to \infty$, hence

$$-K(u(t)) = -6E(u) + 2\|u_r(t)\|^2_{L^2} \to \infty. \tag{13.7}$$

So one can choose $T_1 > 0$ such that $-K(u(t)) \sim \|u_r(t)\|^2_{L^2}$ for $t \ge T_1$. Hence

$$m\|u_r(T_2)\|_{L^2} \ge -m\|u_r(T_1)\|_{L^2} + \frac{c}{\|u_0\|_{L^2}}\int_{T_1}^{T_2}\|u_r(t)\|^2_{L^2}dt \tag{13.8}$$

for $T_2 \ge T_1$ and some absolute constant $c \in (0,1)$. Therefore, defining

$$f(t) := -m\|u_r(T_1)\|_{L^2} + \frac{c}{\|u_0\|_{L^2}}\int_{T_1}^{t}\|u_r(s)\|^2_{L^2}ds, \tag{13.9}$$

we see that for large $t > T_1$, $f(t)$ is positive and $\partial_t f(t) \gtrsim f(t)^2$. Integrating this differential inequality yields a singularity and therefore blow-up in finite time.

13.2. **Scattering after ejection.** In the case of $\Theta(u) = +1$, the proof of scattering uses arguments from [11] with arguments from [19]. Unlike the subcritical case, we have to take account of the scaling parameter and the fact that the maximal time interval of existence might be finite, even though the $\dot{H}^1$ norm is bounded by (3.32).

We recall the following result proved by Keraani [10] using a concentration compactness procedure, cf. [2, 14, 15]. Since we are dealing with radial solutions only, we restrict it to the radial case.

Lemma 13.1 ([10]). *Let $\{v_{0,n}\}_{n\ge1}$ be a bounded sequence in $\dot{H}^1_{\mathrm{radial}}$. Then, passing to a subsequence, there exist sequences $\{V^j\}_{j\ge0} \subset \dot{H}^1_{\mathrm{radial}}$ and $\{(\sigma_{j,n}, t_{j,n})\}_{j\ge0,n\ge1} \subset \mathbb{R}^2$ with the following properties. For each $j \ne j'$*

$$\lim_{n\to\infty} |\sigma_{j,n} - \sigma_{j',n}| + |e^{-2\sigma_{j,n}}(t_{j,n} - t_{j',n})| = \infty. \tag{13.10}$$

For $\gamma_{k,n}(t,x)$ defined by

$$e^{-it\Delta} v_{0,n} = \sum_{j=0}^{k} e^{-i(t+t_{j,n})\Delta} S_{-1}^{-\sigma_{j,n}} V_j + \gamma_{k,n}, \tag{13.11}$$

we have

$$\lim_{k\to\infty} \limsup_{n\to\infty} \|\gamma_{k,n}\|_S = 0. \tag{13.12}$$

For all k and as $n \to \infty$,

$$\|v_{0,n}\|_{\dot{H}^1}^2 = \sum_{j=0}^{k} \|V_j\|_{\dot{H}^1}^2 + \|\gamma_{k,n}(0)\|_{\dot{H}^1}^2 + o(1) \tag{13.13}$$

and, putting $s_{j,n} := t_{j,n} e^{-2\sigma_{j,n}}$,

$$E(v_{0,n}) = \sum_{j=0}^{k} E(e^{-is_{j,n}\Delta} V_j) + E(\gamma_{k,n}(0)) + o(1). \tag{13.14}$$

For any $A < E(W) + \epsilon_S^2$ and any $\delta \in (0, \delta_B]$, let $\mathscr{S}(A,\delta)$ be the collection of solutions of (1.1) such that

$$E(u) \le A, \quad u([0, T_+(u))) \subset \check{\mathcal{H}} \setminus \tilde{B}_\delta(\mathcal{W}), \quad \Theta(u(0)) = +1. \tag{13.15}$$

Since u stays in $\check{\mathcal{H}}$ for $0 \le t < T_+(u)$, $\Theta(u(t)) = +1$ is preserved.

It is well known that $u \in S(0,\infty)$ implies the scattering as $t \to \infty$, see [4] or [11, Remark 2.15]. Define the minimal energy where uniform Strichartz bound fails.

$$\begin{aligned} S(A,\delta) &:= \sup_{u\in\mathscr{S}(A,\delta)} \|u\|_{S(0,T_+(u))}, \\ E_c(\delta) &:= \sup\{A < E(W) + \epsilon_S^2 \mid S(A,\delta) < \infty\}. \end{aligned} \tag{13.16}$$

Notice that $S(A,\delta) \lesssim A^{1/2}$ holds for $0 < A \ll 1$, by the small data scattering (see [4] for example). Moreover, the result of [11] implies $E_c(\delta) \ge E(W)$. If $E_c(\delta) < E(W) + \epsilon_S^2$, there exists a sequence of solutions $u_n \in \mathscr{S}(A_n, \delta)$ for some sequence of numbers $A_n \to E_c(\delta)$ such that

$$\|u_n\|_{S(0,T_+(u_n))} \to \infty. \tag{13.17}$$

Next we prove the existence of a critical element:

Lemma 13.2. *Let $\delta \in (0, \delta_B)$. Suppose that $E_c(\delta) \le E(W) + \epsilon^2$ for some ϵ such that*

$$0 < \epsilon < \min(\epsilon_V(\delta), \epsilon_B(\delta_B)), \quad \epsilon \ll \min(\epsilon_S, \delta, \sqrt{\kappa(\delta)}). \tag{13.18}$$

Let $A_n \to E_c(\delta)$ and $u_n \in \mathscr{S}(A_n, \delta)$ satisfying (13.17). Then there exist $U_c \in \mathscr{S}(E_c(\delta), \delta_B)$ satisfying $E(U_c) = E_c(\delta)$ and $\|U_c\|_{S(0,T_+(U_c))} = \infty$, and $(\sigma_n, s_n) \in \mathbb{R}^2$ such that $e^{is_n\Delta} S_{-1}^{\sigma_n} u_n(0)$ is strongly convergent in $\dot{H}^1_{\mathrm{radial}}$.

Note that once we have $U_c \in \mathscr{S}(E_c(\delta), \delta)$ with the other properties, then a time translation yields another minimal element in $\mathscr{S}(E_c(\delta), \delta_B)$ as a consequence of the ejection and the one-pass lemmas (see the proof below for the detail).

Proof. Notice that, by the small data scattering, cf. [11, Remark 2.7], we must have $\|u_n\|_{\dot{H}^1} \gtrsim 1$, otherwise $\|u_n\|_{S(0,\infty)}$ are uniformly small. Hence using (3.32) as well, we have for all n and $t \in [0, T_+(u_n))$,

$$\|u_n(t)\|_{\dot{H}^1} \sim 1. \tag{13.19}$$

We then apply (13.11) to $v_{0,n} := u_n(0)$. Then we have, up to a subsequence,

$$e^{-it\Delta} u_n(0) = \sum_{j=0}^{k} e^{-it\Delta} S_{-1}^{-\sigma_{j,n}} e^{-is_{j,n}\Delta} V_j + \gamma_{k,n}. \tag{13.20}$$

Let $s_{j,\infty} \in [-\infty, \infty]$ such that $s_{j,n} \to s_{j,\infty}$ (up to a subsequence), and let U_j be the nonlinear profile associated with $(V_j, \{s_{j,m}\}_{m\ge1})$, that is the unique solution of (1.1) around $t = s_{j,\infty}$ satisfying (see [11] for more detail),

$$\lim_{m\to\infty} \|U_j(s_{j,m}) - e^{-is_{j,m}\Delta} V_j\|_{\dot{H}^1} = 0. \tag{13.21}$$

We also define $U_{j,n}(t) := S_{-1}^{-\sigma_{j,n}} U_j((t + t_{j,n})e^{-2\sigma_{j,n}})$. Since $\epsilon < \epsilon_V(\delta)$, Proposition 3.9 implies that $K(u_n(t)) \gtrsim \kappa(\delta) \gg \epsilon^2$ for all $n \ge 1$ and $t \ge 0$. Then using (13.13) and the conservation of G for the free equation, we have

$$\begin{aligned} E(W) - \epsilon^2 &> E(u_n) - K(u_n(0))/6 + \epsilon^2 \\ &= G(u_n(0)) + \epsilon^2 \ge \sum_{j=0}^{k} G(V_j) + G(\gamma_{k,n}). \end{aligned} \tag{13.22}$$

This implies that $G(e^{-it\Delta} V_j) \le E(W) - \epsilon^2$ and $G(\gamma_{j_{k,n}}) \le E(W) - \epsilon^2$. By (2.9), $K(\gamma_{k,n}(t)) \ge 0$ and so $E(\gamma_{k,n}(t)) \ge 0$ for all $t \in \mathbb{R}$. Similarly, $K(e^{-it\Delta} V_j) \ge 0$ for all $t \in \mathbb{R}$, and $K(e^{-it\Delta} V_j) \to \|V_j\|^2_{\dot{H}^1}$ as $t \to \pm\infty$, and both can be zero only if $V_j = 0$. Hence by (13.21), $K(U_j(t)) \ge 0$ in a neighborhood of $s_{j,\infty}$, and $E(U_j) \ge 0$.

By (13.14) and (13.21), we have as $n \to \infty$,

$$E(u_n) = \sum_{j=0}^{k} E(U_j) + E(\gamma_{k,n}(0)) + o(1), \tag{13.23}$$

hence we see that for all $j \ge 0$

$$E(U_j) \le E_c. \tag{13.24}$$

If $E(U_j) < E(W)$, then we conclude from $K(U_j) \ge 0$ in a neighborhood of $s_{j,\infty}$ and [11] that U_j exists globally in time and scatters with $\|U_j\|_{S \cap W^1 \cap L^\infty_t \dot{H}^1} < \infty$.

Assuming that $\|U_j\|_S < \infty$ for all $j = 0, 1, \ldots, k$, we apply Lemma 11.1 to

$$\tilde{u} := \sum_{j=0}^{k} U_{j,n} + \gamma_{k,n}, \quad u := u_n \tag{13.25}$$

from $t_0 := 0$ on $I := \mathbb{R}$. (2.7) and (13.23) imply that $\tilde{u}$ is bounded in $L_t^\infty \dot{H}^1$ as $n \to \infty$ uniformly in k. From (13.13), the orthogonality conditions (13.10) and a similar argument to that in the proof of Proposition 4.2 in [11], $\tilde{u}$ is bounded in S as $n \to \infty$ uniformly in k. (13.22) implies that $\|\tilde{u}(0) - u_n(0)\|_{\dot{H}^1} \to 0$ as $n \to \infty$. Hence, in order to apply the lemma for large n, it suffices to make

$$i\partial_t \tilde{u} - \Delta \tilde{u} - |\tilde{u}|^4 \tilde{u} = \sum_{j=0}^{k} |U_{j,n}|^4 U_{j,n} - |\tilde{u}|^4 \tilde{u} \tag{13.26}$$

small in $N^1(\mathbb{R})$. Indeed, using $\|U_j\|_{S \cap W^1} < \infty$ and the orthogonality conditions (13.10), as well as (13.12), we obtain

$$\lim_{k\to\infty} \limsup_{n\to\infty} \|i\partial_t \tilde{u} - \Delta \tilde{u} - |\tilde{u}|^4 \tilde{u}\|_{N^1(\mathbb{R})} = 0. \tag{13.27}$$

For a proof, we refer again to Proposition 4.2 in [11] and the references therein (in particular [10]). Thus for large k and large n, Lemma 11.1 yields a bound on $\|u_n\|_{S(\mathbb{R})}$ uniform in n, contradicting $\|u_n\|_{S(0,T_+(u_n))} \to \infty$.

This means that there exists at least one U_j such that $E(W) \le E(U_j) \le E_c(\delta)$ and $\|U_j\|_{S(I(U_j))} = \infty$, say U_0. Then, by the orthogonality (13.23) and the positivity of $K(U_j) \ge 0$, we see that $T_+(U_j) = T_-(U_j) = \infty$ for $j = 1, .., k$ and

$$\sum_{j=1}^{k} \|U_j\|_{L_t^\infty \dot{H}^1}^2 + \|\gamma_{k,n}\|_{\dot{H}^1}^2 \lesssim \epsilon^2 \ll 1, \tag{13.28}$$

hence all the U_j, $j = 1, \ldots, k$, are small and scatter. That allows us to apply Lemma 11.1 to $\tilde{u}$ and $u = u_n$ on $I = I_n := e^{2\sigma_{0,n}} I_0 - t_{0,n}$ for any interval $I_0 \subset I(U_0)$ such that $\|U_0\|_{S(I_0)} < \infty$. Then the lemma yields

$$\limsup_{n\to\infty} \|u_n\|_{S(I_n)} < \infty, \quad \lim_{k\to\infty} \limsup_{n\to\infty} \|\tilde{u} - u_n\|_{L_t^\infty \dot{H}^1(I_n)} = 0, \tag{13.29}$$

since $B_3 + \nu \to 0$. Combining the second estimate with (13.28) and the orthogonality (13.10), we see that

$$\limsup_{n\to\infty} \|u_n - U_{0,n}\|_{L_t^\infty \dot{H}^1(I_n)} \lesssim \epsilon. \tag{13.30}$$

Suppose that $s_{0,\infty} = +\infty$. Then by definition U_0 scatters in a neighborhood of ∞, namely $\|U_0\|_{S([T_0,\infty))} < \infty$ for some $T_0 \in I(U_0)$. Hence, choosing $I_0 = [T_0, \infty)$ in the above argument yields a uniform bound on $\|u_n\|_{S(I_n)}$, but $I_n \supset [0, \infty)$ for large n, contradicting $\|u_n\|_{S(0,T_+(u_n))} \to \infty$.

Suppose that $s_{0,\infty} = -\infty$. Then by definition $\|U_0\|_{S(-\infty,t_0)} < \infty$ for some $t_0 \in I(U_0)$. Then $U_0 \notin S(I(U_0))$ implies $\|U_0\|_{S(t_0,T_+(U_0))} = \infty$.

Suppose that $s_{0,\infty} \in \mathbb{R}$. If $\|U_0\|_{S(s_{0,\infty},T_+(U_0))} < \infty$, then the blow-up criterion (see for example [11]) implies $T_+(U_0) = \infty$, and so choosing $I_0 = (s_-, \infty)$ for some $s_- \in (-T_-(U_0), s_{0,\infty})$, we see that $I_n \supset [0, \infty)$ for large n, leading to a contradiction with $\|u_n\|_{S(0,\infty)} \to \infty$ as in the case $s_{0,\infty} = +\infty$ above. Therefore $\|U_0\|_{S(s_{0,\infty},T_+(U_0))} = \infty$.

Thus we have obtained $s_{0,\infty} < +\infty$ and $\|U_0\|_{S(t_0,T_+(U_0))} = \infty$ for some $t_0 \in (s_{0,\infty}, T_+(U_0))$. Since $E(U_0) \le E_c(\delta) \le E(W) + \epsilon^2$ and $\epsilon < \epsilon_B(\delta_B)$, Propositions 3.11 and 3.14 imply that there are only two options for U_0:

(1) There exists $t_+ \in I(U_0)$ such that $\tilde{d}_{\mathcal{W}}(U_0(t)) \ge \delta_B$ for all $t_+ < t < T_+(U_0)$.
(2) $\limsup_{t \nearrow T_+(U_0)} \tilde{d}_{\mathcal{W}}(U_0(t)) \le \epsilon/c_D$.

In the second case, choosing $I_0 = (t_0, t_c) \subset I(U_0)$ such that $\tilde{d}_{\mathcal{W}}(U_0(t_c)) < 2\epsilon/c_D$, we obtain from Proposition 3.6 and (13.30) that

$$\tilde{d}_{\mathcal{W}}(u_n(t_n)) \lesssim \tilde{d}_{\mathcal{W}}(U_0(t_c)) + \|u_n(t_n) - U_{0,n}(t_n)\|_{\dot{H}^1} \lesssim \epsilon \ll \delta, \tag{13.31}$$

where $t_n := e^{2\sigma_{0,n}} t_c - t_{0,n} > 0$ for large n, since $t_c > s_{0,\infty}$. This contradicts $\tilde{d}_{\mathcal{W}}(u_n(t)) \ge \delta$ on $[0, T_+(u_n))$. Therefore $\tilde{d}_{\mathcal{W}}(U_0(t)) \ge \delta_B$ for $t_+ \le t < T_+(U_0)$.

Since $E(U_0) \le E_c(\delta) \le E(W) + \epsilon^2$ and $\epsilon < \epsilon_B(\delta_B)$, $\Theta(U_0(t))$ is constant on $[t_+, T_+(U_0))$. Let $t_n := e^{2\sigma_{0,n}} t_+ - t_{0,n}$, then $\Theta(U_{0,n}(t_n)) = \Theta(U_0(t_+))$ by the invariance of Θ. (13.30) implies for large n that $u_n(t_n)$ is in an $O(\epsilon)$ ball around $U_{0,n}(t_n)$, which is included in $\mathcal{H}^{\epsilon_S} \cap \check{\mathcal{H}}$ because $\epsilon \ll \min(\epsilon_S, c_D\delta)$. Hence

$$+1 = \Theta(u_n(t_n)) = \Theta(U_{0,n}(t_n)) = \Theta(U_0(t_+)). \tag{13.32}$$

Therefore, putting $U_c(t) := U_0(t + \tilde{t})$ with $\tilde{t} := \max(t_0, t_+)$, we obtain $U_c \in \mathscr{S}(E_c(\delta), \delta_B)$ and $\|U_c\|_{S(0,T_+(U_c))} = \infty$. Then the definition of E_c implies $E(U_c) \ge E_c(\delta)$, and so $E(U_c) = E_c(\delta)$. Thus U_c satisfies all the properties in the lemma.

Since $E(u_n) \to E_c = E(U_0)$ and $K(U_j) \ge 0$, by (2.7) and (13.23) we have $U_j = 0$ for all $j \ge 1$ and $\gamma_{k,n} \to 0$ in $\dot{H}^1$. Hence $u_n(0) = S_{-1}^{-\sigma_{0,n}} e^{-is_{0,n}\Delta} V_0 + o(1)$ in $\dot{H}^1$. Hence $e^{is_n\Delta} S_{-1}^{\sigma_n} u_n(0) \to V_0$ with $\sigma_n := \sigma_{0,n}$ and $s_n := s_{0,n}$. □

The following is a corollary of the last part.

Claim 13.3. *There exists $\sigma_c : [0, T_+(U_c)) \to \mathbb{R}$ such that*

$$\mathscr{K} := \{S_{-1}^{-\sigma_c(t)} U_c(t)\}_{0 \le t < T_+(U_c)} \subset \dot{H}^1_{\text{radial}} \tag{13.33}$$

is precompact.

Proof. If there is no such σ_c, then there exists $\{t_n\}_{n \ge 1}$ and $\eta > 0$ such that

$$\inf_{\sigma \in \mathbb{R}} \|S_{-1}^{\sigma} U_c(t_n) - U_c(t_{n'})\|_{\dot{H}^1} \ge \eta \tag{13.34}$$

for all $n \ne n'$. Notice that we must have, after possibly passing to a subsequence, $t_n \to T_+(U_c)$: otherwise, we get a contradiction from (13.34) with $\sigma_0 = 0$ by continuity of $U_c(t)$. Applying Lemma 13.2 to $u_n(t) := U_c(t + t_n)$ yields a sequence $(\sigma_n, s_n) \in \mathbb{R}^2$ such that $e^{is_n\Delta} S_{-1}^{\sigma_n} U_c(t_n)$ is strongly convergent in $\dot{H}^1$.

After possibly passing to a subsequence, we may assume that s_n converges to some $s_\infty \in [-\infty, \infty]$. If $s_\infty \in \mathbb{R}$, then $S_{-1}^{\sigma_n} U_c(t_n)$ is also convergent, contradicting (13.34) for $n, n' \to \infty$. If $s_\infty = -\infty$, then $\|U_c(t + t_n)\|_{S(0,\infty)} \to 0$, and if $s_\infty = \infty$, then $\|U_c(t + t_n)\|_{S(-\infty,0)} \to 0$, since the free solutions with the same data as U_c at $t = t_n$ are vanishing in that way: see [11] for more detail. In either case, it contradicts $\|U_c\|_{S(0,T_+(U_c))} = \infty$. □

We are now ready for the final step of the proof of Proposition 3.15.

Claim 13.4. *U_c does not exist.*

Proof. First we consider the case $T_+(U_c) < \infty$. The local wellposedness theory, together with the precompactness of $\mathscr{K}$, implies that blow-up is possible only by concentration $\sigma_c(t) \to \infty$ as $t \nearrow T_+(U_c) < \infty$, see [11] for a proof. For any $m > 0$ and $t \in I(U_c)$, put

$$y_m(t) := \langle |U_c(t)|^2, \phi_m \rangle. \tag{13.35}$$

Then as $t \nearrow T_+(U_c)$, we have

$$y_m = \langle |S_{-1}^{-\sigma_c} U_c|^2, e^{-2\sigma_c} \phi_{me^{\sigma_c}} \rangle \to 0, \tag{13.36}$$

because $|S_{-1}^{-\sigma_c} U_c|^2$ is precompact in L_x^3 while $e^{-2\sigma_c}\phi_{me^{\sigma_c}} \to 0$ weakly in $L_x^{3/2}$. Using (11.32) and Hardy's inequality, we have

$$|\dot{y}_m(t)| \le 2|\langle \nabla U_c, U_c \nabla \phi_m \rangle| \lesssim \|U_c\|_{\dot{H}^1}^2 \lesssim 1, \tag{13.37}$$

uniformly in $m > 0$. Integrating it on $t < T_+(U_c)$ and sending $m \to \infty$, we obtain $\|U_c(t)\|_{L^2}^2 \lesssim |T_+(U_c) - t|$ and so $U_c(0) \in L_x^2$. Hence by the L^2 conservation, we get $\|U_c(0)\|_{L^2} = \|U_c(t)\|_{L^2} \to 0$ as $t \nearrow T_+(U_c)$. So $U_c = 0$ and it contradicts $T_+(U_c) < \infty$.

Therefore $T_+(U_c) = \infty$. For all $t \in [0,\infty)$, we have $\tilde{d}_{\mathcal{W}}(U_c(t)) \ge \delta_B$, and also $\|U_c(t)\|_{\dot{H}^1} \gtrsim 1$ by the small data scattering. Hence Proposition 3.9 implies that

$$\tilde{\kappa} := \inf_{t \ge 0} K(U_c(t)) > 0. \tag{13.38}$$

Suppose that

$$A := \inf_{0 \le t < \infty} \sigma_c(t) > -\infty. \tag{13.39}$$

Then by precompactness of $\mathscr{K}$ and Hardy's and Sobolev's inequalities, there exists m such that

$$\int_{|x|>m} |\nabla U_c|^2 + |U_c|^6 + |U_c/r|^2 dx \ll \tilde{\kappa} \tag{13.40}$$

for all $t \in [0,\infty)$, while $\mathscr{V}_m(t)$ is bounded for $t \to \infty$. Applying (13.2) to U_c, integrating it on $[0,T]$ with $T \to \infty$, we get a contradiction from $T\tilde{\kappa} \le [\mathscr{V}_m]_0^T$.

Therefore $A = -\infty$. Then by continuity of $U_c(t)$, we deduce that $\sigma_c(t_n) \to -\infty$ along some sequence $t_n \to \infty$ satisfying

$$\min_{0 \le s \le t_n} \sigma_c(s) = \sigma_c(t_n). \tag{13.41}$$

By the precompactness of $\mathscr{K}$, we may assume that $S_{-1}^{-\sigma_c(t_n)} U_c(t_n)$ converges strongly in $\dot{H}^1_{\text{radial}}$. Let U_n and U_ω be the solutions of (1.1) with the initial data

$$U_n(0) = S_{-1}^{-\sigma_c(t_n)} U_c(t_n), \quad U_\omega(0) = \lim_{n\to\infty} U_n(0). \tag{13.42}$$

The local wellposedness theory implies that for any compact $J \subset I(U_\omega)$, $U_n \to U_\omega$ as $n \to \infty$ in $C(J; \dot{H}_x^1) \cap S(J)$. This convergence in S and

$$\|U_n\|_{S(-t_n e^{2\sigma_c(t_n)},0)} = \|U_c\|_{S(0,t_n)} \to \|U_c\|_{S(0,\infty)} = \infty \tag{13.43}$$

imply that for each $t \in (-T_-(U_\omega), 0]$ and large n, we have $|t| < t_n e^{2\sigma_c(t_n)}$. Then putting $s_n := t_n - |t|e^{-2\sigma_c(t_n)} \in (0, t_n]$, we have by the scale invariance,

$$S_{-1}^{\sigma_c(t_n)-\sigma_c(s_n)} U_n(t) = S_{-1}^{-\sigma_c(s_n)} U_c(s_n) \in \mathscr{K} \setminus \tilde{B}_{\delta_B}(\mathcal{W}). \tag{13.44}$$

Since $U_n(t) \to U_\omega(t)$ in $\dot H^1$ and $\mathscr{K}$ is precompact, $\sigma_c(s_n)-\sigma_c(t_n)$ converges to some $\sigma_\omega(t) \in [0,\infty)$ up to a subsequence, where positivity comes from (13.41). Then $\{S_{-1}^{-\sigma_\omega(t)} U_\omega(t)\}_{t\in(-T_-(U_\omega),0]}$ is in the closure of $\mathscr{K}$, hence precompact, and also, $\tilde d_{\mathcal{W}}(U_\omega(t)) \ge \delta_B$ for all $t \in (-T_-(U_\omega),0]$. Moreover $\|U_\omega\|_{S(-T_-(U_\omega),0)} = \infty$, since otherwise the blow-up criterion yields $T_-(U_\omega)=\infty$ and the long-time perturbation for $t<0$ yields a uniform bound on $\|U_n\|_{S(-\infty,0)}$ for large n, contradicting (13.43).

Thus we have obtained another critical element $\bar U_\omega(-t)$, that is the time inversion of U_ω, with the scale bound $\sigma_\omega \ge 0$. Hence the above argument for $A > -\infty$ applied to this new critical element yields a contradiction. $\square$

14. Four sets of dynamics

In this section, we prove Theorem 1.2. Let $0 < \beta \ll \epsilon_\star$ and $R > 0$ be such that $\|\phi_R^C W\|_{\dot H^1}^2 \le \beta^4$. We consider four solutions u around $\mathcal{W}$ with the following initial data at $t=0$ in the coordinate (3.16) with $\vec\lambda := (\lambda_1,\lambda_2)$,

$$\begin{aligned} &\gamma(0) := -\phi_R^C W + \omega(\phi_R^C W, g_-)g_+ - \omega(\phi_R^C W, g_+)g_-, \\ &\vec\lambda(0) = \beta(\pm 1, 0), \beta(0,\pm 1). \end{aligned} \tag{14.1}$$

Note that $\gamma(0)$ is the symplectic projection of $-\phi_R^C W$ to the subspace that is perpendicular (with respect to ω) to $\operatorname{span}\{g_-,g_+\}$. This ensures $u(0) \in L_x^2$ so that we can apply the above blow-up result.

Let $I_E(u) \subset I(u)$ be the maximal interval where $u(t) \in B_{\delta_E}(\mathcal{W})$ so that we can use the coordinate (3.16). For brevity, put $\tilde d(t) := \tilde d_{\mathcal{W}}(u(t))$ on $I_E(u)$. Then by Proposition 3.6 and by the same argument as in (9.8)–(9.10), we get

$$\|\gamma(t)\|_{\dot H^1}^2 + O(\tilde d(t)^4) \lesssim \beta^4 + \int_0^{\tau(t)} (\tilde d^2 \|\gamma\|_{\dot H_x^1} + \tilde d^4) d\tau \tag{14.2}$$

within $I_E(u)$. Hence on any interval $J \subset I_E(u)$ where $|\tau| \le \delta_E^{-1/2}$, we have

$$\|\gamma\|_{L_t^\infty(J;\dot H^1)}^2 \lesssim (1+\delta_E^{-1})\|\vec\lambda\|_{L_t^\infty}^4 \lesssim \|\vec\lambda\|_{L_t^\infty(J)}^3. \tag{14.3}$$

Then from (3.11) and a continuity argument we see that

$$\begin{cases} \vec\lambda(0) = \beta(\pm 1,0) \implies \vec\lambda = \pm\beta(\cosh(\mu\tau),\sinh(\mu\tau))(1+O(\beta^{1/2})) \\ \vec\lambda(0) = \beta(0,\pm 1) \implies \vec\lambda = \pm\beta(\sinh(\mu\tau),\cosh(\mu\tau))(1+O(\beta^{1/2})) \end{cases} \tag{14.4}$$

as long as

$$|\tau| \le \delta_E^{-1/2}, \quad \beta e^{\mu|\tau|} \sim \tilde d(t) < \delta_E. \tag{14.5}$$

Using (8.1) and (9.11), we see that if $\vec\lambda(0) = \pm\beta(1,0)$, then $E(u)-E(W) \sim -\beta^2 < 0$ and $K(u(0)) \sim \mp\beta$. Hence by [11],

$$\begin{cases} \vec\lambda(0) = \beta(1,0) \implies u(0) \in \mathcal{B}_- \cap \mathcal{B}_+, \\ \vec\lambda(0) = -\beta(1,0) \implies u(0) \in \mathcal{S}_- \cap \mathcal{S}_+. \end{cases} \tag{14.6}$$

If $\vec\lambda(0) := \pm\beta(0,1)$, then $0 < E(u)-E(W) \sim \beta^2 \sim \tilde d(0)^2 \ll \epsilon_\star^2$, while near the boundary of the interval (14.5), we have $\tilde d(t) \sim \min(\delta_E, \beta e^{\mu\delta_E^{-1/2}}) \gg \beta$. Therefore Proposition 3.7 applies to u at some $t_+ > 0$ in the forward direction and at some

$t_- < 0$ in the backward direction, both within the interval (14.5), where we have (14.4), and also

$$\|\gamma(t_\pm)\|_{\dot H^1} \ll |\lambda_1(t_\pm)| \sim |\lambda_2(t_\pm)|. \tag{14.7}$$

Hence by (9.11), we have $\operatorname{sign} K(u(t_\pm)) = -\operatorname{sign}\lambda_1(t_\pm) = \Theta(u(t_\pm))$ and Proposition 3.15 yields

$$\begin{cases}\vec\lambda(0) = \beta(0,1) \implies \Theta(u(t_\pm)) = \mp 1, & \implies u(0) \in \mathcal{S}_- \cap \mathcal{B}_+,\\ \vec\lambda(0) = -\beta(0,1) \implies \Theta(u(t_\pm)) = \pm 1, & \implies u(0) \in \mathcal{S}_+ \cap \mathcal{B}_-.\end{cases} \tag{14.8}$$

It is obvious that the above argument is stable for adding small perturbation in $\mathbb{R}^2$ to $\vec\lambda(0)$ and small perturbation in H^1 (in the orthogonal subspace) to $\gamma(0)$. Hence we obtain a small open set in H^1 around each of the four solutions. □

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