# CLASSIFYING SUBCATEGORIES OF MODULES OVER A COMMUTATIVE NOETHERIAN RING

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Abstract. Let $R$ be a quotient ring of a commutative coherent regular ring by a finitely generated ideal. Hovey gave a bijection between the set of coherent subcategories of the category of finitely presented $R$-modules and the set of thick subcategories of the derived category of perfect $R$-complexes. Using this isomorphism, he proved that every coherent subcategory of finitely presented $R$-modules is a Serre subcategory. In this paper, it is proved that this holds whenever $R$ is a commutative noetherian ring. This paper also yields a module version of the bijection between the set of localizing subcategories of the derived category of $R$-modules and the set of subsets of $\operatorname{Spec} R$ which was given by Neeman.

## 1. Introduction

Around 1990, Hopkins [6] and Neeman [9] gave a classification theorem of the thick subcategories of the derived category of perfect complexes (i.e. finite complexes of finitely generated projective modules) over a commutative noetherian ring in terms of the ring spectrum. After that, Thomason [10] generalized this classification theorem to quasi-compact and quasi-separated schemes, in particular, to arbitrary commutative rings. Let $\mathcal{D}_{\mathrm{perf}}(R)$ denote the derived category of perfect complexes over a commutative ring $R$. The classification theorem (for commutative rings) can be stated as follows.

**Theorem** (Hopkins-Neeman-Thomason)**.** *Let $R$ be a commutative ring. Then there is an isomorphism*

$$\begin{aligned} &\{\ \textit{thick subcategories of } \mathcal{D}_{\mathrm{perf}}(R)\ \} \\ \cong &\{\ \textit{complements of intersections of quasi-compact open subsets of } \operatorname{Spec} R\ \} \end{aligned}$$

*of lattices.*

Here we recall the definitions of several subcategories of an abelian category. A coherent subcategory is defined to be a full subcategory which is closed under kernels, cokernels and extensions. A Serre subcategory is defined to be a coherent subcategory which is closed under subobjects. A torsion class is defined to be a Serre subcategory which is closed under arbitrary direct sums. Let $R$ be a commutative ring. We denote by $\operatorname{Mod} R$ the category of $R$-modules and by $\operatorname{mod} R$ the full subcategory of finitely presented $R$-modules. If $R$ is noetherian, then the lattice of

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Serre subcategories of $\operatorname{mod} R$, the lattice of torsion classes of $\operatorname{Mod} R$, and the lattice of subsets of $\operatorname{Spec} R$ which are closed under specialization are isomorphic to each other. Taking advantage of the Hopkins-Neeman-Thomason theorem, Garkusha and Prest [3, 4] gave the following result very recently. (A torsion class $\mathcal{X}$ of $\mathcal{M} = \operatorname{Mod} R$ is said to be of finite type if the inclusion functor $\mathcal{M}/\mathcal{X} \to \mathcal{M}$, where $\mathcal{M}/\mathcal{X}$ denotes the quotient category, preserves arbitrary direct sums.)

**Theorem** (Garkusha-Prest)**.** *Let $R$ be a commutative ring. Then the following hold.*

(1) *One has lattice isomorphisms*

$$\begin{aligned}&\{\text{ thick subcategories of } \mathcal{D}_{\mathrm{perf}}(R)\}\\ \cong\ &\{\text{ torsion classes of finite type of } \operatorname{Mod} R\}\\ \cong\ &\{\text{ complements of intersections of quasi-compact open subsets of } \operatorname{Spec} R\}.\end{aligned}$$

(2) *Suppose that $R$ is coherent. Then one has lattice isomorphisms*

$$\begin{aligned}&\{\text{ thick subcategories of } \mathcal{D}_{\mathrm{perf}}(R)\}\\ \cong\ &\{\text{ Serre subcategories of } \operatorname{mod} R\}\\ \cong\ &\{\text{ complements of intersections of quasi-compact open subsets of } \operatorname{Spec} R\}.\end{aligned}$$

Also by using the Hopkins-Neeman-Thomason theorem, Hovey [7] proved the following classification theorem of coherent subcategories.

**Theorem** (Hovey)**.** *Let $R$ be a quotient ring of a commutative coherent regular ring by a finitely generated ideal. Then the following hold.*

(1) *One has a lattice isomorphism*

$$\begin{aligned}&\{\text{ thick subcategories of } \mathcal{D}_{\mathrm{perf}}(R)\}\\ \cong\ &\{\text{ coherent subcategories of } \operatorname{mod} R\}.\end{aligned}$$

(2) *Every coherent subcategory of $\operatorname{mod} R$ is a Serre subcategory.*

As Hovey pointed it out as an interesting fact, there was no direct proof of the second assertion of the above theorem; it could not be proved without resorting to the rather difficult classification of thick subcategories of the derived category, namely the Hopkins-Neeman-Thomason theorem. Hovey conjectures that the isomorphism stated in the above theorem always holds for commutative coherent rings. (Recall that a commutative ring is called coherent if every finitely generated ideal is finitely presented.)

**Conjecture** (Hovey)**.** Let $R$ be a commutative coherent ring. Then one has a lattice isomorphism

$$\begin{aligned}&\{\text{ thick subcategories of } \mathcal{D}_{\mathrm{perf}}(R)\}\\ \cong\ &\{\text{ coherent subcategories of } \operatorname{mod} R\}.\end{aligned}$$

One of the main purposes of this paper is to prove that this conjecture is true if $R$ is noetherian. First of all, we will directly prove that every coherent subcategory of $\operatorname{mod} R$ is Serre; in the proof, we will not apply the Hopkins-Neeman-Thomason theorem. Actually, we will not use the notion of a derived category in the proof. Furthermore, the proof we will give is much simpler than Hovey's. After that, we

will prove that if $R$ is noetherian, then all the lattices appearing in the above part are isomorphic to each other. Our first main theorem is the following.

**Theorem A.** *Let $R$ be a commutative noetherian ring. Then every coherent subcategory of* $\operatorname{mod} R$ *is a Serre subcategory, and one has the following isomorphisms of lattices:*

$$\begin{aligned}
&\{\, \textit{thick subcategories of } \mathcal{D}_{\mathrm{perf}}(R) \,\} \\
\cong &\{\, \textit{coherent subcategories of } \operatorname{mod} R \,\} \\
= &\{\, \textit{Serre subcategories of } \operatorname{mod} R \,\} \\
\cong &\{\, \textit{torsion classes of } \operatorname{Mod} R \,\} \\
\cong &\{\, \textit{subsets of } \operatorname{Spec} R \textit{ closed under specialization} \,\} \\
= &\{\, \textit{complements of intersections of quasi-compact open subsets of } \operatorname{Spec} R \,\}.
\end{aligned}$$

On the other hand, Neeman [9] showed the following theorem.

**Theorem** (Neeman)**.** *Let $R$ be a commutative noetherian ring. Then one has an isomorphism*

$$\begin{aligned}
&\{\, \textit{localizing subcategories of } \mathcal{D}(R) \,\} \\
\cong &\{\, \textit{subsets of } \operatorname{Spec} R \,\}
\end{aligned}$$

*of lattices. Moreover, this induces an isomorphism*

$$\begin{aligned}
&\{\, \textit{smashing subcategories of } \mathcal{D}(R) \,\} \\
\cong &\{\, \textit{subsets of } \operatorname{Spec} R \textit{ closed under specialization} \,\}
\end{aligned}$$

*of lattices.*

Here, $\mathcal{D}(R)$ denotes the derived category of $\operatorname{Mod} R$. A localizing subcategory is defined as a full triangulated subcategory which is closed under arbitrary direct sums, and a smashing subcategory is defined as a localizing subcategory such that Bousfield localization commutes with arbitrary direct sums. The second main purpose of this paper is to construct a module version of the above Neeman's theorem:

**Theorem B.** *Let $R$ be a commutative noetherian ring. Then one has an isomorphism*

$$\begin{aligned}
&\{\, \textit{full subcategories of } \operatorname{mod} R \textit{ closed under submodules and extensions} \,\} \\
\cong &\{\, \textit{subsets of } \operatorname{Spec} R \,\}
\end{aligned}$$

*of lattices. Moreover, this induces the isomorphism*

$$\begin{aligned}
&\{\, \textit{Serre subcategories of } \operatorname{mod} R \,\} \\
\cong &\{\, \textit{subsets of } \operatorname{Spec} R \textit{ closed under specialization} \,\}
\end{aligned}$$

*of lattices given in Theorem A.*

In Section 2, we will give the precise definitions of subcategories which are stated above, and study several basic properties of them. In Sections 3 and 4, we shall give proofs of Theorems A and B, respectively.

**Remark.** It is known that Hovey's paper [7] contains an error, but it is not relevant to the results and arguments in this paper. The error has recently been corrected by Krause [8].

## 2. Basic properties

In this section, we will give some definitions and several basic results most of which are necessary to state and prove the main results of this paper.

Let $\mathcal{A}$ be an additive category. A full subcategory $\mathcal{X}$ of $\mathcal{A}$ is said to be *closed under isomorphisms* (or *replete*) provided that if $X$ is an object of $\mathcal{X}$ and $Y$ is an object of $\mathcal{A}$ which is isomorphic to $X$, then $Y$ is also an object of $\mathcal{X}$. In this paper, by a *subcategory* we always mean a nonempty full subcategory which is closed under isomorphisms.

First of all, we recall the definitions of various types of closedness of a subcategory of an additive category.

**Definition 2.1.** Let $\mathcal{A}$ be an additive category, and let $\mathcal{X}$ be a full subcategory of $\mathcal{A}$. We say that

(1) $\mathcal{X}$ is *closed under subobjects* (resp. *closed under quotient objects*) provided that if $X$ is an object of $\mathcal{X}$ and $Y \in \mathcal{A}$ is a subobject (resp. a quotient object) of $X$, then $Y$ is also an object of $\mathcal{X}$.
(2) $\mathcal{X}$ is *closed under direct summands* (or *closed under retracts*) provided that if $X$ is an object of $\mathcal{X}$ and $Y \in \mathcal{A}$ is a direct summand of $X$, then $Y$ is also an object of $\mathcal{X}$.
(3) $\mathcal{X}$ is *closed under finite direct sums* (resp. *closed under arbitrary direct sums*) if all finite direct sums (resp. arbitrary direct sums) of objects of $\mathcal{X}$ are objects of $\mathcal{X}$.
(4) $\mathcal{X}$ is *closed under extensions* provided that for any exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$, if both $A$ and $C$ are objects of $\mathcal{X}$, then so is $B$.
(5) $\mathcal{X}$ is *closed under kernels* (resp. *closed under images*, *closed under cokernels*) if the kernel (resp. the image, the cokernel) of every morphism of objects of $\mathcal{X}$ is also an object of $\mathcal{X}$.
(6) $\mathcal{X}$ is *closed under homologies* if the homologies of every chain complex of objects of $\mathcal{X}$ are objects of $\mathcal{X}$.
(7) $\mathcal{X}$ is *closed under direct limits* provided that if $\{X_\lambda\}_{\lambda\in\Lambda}$ is a direct system of objects of $\mathcal{X}$ then the direct limit $\varinjlim_{\lambda\in\Lambda} X_\lambda$ is an object of $\mathcal{X}$.

Here we study the relationships among the closed properties of a subcategory which are defined above.

**Proposition 2.2.** *Let $\mathcal{A}$ be an abelian category and $\mathcal{X}$ a subcategory of $\mathcal{A}$. Then the following hold.*

(1) *If $\mathcal{X}$ is closed under kernels or cokernels, then $\mathcal{X}$ is closed under direct summands and contains the zero object of $\mathcal{A}$.*
(2) *If $\mathcal{X}$ is closed under extensions, then $\mathcal{X}$ is closed under finite direct sums.*
(3) *If $\mathcal{X}$ is closed under kernels and cokernels, then $\mathcal{X}$ is closed under images and homologies.*
(4) *If $\mathcal{X}$ is closed under subobjects and cokernels, then $\mathcal{X}$ is closed under quotient objects.*
(5) *If $\mathcal{X}$ is closed under arbitrary direct sums and quotient objects, then $\mathcal{X}$ is closed under direct limits.*

*Proof.* (1) Let $X$ be an object of $\mathcal{X}$ and $Y$ an object of $\mathcal{A}$ which is a direct summand of $X$. Then we can write $X = Y \oplus Z$ for some object $Z$ of $\mathcal{A}$. Considering the morphism $f : X \to X$ given by $(y, z) \mapsto (0, z)$ for $y \in Y$ and $z \in Z$, we see that both the kernel and the cokernel of $f$ are isomorphic to $Y$. By the assumption that $\mathcal{X}$ is closed under kernels or cokernels, the object $Y$ is in $\mathcal{X}$. Therefore $\mathcal{X}$ is closed under direct summands.

On the other hand, since $\mathcal{X}$ is nonempty, there exists an object $W \in \mathcal{X}$. Let $g : W \to W$ be the identity morphism. Then both the kernel and the cokernel of $g$ are the zero object. Since $\mathcal{X}$ is closed under kernels or cokernels, $\mathcal{X}$ contains the zero object.

(2) Let $X$ and $Y$ be two objects of $\mathcal{X}$. Then there exists a natural split exact sequnece

$$0 \to X \to X \oplus Y \to Y \to 0$$

in $\mathcal{A}$. Since $\mathcal{X}$ is closed under extensions, we see from this exact sequence that the direct sum $X \oplus Y$ is an object of $\mathcal{X}$. This argument shows that $\mathcal{X}$ is closed under finite direct sums.

(3) Let $f : X \to Y$ be a morphism of objects of $\mathcal{X}$. Then there is an exact sequence

$$0 \to \operatorname{Im} f \to Y \xrightarrow{\pi} \operatorname{Coker} f \to 0$$

in $\mathcal{A}$. Since $\mathcal{X}$ is closed under cokernels, the object $\operatorname{Coker} f$ is in $\mathcal{X}$. Noting that the object $\operatorname{Im} f$ is the kernel of $\pi$ and $\mathcal{X}$ is closed under kernels, we see that $\operatorname{Im} f$ is in $\mathcal{X}$. Therefore $\mathcal{X}$ is closed under images. That $\mathcal{X}$ is closed under kernels and images implies that $\mathcal{X}$ is closed under homologies.

(4) Let $X$ be an object of $\mathcal{X}$ and $Y \in \mathcal{A}$ a quotient object of $X$. Then there exists a subobject $Z \in \mathcal{A}$ of $X$ such that $Y = X/Z$. Since $\mathcal{X}$ is closed under subobjects, $Z$ is an object of $\mathcal{X}$. Let $i : Z \to X$ be the natural inclusion. Since $Y$ coincides with the cokernel of $i$ and $\mathcal{X}$ is closed under cokernels, we have $Y \in \mathcal{X}$. This says that $\mathcal{X}$ is closed under quotient objects.

(5) Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be a direct system of objects of $\mathcal{X}$, and let $X = \varinjlim_{\lambda \in \Lambda} X_\lambda$ be the direct limit. Then, by definition, $X$ is a quotient object of $Y := \bigoplus_{\lambda \in \Lambda} X_\lambda$. Since $\mathcal{X}$ is closed under arbitrary direct sums, $Y$ is an object of $\mathcal{X}$. Since $\mathcal{X}$ is closed under quotient objects, $X$ is an object of $\mathcal{X}$. Consequently, $\mathcal{X}$ is closed under direct limits. □

Next, we recall the definitions of a coherent subcategory, a Serre subcategory and a torsion class, which will play important roles throughout this paper.

**Definition 2.3.** Let $\mathcal{A}$ be an abelian category, and let $\mathcal{X}$ be a subcategory of $\mathcal{A}$. Then

(1) $\mathcal{X}$ is called a *coherent subcategory* of $\mathcal{A}$ if it is closed under kernels, cokernels and extensions.
(2) $\mathcal{X}$ is called a *Serre subcategory* of $\mathcal{A}$ if it is a coherent subcategory which is closed under subobjects.
(3) $\mathcal{X}$ is called a *(hereditary) torsion class* of $\mathcal{A}$ if it is a Serre subcategory which is closed under arbitrary direct sums.

**Remark 2.4.** The original definition of a coherent subcategory is as follows: let $\mathcal{A}$ be an abelian category, and let $\mathcal{X}$ be a subcategory of $\mathcal{A}$. It is said that $\mathcal{X}$ is

coherent provided that for any exact sequence

$$A \to B \to C \to D \to E$$

in $\mathcal{A}$, if $A$, $B$, $D$ and $E$ are in $\mathcal{A}$, then so is $C$. One can easily check that this definition is equivalent to our definition.

A coherent subcategory, a Serre subcategory and a torsion class have the following properties, which immediately follow from Proposition 2.2.

**Corollary 2.5.** *Let $\mathcal{A}$ be an abelian category.*

(1) *Let $\mathcal{X}$ be a coherent subcategory of $\mathcal{A}$. Then $\mathcal{X}$ contains the zero object of $\mathcal{A}$, and $\mathcal{X}$ is closed under finite direct sums, direct summands, images and homologies.*
(2) *Let $\mathcal{X}$ be a Serre subcategory of $\mathcal{A}$. Then $\mathcal{X}$ is closed under quotient objects.*
(3) *Let $\mathcal{X}$ be a torsion class of $\mathcal{A}$. Then $\mathcal{X}$ is closed under direct limits.*

Throughout the rest of this paper, let $R$ be a commutative ring. We denote by $\operatorname{Mod} R$ the category of $R$-modules, and by $\operatorname{mod} R$ the full subcategory of $\operatorname{Mod} R$ consisting of finitely presented $R$-modules. Let us recall the definitions of a lattice and a homomorphism of lattices.

**Definition 2.6.** (1) Let $\mathcal{L}$ be an ordered set.
  (i) Let $x, y \in \mathcal{L}$ be elements. If the supremum (resp. infimum) of the set $\{x, y\}$ exists, then it is called the *join* (resp. *meet*) of $x$ and $y$, and denoted by $x \vee y$ (resp. $x \wedge y$).
  (ii) It is said that $\mathcal{L}$ is a *lattice* if any two elements of $\mathcal{L}$ have both the join and the meet.
(2) A map $f : \mathcal{L} \to \mathcal{L}'$ of lattices is called a *(lattice) homomorphism* if $f(x \vee y) = f(x) \vee f(y)$ and $f(x \wedge y) = f(x) \wedge f(y)$ for all $x, y \in \mathcal{L}$. A bijective lattice homomorphism is called a *(lattice) isomorphism*.

This paper will deal with the following lattices of subcategories of modules.

**Definition 2.7.** (1) Let $\operatorname{coh}(R)$ be the coherent subcategory of $\operatorname{Mod} R$ generated by $R$. We denote by $\mathcal{L}_{\mathsf{coh}}(\operatorname{coh}(R))$ the lattice of all coherent subcategories of $\operatorname{coh}(R)$.
(2) Let $\operatorname{Serre}(R)$ be the Serre subcategory of $\operatorname{Mod} R$ generated by $R$. We denote by $\mathcal{L}_{\mathsf{Serre}}(\operatorname{Serre}(R))$ the lattice of all Serre subcategories of $\operatorname{Serre}(R)$.
(3) We denote by $\mathcal{L}_{\mathsf{tors}}(\operatorname{Mod} R)$ the lattice of all torsion classes of $\operatorname{Mod} R$.

**Remark 2.8.** If $R$ is noetherian, then one has $\operatorname{coh}(R) = \operatorname{Serre}(R) = \operatorname{mod} R$; see the first sentence and the latter half of Page 3185 of [7]. Hence, whenever $R$ is a noetherian ring, one has $\mathcal{L}_{\mathsf{coh}}(\operatorname{coh}(R)) = \mathcal{L}_{\mathsf{coh}}(\operatorname{mod} R)$ and $\mathcal{L}_{\mathsf{Serre}}(\operatorname{Serre}(R)) = \mathcal{L}_{\mathsf{Serre}}(\operatorname{mod} R)$.

A *perfect* $R$-complex $P_\bullet$ is defined to be an $R$-complex of the form

$$P_\bullet = (0 \to P_s \to P_{s-1} \to \cdots \to P_{t+1} \to P_t \to 0),$$

where each $P_i$ is a finitely generated projective $R$-module. We denote by $\mathcal{D}(R)$ the derived category of the category $\operatorname{Mod} R$, and by $\mathcal{D}_{\mathrm{perf}}(R)$ the full subcategory of $\mathcal{D}(R)$ consisting of $R$-complexes which are isomorphic to perfect $R$-complexes.

**Remark 2.9.** Recall that an object $X$ of an additive category $\mathcal{A}$ is called *small* (or *compact*) if the functor $\mathrm{Hom}_{\mathcal{A}}(X, -)$ preserves arbitrary direct sums. It is well-known that a complex of $R$-modules is quasi-isomorphic to a perfect complex if and only if it is a small object of $\mathcal{D}(R)$ (cf. [2, 3.7]).

**Definition 2.10.** (1) Let $\mathcal{T}$ be a triangulated category and $\mathcal{X}$ a subcategory of $\mathcal{T}$. Then we say that $\mathcal{X}$ is a *thick subcategory* if it satisfies the following two conditions.
   (a) $\mathcal{X}$ is closed under direct summands.
   (b) For any exact triangle $A \to B \to C \to \Sigma A$ in $\mathcal{T}$, if two of the objects $A, B, C$ are in $\mathcal{X}$, then so is the third.

(2) We denote by $\mathcal{L}_{\mathsf{thick}}(\mathcal{D}_{\mathrm{perf}}(R))$ the lattice of all of the thick subcategories of $\mathcal{D}(R)$ whose objects are small, namely, the lattice of all thick subcategories of $\mathcal{D}_{\mathrm{perf}}(R)$.

Recall that a subset $S$ of $\operatorname{Spec} R$ is said to be *closed under specialization* provided that if $\mathfrak{p}$ is a prime ideal in $S$ and $\mathfrak{q}$ is a prime ideal containing $\mathfrak{p}$ then $\mathfrak{q}$ is also in $S$. Dually, $S$ is said to be *closed under generalization* provided that if $\mathfrak{p}$ is a prime ideal in $S$ and $\mathfrak{q}$ is a prime ideal contained in $\mathfrak{p}$ then $\mathfrak{q}$ is also in $S$. Note that every union of closed subsets of $\operatorname{Spec} R$ is closed under specialization. Similarly, every intersection of open subsets of $\operatorname{Spec} R$ is closed under generalization.

**Definition 2.11.** We denote by $\mathcal{L}_{\mathsf{spcl}}(\operatorname{Spec} R)$ the lattice of all subsets of $\operatorname{Spec} R$ that are closed under specialization, and by $\mathcal{L}^0_{\mathsf{spcl}}(\operatorname{Spec} R)$ the sublattice of $\mathcal{L}_{\mathsf{spcl}}(\operatorname{Spec} R)$ consisting of all complements of arbitrary intersections of quasi-compact open subsets of $\operatorname{Spec} R$.

**Remark 2.12.** An open subset $U$ of $\operatorname{Spec} R$ is quasi-compact if and only if $U = D(I) := \{\mathfrak{p} \in \operatorname{Spec} R \mid I \nsubseteq \mathfrak{p}\}$ for some finitely generated ideal $I$ of $R$; see the argument on the top of [5, Page 72]. Therefore, if $R$ is noetherian, then every open subset of $\operatorname{Spec} R$ is quasi-compact, and $\mathcal{L}^0_{\mathsf{spcl}}(\operatorname{Spec} R)$ consists of all unions of closed subsets of $\operatorname{Spec} R$.

Hovey [7] constructs the following order-preserving maps among the lattices which we defined above:

$$\begin{tikzcd}
\mathcal{L}_{\mathsf{spcl}}(\operatorname{Spec} R) \arrow[d, shift left, "\sigma"] \\
\mathcal{L}_{\mathsf{tors}}(\operatorname{Mod} R) \arrow[u, shift left, "\tau"] \arrow[d, shift left, "\mu"] \\
\mathcal{L}_{\mathsf{Serre}}(\operatorname{Serre}(R)) \arrow[u, shift left, "\nu"] \arrow[d, shift left, "\alpha"] \\
\mathcal{L}_{\mathsf{coh}}(\operatorname{coh}(R)) \arrow[u, shift left, "\beta"] \arrow[d, shift left, "f"] \\
\mathcal{L}_{\mathsf{thick}}(\mathcal{D}_{\mathrm{perf}}(R)) \arrow[u, shift left, "g"]
\end{tikzcd}$$

The above maps are defined as follows.

(1) For $S \in \mathcal{L}_{\mathsf{spcl}}(\operatorname{Spec} R)$, let $\sigma(S)$ be the full subcategory of $\operatorname{Mod} R$ consisting of all $R$-modules $M$ with $\operatorname{Supp} M \subseteq S$. For $\mathcal{X} \in \mathcal{L}_{\mathsf{tors}}(\operatorname{Mod} R)$, let $\tau(\mathcal{X})$ be the union $\bigcup_{M \in \mathcal{X}} \operatorname{Supp} M$.
(2) For $\mathcal{X} \in \mathcal{L}_{\mathsf{tors}}(\operatorname{Mod} R)$, let $\mu(\mathcal{X})$ be the full subcategory of $\operatorname{Mod} R$ consisting of all $R$-modules $M$ in the intersection of $\mathcal{X}$ and $\operatorname{Serre}(R)$. For $\mathcal{Y} \in \mathcal{L}_{\mathsf{Serre}}(\operatorname{Serre}(R))$, let $\nu(\mathcal{Y})$ be the torsion class of $\operatorname{Mod} R$ generated by $\mathcal{Y}$.
(3) For $\mathcal{Y} \in \mathcal{L}_{\mathsf{Serre}}(\operatorname{Serre}(R))$, let $\alpha(\mathcal{Y})$ be the full subcategory of $\operatorname{Mod} R$ consisting of all $R$-modules $M$ in the intersection of $\mathcal{Y}$ and $\operatorname{coh}(R)$. For $\mathcal{Z} \in \mathcal{L}_{\mathsf{coh}}(\operatorname{coh}(R))$, let $\beta(\mathcal{Z})$ be the Serre subcategory of $\operatorname{Mod} R$ generated by $\mathcal{Z}$.
(4) For $\mathcal{Z} \in \mathcal{L}_{\mathsf{coh}}(\operatorname{coh}(R))$, let $f(\mathcal{Z})$ be the full subcategory of $\mathcal{D}(R)$ consisting of all complexes $X_\bullet \in \mathcal{D}_{\mathrm{perf}}(R)$ such that the $i$th homology $H_i(X_\bullet)$ is in $\mathcal{Z}$ for any $i \in \mathbb{Z}$. For $\mathcal{W} \in \mathcal{L}_{\mathsf{thick}}(\mathcal{D}_{\mathrm{perf}}(R))$, let $g(\mathcal{W})$ be the coherent subcategory of $\operatorname{Mod} R$ generated by $\{\, H_i(X_\bullet) \mid X_\bullet \in \mathcal{W}, i \in \mathbb{Z} \,\}$.

We recall here the definition of an adjoint pair of order-preserving maps. Let $\phi : A \to B$ and $\psi : B \to A$ be two order-preserving maps between ordered sets. Then the pair $(\phi, \psi)$ is said to be an *adjoint pair* provided that $\phi(a) \leq b$ if and only if $a \leq \psi(b)$ for any $a \in A$ and $b \in B$. Concerning the above order-preserving maps among lattices, the following proposition holds.

**Proposition 2.13.** (1) (a) *The pair $(\tau, \sigma)$ is an adjoint pair.*
(b) *The composite map $\tau\sigma$ is the identity map.*
(c) *If $R$ is noetherian, then $\sigma$ is a lattice isomorphism and $\tau$ is the inverse homomorphism.*
(2) (a) *The pair $(\nu, \mu)$ is an adjoint pair.*
(b) *The composite map $\nu\mu$ is the identity map.*
(c) *If $R$ is noetherian, then $\mu$ is a lattice isomorphism and $\nu$ is the inverse homomorphism.*
(3) (a) *The pair $(\beta, \alpha)$ is an adjoint pair.*
(b) *If $R$ is noetherian, then the composite map $\beta\alpha$ is the identity map.*
(4) (a) *The pair $(g, f)$ is an adjoint pair.*
(b) *The composite map $fg$ is the identity map.*

*Proof.* It is easy to check that every order-preserving bijective map between two lattices is a lattice isomorphism. Hence we see from the arguments in Page 3185 of [7] that the assertions (1), (2) and (3) hold. (Here, note that derived categories do not appear in those arguments.)

The assertion (a) and (b) in (4) are shown in [7, Proposition 1.4] and [7, Corollary 2.2], respectively. □

The above proposition especially says that one has the following relationships between two modules whose supports have inclusion relation.

**Corollary 2.14.** *Let $R$ be a noetherian ring, and let $M, N$ be $R$-modules with $\operatorname{Supp} M \subseteq \operatorname{Supp} N$. Then $M$ belongs to the torsion class of $\operatorname{Mod} R$ generated by $N$. If $M$ and $N$ are finitely generated, then $M$ belongs to the Serre subcategory of $\operatorname{mod} R$ generated by $N$.*

*Proof.* Proposition 2.13 says that the maps $\sigma, \mu$ are isomorphisms whose inverse maps are $\tau, \nu$ respectively. Let $\mathcal{T}$ be the torsion class of $\operatorname{Mod} R$ generated by $N$.

We have $\mathcal{T} = \sigma\tau(\mathcal{T})$, which is the full subcategory of $\operatorname{Mod} R$ consisting of all $R$-modules $K$ with $\operatorname{Supp} K \subseteq \bigcup_{L\in\mathcal{T}} \operatorname{Supp} L$. Since $N$ is in $\mathcal{T}$, we get $\operatorname{Supp} M \subseteq \operatorname{Supp} N \subseteq \bigcup_{L\in\mathcal{T}} \operatorname{Supp} L$. Therefore $M$ is in $\mathcal{T}$.

Suppose that both $M$ and $N$ are finitely generated. Let $\mathcal{S}$ be the Serre subcategory of $\operatorname{mod} R$ generated by $N$. Then we have $\mathcal{S} = \mu\sigma\tau\nu(\mathcal{S})$, which consists of all finitely generated $R$-modules $K$ with $\operatorname{Supp} K \subseteq \bigcup_{L\in\mathcal{T}} \operatorname{Supp} L (= \bigcup_{L\in\mathcal{S}} \operatorname{Supp} L)$. Hence $M$ is in $\mathcal{S}$. □

## 3. Coherent subcategories are Serre

Throughout this section, let $R$ be a commutative ring. We begin with proving the following theorem.

**Theorem 3.1.** *Let $R$ be a noetherian ring. Let $\mathcal{X}$ be a full subcategory of* $\operatorname{mod} R$ *which is closed under finite direct sums, kernels and cokernels. Then $\mathcal{X}$ is closed under submodules and quotient modules.*

*Proof.* According to Proposition 2.2(4), it is enough to prove that $\mathcal{X}$ is closed under submodules. Assume that $\mathcal{X}$ is not closed under submodules. Then there exist an $R$-module $X$ in $\mathcal{X}$ and an $R$-submodule $M$ of $X$ such that $M$ does not belong to $\mathcal{X}$. Since $R$ is noetherian and $X$ is a finitely generated $R$-module, $X$ is a noetherian $R$-module. Hence we can choose $M$ to be a maximal element, with respect to the inclusion relation, of the set of $R$-submodules $M'$ of $X$ such that $M'$ does not belong to $\mathcal{X}$. Since $M$ does not coincide with $X$, there is an element $x \in X - M$. Set $Y = M + Rx$. Note that $Y$ is an $R$-submodule of $X$ strictly containing $M$. By the maximality of $M$, the module $Y$ is in $\mathcal{X}$. Put $I = (M : x) := \{a \in R \mid ax \in M\}$. This is an ideal of $R$, and we easily see that the quotient $R$-module $Y/M$ is isomorphic to $R/I$. There is an exact sequence

$$0 \to M \to Y \xrightarrow{\pi} R/I \to 0$$

of $R$-modules. Since $M \notin \mathcal{X}$ and $Y \in \mathcal{X}$ and $\mathcal{X}$ is closed under kernels, we see from this exact sequence that $R/I$ must not be in $\mathcal{X}$.

On the other hand, the map $\pi$ in the exact sequence induces a surjective homomorphism

$$\overline{\pi} : Y/IY \to R/I$$

of $R/I$-modules, which sends the residue class of $y \in Y$ in $Y/IY$ to $\pi(y)$. Of course $R/I$ is a projective $R/I$-module, so $\overline{\pi}$ is a split epimorphism. Therefore $R/I$ is isomorphic to a direct summand of $Y/IY$. The noetherian property of $R$ implies that the ideal $I$ is finitely generated; write $I = (a_1, a_2, \dots, a_n)R$ for some elements $a_1, a_2, \dots, a_n \in R$. There is an exact sequence

$$R^{\oplus n} \xrightarrow{(a_1,\dots,a_n)} R \longrightarrow R/I \longrightarrow 0$$

of $R$-modules. Tensoring the $R$-module $Y$ with this exact sequence yields another exact sequence of $R$-modules:

$$Y^{\oplus n} \xrightarrow{(a_1,\dots,a_n)} Y \longrightarrow Y/IY \longrightarrow 0.$$

The assumption of the theorem says that $\mathcal{X}$ is closed under finite direct sums, cokernels, and direct summands; see Proposition 2.2(1). Hence the direct sum $Y^{\oplus n}$ belongs to $\mathcal{X}$, and so does the module $Y/IY$, and therfore so does $R/I$. This is a contradiction, which says that $\mathcal{X}$ is closed under submodules. Thus the proof of the theorem is completed. □

Corollary 2.5(1) says that any coherent subcategory of $\operatorname{mod} R$ is closed under finite direct sums, kernels and cokernels. Thus, according to Theorem 3.1, we obtain the following result, which is the former half part of Theorem A in the first section of this paper.

**Corollary 3.2.** *Let $R$ be a noetherian ring. Then every coherent subcategory of* $\operatorname{mod} R$ *is a Serre subcategory of* $\operatorname{mod} R$.

Now, let us check that the subset $\mathcal{L}_{\mathsf{spcl}}(\operatorname{Spec} R)$ coincides with $\mathcal{L}^0_{\mathsf{spcl}}(\operatorname{Spec} R)$ if $R$ is a noetherian ring.

**Proposition 3.3.** (1) *Let $Z$ be a subset of* $\operatorname{Spec} R$ *which is closed under specialization. Then one has*

$$Z = \bigcup_{\mathfrak{p} \in Z} V(\mathfrak{p}).$$

(2) *Let $R$ be a noetherian ring. Then*

$$\mathcal{L}_{\mathsf{spcl}}(\operatorname{Spec} R) = \mathcal{L}^0_{\mathsf{spcl}}(\operatorname{Spec} R).$$

*Proof.* (1) Let $\mathfrak{q}$ be a prime ideal in $Z$. Then $\mathfrak{q}$ is in $V(\mathfrak{q})$, which is contained in $\bigcup_{\mathfrak{p} \in Z} V(\mathfrak{p})$. As to the opposite inclusion relation, take a prime ideal $\mathfrak{q}$ in $\bigcup_{\mathfrak{p} \in Z} V(\mathfrak{p})$. Then $\mathfrak{q}$ is in $V(\mathfrak{p})$ for some $\mathfrak{p} \in Z$. Since $Z$ is closed under specialization, we get $\mathfrak{q} \in Z$, as required.

(2) This immediately follows from Remark 2.12 and the assertion (1). □

To prove our next result, we prepare here the following two lemmas.

**Lemma 3.4.** *Let $R$ be a noetherian ring and $\mathcal{M}$ a subcategory of* $\operatorname{mod} R$ *which is closed under finite direct sums and cokernels. Let $M$ be a cyclic $R$-module in $\mathcal{M}$. Then there exists a perfect $R$-complex $X_\bullet$ such that $H_0(X_\bullet)$ is isomorphic to $M$ and that $H_j(X_\bullet)$ belongs to $\mathcal{M}$ for any $j \in \mathbb{Z}$.*

*Proof.* Since $M$ is cyclic, there exists an ideal $I$ of $R$ such that $M$ is isomorphic to $R/I$. The noetherian property of $R$ implies that the ideal $I$ is finitely generated; let $\boldsymbol{x} = x_1, x_2, \dots, x_r$ be a system of generators of $I$. Consider the Koszul complex $K_\bullet := K_\bullet(\boldsymbol{x}, R)$ of the sequence $\boldsymbol{x}$. The complex $K_\bullet$ is a perfect $R$-complex, and the zeroth homology $H_0(K_\bullet)$ is equal to $R/I$, which is isomorphic to $M$. Thus, to show the lemma, it suffices to check that the homology $H_j(K_\bullet)$ belongs to $\mathcal{M}$ for each $j \in \mathbb{Z}$. Note from [1, Proposition 1.6.5(b)] that the $R$-module $H_j(K_\bullet) = H_j(\boldsymbol{x}, R)$ is annihilated by the ideal $I = \boldsymbol{x}R$. Hence $H_j(K_\bullet)$ can be regarded as an $R/I$-module. Since $R$ is noetherian, $H_j(K_\bullet)$ is finitely generated, hence finitely presented as an $R/I$-module. It follows that there is an exact sequence

$$(R/I)^{\oplus n} \to (R/I)^{\oplus m} \to H_j(K_\bullet) \to 0$$

of $R/I$-modules. Since $\mathcal{M}$ is closed under finite direct sums, the sum $(R/I)^{\oplus i} \cong M^{\oplus i}$ is an object of $\mathcal{M}$ for any $i \geq 0$. Since $\mathcal{M}$ is closed under cokernels, the module $H_j(K_\bullet)$ is in $\mathcal{M}$, as desired. □

**Lemma 3.5.** *Let $M$ be a finitely generated $R$-module. Then there exist exact sequences*

$$\begin{cases} 0 \to R/I_1 \to M_0 \to M_1 \to 0, \\ 0 \to R/I_2 \to M_1 \to M_2 \to 0, \\ \qquad \vdots \\ 0 \to R/I_{n-1} \to M_{n-2} \to M_{n-1} \to 0, \\ 0 \to R/I_n \to M_{n-1} \to M_n \to 0 \end{cases}$$

*of $R$-modules such that $I_1, I_2, \dots, I_{n-1}, I_n$ are ideals of $R$, and $M_0 = M$ and $M_n = 0$.*

*Proof.* Let $x_1, x_2, \dots, x_n$ be a system of generators of $M$. Set $M_0 = M$. We have an isomorphism $Rx_1 \cong R/I_1$, where $I_1 = \operatorname{Ann}(x_1)$. Putting $M_1 = M_0/Rx_1$, we have an exact sequence

$$0 \to R/I_1 \to M_0 \to M_1 \to 0$$

of $R$-modules. Note that the $R$-module $M_1$ is generated by $n-1$ elements. By induction on $n$, we can obtain such a system of exact sequences as in the lemma. □

Now we are in a position to prove the following theorem, which is the latter half part of Theorem A in the first section. In the proof, we should note that all the isomorphisms in the theorem except $\mathcal{L}_{\mathsf{coh}}(\operatorname{coh}(R)) \cong \mathcal{L}_{\mathsf{thick}}(\mathcal{D}_{\mathrm{perf}}(R))$ are obtained without using derived categories.

**Theorem 3.6.** *Let $R$ be a noetherian ring. Then the homomorphisms $\sigma, \mu, \alpha, f$ (defined in the previous section) are lattice isomorphisms, and $\tau, \nu, \beta, g$ are their inverse homomorphisms, respectively. Consequently, one has*

$$\begin{aligned} \mathcal{L}^0_{\mathsf{spcl}}(\operatorname{Spec} R) &= \mathcal{L}_{\mathsf{spcl}}(\operatorname{Spec} R) \\ &\cong \mathcal{L}_{\mathsf{tors}}(\operatorname{Mod} R) \\ &\cong \mathcal{L}_{\mathsf{Serre}}(\operatorname{Serre}(R)) \\ &= \mathcal{L}_{\mathsf{coh}}(\operatorname{coh}(R)) \\ &\cong \mathcal{L}_{\mathsf{thick}}(\mathcal{D}_{\mathrm{perf}}(R)). \end{aligned}$$

*Proof.* The equality $\mathcal{L}^0_{\mathsf{spcl}}(\operatorname{Spec} R) = \mathcal{L}_{\mathsf{spcl}}(\operatorname{Spec} R)$ is already shown in Proposition 3.3. Proposition 2.13 says that the homomorphisms $\sigma, \mu$ are lattice isomorphisms and $\tau, \nu$ are the inverse homomorphisms of $\sigma, \mu$ respectively. Hence we have $\mathcal{L}_{\mathsf{spcl}}(\operatorname{Spec} R) \cong \mathcal{L}_{\mathsf{tors}}(\operatorname{Mod} R) \cong \mathcal{L}_{\mathsf{Serre}}(\operatorname{Serre}(R))$. It is seen from Corollary 3.2 and Remark 2.8 that both of the homomorphisms $\alpha$ and $\beta$ are the identity maps, and we have $\mathcal{L}_{\mathsf{Serre}}(\operatorname{Serre}(R)) = \mathcal{L}_{\mathsf{coh}}(\operatorname{coh}(R))$.

It remains to prove that $f$ is an isomorphism with the inverse homomorphism $g$. The composite map $fg$ is the identity homomorphism by Proposition 2.13(4)(b), and the subcategory $gf(\mathcal{X})$ is contained in $\mathcal{X}$ for every $\mathcal{X} \in \mathcal{L}_{\mathsf{coh}}(\operatorname{coh}(R))$ by Proposition 2.13(4)(a). Let us show that $\mathcal{X}$ is contained in $gf(\mathcal{X})$. Let $M$ be an $R$-module in $\mathcal{X}$. Since Remark 2.8 guarantees that $\mathcal{X}$ is a subcategory of $\operatorname{mod} R$, $M$ is a finitely generated $R$-module, and we have a system of exact sequences as in Lemma 3.5. (In the following, we use the same notation.) Since $\mathcal{X}$ is a coherent subcategory of $\operatorname{mod} R$ by Remark 2.8, we see from Corollary 3.2 that $\mathcal{X}$ is a Serre subcategory of $\operatorname{mod} R$. Hence $\mathcal{X}$ is closed under submodules and quotient modules in $\operatorname{mod} R$ by Corollary 2.5(2). From the above exact sequences we easily see that the cyclic

$R$-module $R/I_i$ belongs to $\mathcal{X}$ for every $1 \leq i \leq n$. Note by Corollary 2.5(1) that $\mathcal{X}$ is closed under finite direct sums. Hence Lemma 3.4 shows that for each integer $1 \leq i \leq n$ there exists a perfect $R$-complex $X_\bullet^{(i)}$ such that $R/I_i$ is isomorphic to $H_0(X_\bullet^{(i)})$ and that $H_j(X_\bullet^{(i)})$ belongs to $\mathcal{X}$ for any $j \in \mathbb{Z}$. It easily follows from the definitions of the homomorphisms $f, g$ that the $R$-module $R/I_i$ belongs to $gf(\mathcal{X})$. Since $gf(\mathcal{X})$ is a coherent subcategory, it is closed under extensions. Hence from the system of exact sequences, we see that $M$ belongs to $gf(\mathcal{X})$. Therefore $\mathcal{X}$ is contained in $gf(\mathcal{X})$, and thus $gf$ is the identity homomorphism. This completes the proof of our theorem. $\square$

## 4. In relation to Neeman's classification

In this section, we shall give a module version of Neeman's classification theorem of localizing categories and smashing categories, which we stated in the first section of this paper. Throughout this section, let $R$ be a commutative noetherian ring.

We denote by $\mathcal{L}_{\mathsf{subext}}(\operatorname{mod} R)$ the lattice of all subcategories of $\operatorname{mod} R$ which are closed under submodules and extensions, and by $\mathcal{L}(\operatorname{Spec} R)$ the lattice of all subsets of $\operatorname{Spec} R$. We define maps

$$\begin{cases} \Phi : \mathcal{L}(\operatorname{Spec} R) \to \mathcal{L}_{\mathsf{subext}}(\operatorname{mod} R) \\ \Psi : \mathcal{L}_{\mathsf{subext}}(\operatorname{mod} R) \to \mathcal{L}(\operatorname{Spec} R) \end{cases}$$

by $\Phi(S) = \{\, M \in \operatorname{mod} R \mid \operatorname{Ass} M \subseteq S \,\}$ and $\Psi(\mathcal{M}) = \bigcup_{M \in \mathcal{M}} \operatorname{Ass} M$. It is easy to check that these maps are lattice homomorphisms. Let $\phi : \mathcal{L}_{\mathsf{spcl}}(\operatorname{Spec} R) \to \mathcal{L}_{\mathsf{Serre}}(\operatorname{mod} R)$ and $\psi : \mathcal{L}_{\mathsf{Serre}}(\operatorname{mod} R) \to \mathcal{L}_{\mathsf{spcl}}(\operatorname{Spec} R)$ be the composite maps $\mu\sigma$ and $\tau\nu$, where $\sigma$, $\tau$, $\mu$, $\nu$ are the homomorphisms defined in Section 2. Note that the maps $\phi, \psi$ are given by $\phi(S) = \{\, M \in \operatorname{mod} R \mid \operatorname{Supp} M \subseteq S \,\}$ and $\psi(\mathcal{M}) = \bigcup_{M \in \mathcal{M}} \operatorname{Supp} M$ for $S \in \mathcal{L}_{\mathsf{spcl}}(\operatorname{Spec} R)$ and $\mathcal{M} \in \mathcal{L}_{\mathsf{Serre}}(\operatorname{mod} R)$.

Recall that $\phi$ is an isomorphism and $\psi$ is its inverse homomorphism since $R$ is noetherian. This section is mainly devoted to proving the following theorem.

**Theorem 4.1.** *Let $R$ be a commutative noetherian ring. Then the homomorphism $\Phi$ is an isomorphism and $\Psi$ is its inverse homomorphism. Moreover, $\Phi$ and $\Psi$ induce the isomorphisms $\phi$ and $\psi$, respectively. Thus one has the following commutative diagram.*

$$\begin{tikzcd}
\mathcal{L}_{\mathsf{subext}}(\operatorname{mod} R) \arrow[r, "\Psi", "\cong"'] & \mathcal{L}(\operatorname{Spec} R) \arrow[r, "\Phi", "\cong"'] & \mathcal{L}_{\mathsf{subext}}(\operatorname{mod} R) \\
\mathcal{L}_{\mathsf{Serre}}(\operatorname{mod} R) \arrow[r, "\psi", "\cong"'] \arrow[u, "\subseteq"'] & \mathcal{L}_{\mathsf{spcl}}(\operatorname{Spec} R) \arrow[r, "\phi", "\cong"'] \arrow[u, "\subseteq"'] & \mathcal{L}_{\mathsf{Serre}}(\operatorname{mod} R) \arrow[u, "\subseteq"']
\end{tikzcd}$$

This theorem will be proved after showing the following two lemmas.

**Lemma 4.2.** *Let $\mathcal{M}$ be a subcategory of* $\operatorname{mod} R$ *which is closed under submodules and extensions, and let $M$ be a finitely generated $R$-module. Suppose that $M$ has a unique associated prime $\mathfrak{p}$. If $R/\mathfrak{p}$ is in $\mathcal{M}$, then so is $M$.*

*Proof.* Assume that $M$ is not in $\mathcal{M}$. Set $M_0 = M$, and let $f_{0,1}, \dots, f_{0,s_0}$ be a system of generators of the $R$-module $\operatorname{Hom}_R(M_0, R/\mathfrak{p})$. There is an exact sequence

$$0 \longrightarrow M_1 \longrightarrow M_0 \xrightarrow{\begin{pmatrix} f_{0,1} \\ \vdots \\ f_{0,s_0} \end{pmatrix}} (R/\mathfrak{p})^{\oplus s_0}$$

of $R$-modules. Since $M_0 = M$ is not in $\mathcal{M}$ and $(R/\mathfrak{p})^{\oplus s_0}$ is in $\mathcal{M}$ and $\mathcal{M}$ is closed under submodules and extensions, it is easily seen that $M_1$ must not be in $\mathcal{M}$. In particular, $M_1 \neq 0$ and hence $\mathfrak{p}$ is the unique associated prime of $M_1$. Letting $f_{1,1}, \dots, f_{1,s_1}$ be a system of generators of the $R$-module $\operatorname{Hom}_R(M_1, R/\mathfrak{p})$, we have an exact sequence

$$0 \longrightarrow M_2 \longrightarrow M_1 \xrightarrow{\begin{pmatrix} f_{1,1} \\ \vdots \\ f_{1,s_1} \end{pmatrix}} (R/\mathfrak{p})^{\oplus s_1}.$$

Since $M_1$ is not in $\mathcal{M}$ and $(R/\mathfrak{p})^{\oplus s_1}$ is in $\mathcal{M}$, we see that $M_2$ is not in $\mathcal{M}$, and that $\mathfrak{p}$ is the unique associated prime of $M_2$. Iterating this procedure, for each integer $i \geq 0$ we obtain an exact sequence

$$0 \longrightarrow M_{i+1} \longrightarrow M_i \xrightarrow{\begin{pmatrix} f_{i,1} \\ \vdots \\ f_{i,s_i} \end{pmatrix}} (R/\mathfrak{p})^{\oplus s_i},$$

where $f_{i,1}, \dots, f_{i,s_i}$ is a system of generators of the $R$-module $\operatorname{Hom}_R(M_i, R/\mathfrak{p})$ and $\mathfrak{p}$ is the unique associated prime of $M_i$. Localizing the descending chain $M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots$ at $\mathfrak{p}$ yields a descending chain

$$M_{\mathfrak{p}} = (M_0)_{\mathfrak{p}} \supseteq (M_1)_{\mathfrak{p}} \supseteq (M_2)_{\mathfrak{p}} \supseteq \cdots$$

of $R_{\mathfrak{p}}$-modules. Since the $R_{\mathfrak{p}}$-module $(M_i)_{\mathfrak{p}}$ has finite length for every $i$, there exists an integer $t$ such that $(M_t)_{\mathfrak{p}} = (M_{t+1})_{\mathfrak{p}} = (M_{t+2})_{\mathfrak{p}} = \cdots$. The exact sequence

$$0 \longrightarrow (M_{t+1})_{\mathfrak{p}} \xrightarrow{=} (M_t)_{\mathfrak{p}} \xrightarrow{\begin{pmatrix} (f_{t,1})_{\mathfrak{p}} \\ \vdots \\ (f_{t,s_t})_{\mathfrak{p}} \end{pmatrix}} \kappa(\mathfrak{p})^{\oplus s_t},$$

shows that $\operatorname{Hom}_{R_{\mathfrak{p}}}((M_t)_{\mathfrak{p}}, \kappa(\mathfrak{p})) = R_{\mathfrak{p}}(f_{t,1})_{\mathfrak{p}} + \cdots + R_{\mathfrak{p}}(f_{t,s_t})_{\mathfrak{p}} = 0$. Therefore $(M_t)_{\mathfrak{p}} = 0$. This is a contradiction since $\mathfrak{p} \in \operatorname{Ass} M_t \subseteq \operatorname{Supp} M_t$. Thus we conclude that $M$ is in $\mathcal{M}$. □

**Lemma 4.3.** *Let $\mathcal{M}$ be a subcategory of* $\operatorname{mod} R$ *which is closed under submodules and extensions. Let $M$ be a finitely generated $R$-module. Suppose that $R/\mathfrak{p}$ belongs to $\mathcal{M}$ for every $\mathfrak{p} \in \operatorname{Ass} M$. Then $M$ also belongs to $\mathcal{M}$.*

*Proof.* Let $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ be the associated primes of $M$, and let

$$0 = N_1 \cap \cdots \cap N_s$$

be an irredundant primary decomposition of the zero submodule 0 of $M$, where $N_i$ is a $\mathfrak{p}_i$-primary submodule of $M$ for $1 \leq i \leq s$. Then the natural homomorphism

$$M = M/N_1 \cap \cdots \cap N_s \to M/N_1 \oplus \cdots \oplus M/N_s$$

is injective. Since $\mathfrak{p}_i$ is the unique associated prime of the $R$-module $M/N_i$, Lemma 4.2 implies that $M/N_i$ belongs to $\mathcal{M}$ for $1 \leq i \leq s$. Hence $M/N_1 \oplus \cdots \oplus M/N_s$ belongs to $\mathcal{M}$, and so does $M$. □

Now we can achieve the main purpose of this section.

*Proof of Theorem 4.1.* Let $S$ be a subset of $\operatorname{Spec} R$. The set $\Psi\Phi(S)$ is the union of $\operatorname{Ass} M$ where $M$ runs through finitely generated $R$-modules all of whose associated primes are in $S$. It is trivial that this set is contained in $S$. For a prime ideal $\mathfrak{p} \in S$, we have $\operatorname{Ass}_R R/\mathfrak{p} = \{\mathfrak{p}\} \subseteq S$. Hence $\mathfrak{p}$ belongs to $\Psi\Phi(S)$, and therefore $\Psi\Phi(S) = S$. Let $\mathcal{M}$ be a subcategory of $\operatorname{mod} R$ which is closed under submodules and extensions. We have that $\Phi\Psi(\mathcal{M})$ is the subcategory of $\operatorname{mod} R$ consisting of all finitely generated $R$-modules $N$ with $\operatorname{Ass} N \subseteq \bigcup_{M \in \mathcal{M}} \operatorname{Ass} M$, and it is obvious that $\Phi\Psi(\mathcal{M})$ contains $\mathcal{M}$. Let $N$ be a finitely generated $R$-module with $\operatorname{Ass} N \subseteq \bigcup_{M \in \mathcal{M}} \operatorname{Ass} M$. Fix a prime ideal $\mathfrak{p} \in \operatorname{Ass} N$. Then there exists an $R$-module $M \in \mathcal{M}$ with $\mathfrak{p} \in \operatorname{Ass} M$. There is an injective homomorphism $R/\mathfrak{p} \to M$, and $R/\mathfrak{p}$ belongs to $\mathcal{M}$ since $\mathcal{M}$ is closed under submodules. It follows from Lemma 4.3 that $N$ is in $\mathcal{M}$. Hence $\Phi\Psi(\mathcal{M}) = \mathcal{M}$. Thus we conclude that $\Phi$ is an isomorphism whose inverse homomorphism is $\Psi$.

On the other hand, let $S$ be a subset of $\operatorname{Spec} R$ which is closed under specialization. Let $M$ be a finitely generated $R$-module such that $\operatorname{Ass} M$ is contained in $S$, and take $\mathfrak{p} \in \operatorname{Supp} M$. Then there is a prime ideal $\mathfrak{q} \in \operatorname{Min} M \subseteq \operatorname{Ass} M$ that is contained in $\mathfrak{p}$. Since $\mathfrak{q}$ is in $S$ and $S$ is closed under specialization, $\mathfrak{p}$ is also in $S$. Thus $\Phi(S) = \{\, M \in \operatorname{mod} R \mid \operatorname{Ass} M \subseteq S \,\}$ coincides with $\phi(S) = \{\, M \in \operatorname{mod} R \mid \operatorname{Supp} M \subseteq S \,\}$. Let $\mathcal{M}$ be a Serre subcategory of $\operatorname{mod} R$. Let $N \in \mathcal{M}$ and $\mathfrak{p} \in \operatorname{Supp} N$. Choose a prime ideal $\mathfrak{q} \in \operatorname{Min} N$ which is contained in $\mathfrak{p}$. Then $\mathfrak{q}$ is an associated prime of $N$, so there is an injective homomorphism $R/\mathfrak{q} \to N$. Since $\mathcal{M}$ is closed under submodules, the module $R/\mathfrak{q}$ is in $\mathcal{M}$. Noting that there is a surjective homomorphism $R/\mathfrak{q} \to R/\mathfrak{p}$ and that $\mathcal{M}$ is closed under quotient modules, $R/\mathfrak{p}$ is also in $\mathcal{M}$. Hence we get $\mathfrak{p} \in \operatorname{Ass} R/\mathfrak{p} \subseteq \bigcup_{M \in \mathcal{M}} \operatorname{Ass} M$. Therefore the set $\psi(\mathcal{M}) = \bigcup_{M \in \mathcal{M}} \operatorname{Supp} M$ is contained in $\Psi(\mathcal{M}) = \bigcup_{M \in \mathcal{M}} \operatorname{Ass} M$, and we see that $\Psi(\mathcal{M}) = \psi(\mathcal{M})$. It follows that $\phi$ and $\psi$ are induced from $\Phi$ and $\Psi$, respectively. □

Here, let us check that an analogous result to Corollary 2.14 holds. This actually follows from Lemma 4.3.

**Corollary 4.4.** *Let $R$ be a noetherian ring. Let $M$ and $N$ be finitely generated $R$-modules with $\operatorname{Ass} M \subseteq \operatorname{Ass} N$. Then $M$ is in the full subcategory of $\operatorname{mod} R$ closed under submodules and extensions which is generated by $N$.*

*Proof.* Let $\mathcal{E}$ be the full subcategory of $\operatorname{mod} R$ closed under submodules and extensions which is generated by $N$. According to Lemma 4.3, we have only to show that the $R$-module $R/\mathfrak{p}$ is in $\mathcal{E}$ for every $\mathfrak{p} \in \operatorname{Ass} M$. Let $\mathfrak{p}$ be a prime ideal in $\operatorname{Ass} M$. The assumption says that $\mathfrak{p}$ is in $\operatorname{Ass} N$. Hence there exists an injective homomorphism $R/\mathfrak{p} \to N$ of $R$-modules. Since $N$ is in $\mathcal{E}$ and $\mathcal{E}$ is closed under submodules, $R/\mathfrak{p}$ is also in $\mathcal{E}$, as required. □

In the following example, we will give several correspondences between subcategories of $\operatorname{mod} R$ which are closed under submodules and extensions and subsets of $\operatorname{Spec} R$, which are made by the isomorphisms $\Phi$ and $\Psi$. Before that, we

need to prepare some notation. Let $I$ be an ideal of $R$, and let $M, N$ be $R$-modules. We denote by $\Gamma_I(M)$ the *$I$-torsion submodule* of $M$, namely, the set of elements of $M$ which are annihilated by some power of $I$. Recall that an $R$-module $M$ is called *$I$-torsion* if $\Gamma_I(M) = M$, and that $M$ is called *$I$-torsionfree* if $\Gamma_I(M) = 0$. It is well-known and easy to see that $M$ is $I$-torsion if and only if $\operatorname{Ass} M \subseteq V(I)$, and that $M$ is $I$-torsionfree if and only if $\operatorname{Ass} M \cap V(I) = \emptyset$. We set $\operatorname{grade}(N, M) = \inf\{\, i \mid \operatorname{Ext}^i_R(N, M) \neq 0 \,\}$, $\operatorname{grade}(I, M) = \operatorname{grade}(R/I, M)$, $\operatorname{grade} I = \operatorname{grade}(I, R)$ and $\operatorname{grade} M = \operatorname{grade}(\operatorname{Ann} M, R)$.

**Example 4.5.** The isomorphisms $\Phi$ and $\Psi$ make the following correspondences. Let $n$ be a nonnegative integer, $I$ an ideal of $R$ and $X$ a finitely generated $R$-module.

(1) $\{\, M \in \operatorname{mod} R \mid M \text{ is } I\text{-torsion} \,\} \leftrightarrow V(I)$.

(2) $\{\, M \in \operatorname{mod} R \mid \operatorname{grade}(X, M) > 0 \,\} \leftrightarrow \operatorname{Spec} R \setminus \operatorname{Supp} X$.

(3) $\{\, M \in \operatorname{mod} R \mid M \text{ is } I\text{-torsionfree} \,\}$
$= \{\, M \in \operatorname{mod} R \mid \operatorname{grade}(I, M) > 0 \,\} \leftrightarrow D(I)$.

(4) $\{\, M \in \operatorname{mod} R \mid \operatorname{grade}(M, X) \geq n \,\} \leftrightarrow \{\, \mathfrak{p} \in \operatorname{Spec} R \mid \operatorname{grade}(\mathfrak{p}, X) \geq n \,\}$.

(5) $\{\, M \in \operatorname{mod} R \mid \operatorname{rank} M = 0 \,\} = \{\, M \in \operatorname{mod} R \mid \operatorname{grade} M > 0 \,\}$
$\leftrightarrow \{\, \mathfrak{p} \in \operatorname{Spec} R \mid \operatorname{grade} \mathfrak{p} > 0 \,\}$.

(6) $\{\, M \in \operatorname{mod} R \mid \text{every } X\text{-regular element is } M\text{-regular} \,\}$
$\leftrightarrow \{\, \mathfrak{p} \in \operatorname{Spec} R \mid \operatorname{grade}(\mathfrak{p}, X) = 0 \,\}$.

(7) $\{\, M \in \operatorname{mod} R \mid M \text{ is torsionfree} \,\} \leftrightarrow \{\, \mathfrak{p} \in \operatorname{Spec} R \mid \operatorname{grade} \mathfrak{p} = 0 \,\}$.

(8) $\{\, M \in \operatorname{mod} R \mid \operatorname{ht} \operatorname{Ann} M \geq n \,\} \leftrightarrow \{\, \mathfrak{p} \in \operatorname{Spec} R \mid \operatorname{ht} \mathfrak{p} \geq n \,\}$.

(9) $\{\, M \in \operatorname{mod} R \mid \dim M \leq n \,\} \leftrightarrow \{\, \mathfrak{p} \in \operatorname{Spec} R \mid \dim R/\mathfrak{p} \leq n \,\}$.

(10) $\{\, M \in \operatorname{mod} R \mid \ell(M) < \infty \,\} \leftrightarrow \operatorname{Max} R$.

*Proof.* In each correspondence, we denote by $\mathcal{M}$ the left-hand subcategory of $\operatorname{mod} R$, and by $S$ the right-hand subset of $\operatorname{Spec} R$. Note that it is enough to check either that $\Psi(\mathcal{M}) = S$ or that $\Phi(S) = \mathcal{M}$ since $\Phi$ is an isomorphism with the inverse homomorphism $\Psi$.

(1) The subcategory $\mathcal{M}$ consists of all finitely generated $R$-modules $M$ with $\operatorname{Ass} M \subseteq V(I)$, which coincides with $\Phi(V(I))$. Hence $\Phi(S) = \mathcal{M}$.

(2) Let $M$ be a finitely generated $R$-module. Note that $\operatorname{Ass} M \subseteq \operatorname{Spec} R \setminus \operatorname{Supp} X$ if and only if $\operatorname{Ass} M \cap \operatorname{Supp} X = \emptyset$, if and only if $\operatorname{Ass} \operatorname{Hom}(X, M) = \emptyset$ (cf. [1, Exercise 1.2.27]), if and only if $\operatorname{Hom}(X, M) = 0$. Thus we have $\Phi(S) = \mathcal{M}$.

(3) The equality is well-known. Putting $X = R/I$ in (2), we obtain the correspondence.

(4) Let $M$ be a finitely generated $R$-module with $\operatorname{grade}(M, X) \geq n$, and let $\mathfrak{p} \in \operatorname{Ass} M$. Then there is an $X$-regular sequence $\boldsymbol{a} = a_1, \dots, a_n$ in $\operatorname{Ann} M$, and $\mathfrak{p} \in \operatorname{Supp} M$. Hence $\boldsymbol{a}$ is an $X$-regular sequence in $\mathfrak{p}$, and we have $\operatorname{grade}(\mathfrak{p}, X) \geq n$. Therefore $\Psi(\mathcal{M})$ is contained in $S$. Conversely, if $\mathfrak{p}$ is a prime ideal with

$\operatorname{grade}(\mathfrak{p}, X) \geq n$, then $R/\mathfrak{p} \in \mathcal{M}$ and $\mathfrak{p} \in \operatorname{Ass} R/\mathfrak{p}$. Hence $\mathfrak{p}$ is in $\Psi(\mathcal{M})$. Therefore $S$ is contained in $\Psi(\mathcal{M})$, and thus we get $\Psi(\mathcal{M}) = S$.

(5) Let $M$ be an $R$-module. We have that $\operatorname{rank} M = 0$ if and only if $M_{\mathfrak{p}} = 0$ for every $\mathfrak{p} \in \operatorname{Ass} R$, if and only if $\operatorname{Supp} M \cap \operatorname{Ass} R = \emptyset$, if and only if $\operatorname{Ass} \operatorname{Hom}(M, R) = \emptyset$, if and only if $\operatorname{Hom}(M, R) = 0$, namely $\operatorname{grade} M > 0$. Thus the equality holds. For the correspondence, put $X = R$ and $n = 1$ in (4).

(6) Let $\mathfrak{p}$ be a prime ideal in $\Psi(\mathcal{M})$. Then there exists an $R$-module $M \in \mathcal{M}$ of which $\mathfrak{p}$ is an associated prime. Assume that $\operatorname{grade}(\mathfrak{p}, X) > 0$. Then there is an $X$-regular element $a \in \mathfrak{p}$, and this is also $M$-regular. This is a contradiction since $\mathfrak{p} \in \operatorname{Ass} M$. Thus $\operatorname{grade}(\mathfrak{p}, X) = 0$, namely $\mathfrak{p}$ belongs to $S$. On the contrary, let $\mathfrak{p}$ be a prime ideal with $\operatorname{grade}(\mathfrak{p}, X) = 0$. Then there exists an associated prime $\mathfrak{q}$ of $X$ which contains $\mathfrak{p}$. Let $a$ be an $X$-regular element. Then $a$ is not in $\mathfrak{q}$, so is not in $\mathfrak{p}$. Hence $a$ is an $R/\mathfrak{p}$-regular element, and $R/\mathfrak{p}$ belongs to $\mathcal{M}$. Since $\mathfrak{p} \in \operatorname{Ass} R/\mathfrak{p}$, the prime ideal $\mathfrak{p}$ is in $\Psi(\mathcal{M})$, and it holds that $\Psi(\mathcal{M}) = S$.

(7) Put $X = R$ in (6), and we get this correspondence.

(8) If $M$ is a finitely generated $R$-module with $\operatorname{ht} \operatorname{Ann} M \geq n$, then $\operatorname{ht} \mathfrak{p} \geq n$ for all $\mathfrak{p} \in \operatorname{Supp} M$, hence for all $\mathfrak{p} \in \operatorname{Ass} M$. Therefore $\Psi(\mathcal{M})$ is contained in $S$. If $\mathfrak{p}$ is a prime ideal of height at least $n$, then the ideal $\operatorname{Ann} R/\mathfrak{p}$ of $R$ also has height at least $n$ and $\mathfrak{p} \in \operatorname{Ass} R/\mathfrak{p}$. Hence $\Psi(\mathcal{M})$ contains $S$, and thus $\Psi(\mathcal{M}) = S$.

(9) Let $M$ be a finitely generated $R$-module. Then $\operatorname{Ass} M$ is contained in $S$ if and only if $\dim R/\mathfrak{p} \leq n$ for every $\mathfrak{p} \in \operatorname{Ass} M$, if and only if $\dim M \leq n$. Therefore $\Phi(S) = \mathcal{M}$.

(10) Putting $n = 0$ in (9) yields this correspondence. □

Note that in the correspondences (1), (4), (5), (8), (9) and (10) in the above example, the left-hand subcategories of $\operatorname{mod} R$ are Serre subcategories and the right-hand subsets of $\operatorname{Spec} R$ are closed under specializations, hence those correspondences are in fact obtained by the isomorphisms $\phi$ and $\psi$.

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