AUSLANDER-REITEN CONJECTURE AND FINITE INJECTIVE DIMENSION OF HOM

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Abstract. For a finitely generated module M over a commutative Noetherian ring R, we settle the Auslander-Reiten conjecture when at least one of $\operatorname{Hom}_R(M,R)$ and $\operatorname{Hom}_R(M,M)$ has finite injective dimension. A number of new characterizations of Gorenstein local rings are also obtained in terms of vanishing of certain Ext and finite injective dimension of Hom.

1. Introduction

Setup 1.1. Unless otherwise specified, R is a commutative Noetherian local ring of dimension d. All R-modules are assumed to be finitely generated.

The vanishing of Ext modules, and their consequences are actively studied subjects in commutative algebra. A bunch of criteria for a given module to be projective, and criteria for a local ring to be Gorenstein have been described in terms of the vanishing of Ext. The purpose of this article is to provide such criteria in terms of properties of $\operatorname{Hom}_R(M,N)$ for R-modules M and N when the vanishing of $\operatorname{Ext}_R^{1\leqslant i\leqslant n}(M,N)$ is given for some $n \geqslant 1$. We prove the following results.

Theorem 2.5. *Let M and N be nonzero R-modules such that* $\operatorname{depth}(N) = d$ *and* $\operatorname{Ext}_R^i(M,N) = 0$ *for all* $1 \leqslant i \leqslant d$. *Then* $\operatorname{Hom}_R(M,N)$ *has finite injective dimension if and only if M is free and N has finite injective dimension.*

Theorem 2.15. *Let M be a nonzero R-module such that* $\operatorname{Ext}_R^i(M,R) = 0$ *and* $\operatorname{Ext}_R^j(M,M) = 0$ *for all* $1 \leqslant i \leqslant 2d+1$ *and* $1 \leqslant j \leqslant \max\{1, d-1\}$. *If* $\operatorname{Hom}_R(M,M)$ *has finite injective dimension, then M is free, and R is Gorenstein.*

One of the most celebrated long-standing conjectures in commutative algebra is the Auslander-Reiten conjecture:

Conjecture 1.2. [4] For an R-module M, if $\operatorname{Ext}_R^i(M, M \oplus R) = 0$ for all $i \geqslant 1$, then M is projective.

The conjecture is known to hold true in the following cases: (1) R is complete intersection [3, 1.9]. (2) M has finite complete intersection dimension [2, Thm. 4.3]. (3) R is a deformation of a CM local ring of minimal multiplicity [8]. (4) R is a locally excellent Cohen-Macaulay (in short, CM) normal ring containing $\mathbb{Q}$ [14, Thm. 0.1]. (5) R is a Gorenstein normal ring [1, Cor. 4]. (6) R is a fiber product of two local rings of the same residue field [19, 1.2]. (7) R is CM

Date: May 19, 2022.

2010 *Mathematics Subject Classification.* Primary 13D07; Secondary 13D05, 13H10.

Key words and phrases. Auslander-Reiten conjecture; Vanishing of Ext; Injective dimension; Gorenstein rings.

of dimension $d \geqslant 1$, M is maximal Cohen-Macaulay (in short, MCM) such that $\operatorname{Ext}_R^{1\leqslant i\leqslant d}(M, \operatorname{Hom}_R(M,M)) = 0$ and M has rank 1 [11, Thm. 1.5]. (8) R is a CM normal domain, M is MCM and $\operatorname{Hom}_R(M,M)$ is free [7, Thm. 3.16]. (9) R is normal and $\operatorname{Hom}_R(M,M)$ has finite Gorenstein dimension [22, Cor. 1.6]. (10) R is CM normal [17, Cor. 1.3]. (11) R is CM such that $e(R) \leqslant (7/4)\operatorname{codim}(R)+1$, or R is Gorenstein such that $e(R) \leqslant \operatorname{codim}(R)+6$ [18, Thm. C]. (There is some overlapping among these eleven conditions, e.g., (1) is included in (2) as well as in (3); while (4), (5) and (8) are included in (10).) However the conjecture is widely open even for Gorenstein local rings. In the present study, as applications of Theorems 2.5 and 2.15, we obtain the following.

Corollary 1.3 (=2.12 and 2.16)**.** *The Auslander-Reiten conjecture holds true for a (finitely generated) module M over a commutative Noetherian ring R when at least one of $\operatorname{Hom}_R(M,R)$ and $\operatorname{Hom}_R(M,M)$ has finite injective dimension.*

More precise statements (about criteria for a module to be free over a local ring) can be seen in Corollaries 2.10.(2), 2.13 and 2.14, and Theorem 2.15.

We also provide some new characterizations of Gorenstein local rings in terms of vanishing of certain Ext and finite injective dimension of Hom.

Theorem 3.6. *The following statements are equivalent:*

(1) *R is Gorenstein.*
(2) *R admits a module M such that $\operatorname{Ext}_R^i(M,R) = 0$ for all $1 \leqslant i \leqslant d-1$ and M^* is nonzero and of finite injective dimension.*
(3) *R admits a nonzero module M such that $\operatorname{Ext}_R^i(M,R) = \operatorname{Ext}_R^j(M,M) = 0$ for all $1 \leqslant i \leqslant 2d+1$ and $1 \leqslant j \leqslant \max\{1, d-1\}$, and $\operatorname{injdim}_R(\operatorname{Hom}_R(M,M)) < \infty$.*
(4) *R admits a module M such that $\operatorname{depth}(M) = d$, $\operatorname{Ext}_R^j(M,M) = 0$ for all $1 \leqslant j \leqslant d$ and $\operatorname{injdim}_R(\operatorname{Hom}_R(M,M)) < \infty$.*
(5) *R admits a module M such that*
$$\operatorname{depth}(\operatorname{Hom}_R(M,M)) = \operatorname{depth}(R) \text{ and } \operatorname{injdim}_R(\operatorname{Hom}_R(M,M)) < \infty.$$

The characterizations given in Theorem 3.6 are motivated by the criteria for Gorenstein local rings due to Ulrich, Hanes-Huneke, Jorgensen-Leuschke (see 3.4 for the details), and the following classical results.

Theorem 1.4 (Peskine-Szpiro)**.** [20, Chapitre II, Théorème (5.5)] *The ring R is Gorenstein if and only if it has a nonzero cyclic module of finite injective dimension.*

Theorem 1.5 (Foxby)**.** [16] *The ring R is Gorenstein if and only if it has a nonzero (finitely generated) module M for which $\operatorname{injdim}_R(M) < \infty$ and $\operatorname{projdim}_R(M) < \infty$.*

Theorem 1.6 (Bass' Conjecture)**.** [20, 21] *If R admits a nonzero (finitely generated) module of finite injective dimension, then R is CM.*

2. Criteria for a module to be free

2.1. Let M be an R-module. Set $M^* := \operatorname{Hom}_R(M,R)$. The minimal number of generators of M is denoted by $\nu(M)$, i.e., $\nu(M) = \dim_k(M \otimes_R k)$. The type of M is defined to be $\operatorname{type}(M) = \dim_k(\operatorname{Ext}_R^t(k,M))$, where $t = \operatorname{depth}(M)$.

Remark 2.2. If M and N are nonzero R-modules, then

$$0 \leqslant \operatorname{depth}(\operatorname{ann}_R(M), N) = \min\{i : \operatorname{Ext}_R^i(M,N) \neq 0\} \leqslant d. \tag{2.1}$$

Hence, if $\operatorname{Ext}_R^{1\leqslant i\leqslant d}(M,N) = 0$, then $\operatorname{Hom}_R(M,N)$ is a nonzero module.

We deduce our first main result from the following theorem, where we study the consequences of $\operatorname{Hom}_R(M,N)$ having finite injective dimension under the condition that $\operatorname{Ext}_R^{1\leqslant i\leqslant d-1}(M,N)=0$.

Theorem 2.3. *Let M and N be R-modules such that* $\operatorname{depth}(N)=d$,

$$\operatorname{Hom}_R(M,N)\neq 0 \text{ and } \operatorname{Ext}_R^i(M,N)=0 \text{ for all } 1\leqslant i\leqslant d-1.$$

Suppose that $\operatorname{Hom}_R(M,N)$ has finite injective dimension. Then, R is CM, N is MCM and of finite injective dimension, and $M\cong\Gamma_{\mathfrak{m}}(M)\oplus R^r$ for some $r\geqslant 0$.

Proof. By Theorem 1.6, R is CM, and hence N is MCM. We need to show that

$$\operatorname{injdim}_R(N)<\infty \text{ and } M\cong\Gamma_{\mathfrak{m}}(M)\oplus R^r \text{ for some } r\geqslant 0. \tag{2.2}$$

We may assume that R is complete. Then R admits a canonical module ω.

(1) We first prove the assertion (2.2) when N is indecomposable. Let $\mathbb{F}:\cdots\to F_2\to F_1\to F_0\to 0$ be a free resolution of M. Since $\operatorname{Ext}_R^i(M,N)=0$ for all $1\leqslant i\leqslant d-1$, the resolution $\mathbb{F}$ induces an exact sequence

$$0\to\operatorname{Hom}_R(M,N)\to\operatorname{Hom}_R(F_0,N)\xrightarrow{f}\operatorname{Hom}_R(F_1,N)\to\cdots\xrightarrow{g}\operatorname{Hom}_R(F_e,N),$$

where $e=\max\{1,d\}$. Set $C:=\operatorname{Image}(f)$ and $D:=\operatorname{Coker}(g)$. Hence there are two exact sequences:

$$0\to\operatorname{Hom}_R(M,N)\to\operatorname{Hom}_R(F_0,N)\to C\to 0 \quad\text{and} \tag{2.3}$$

$$0\to C\to\operatorname{Hom}_R(F_1,N)\to\operatorname{Hom}_R(F_2,N)\to\cdots\xrightarrow{g}\operatorname{Hom}_R(F_e,N)\to D\to 0. \tag{2.4}$$

Note that each $\operatorname{Hom}_R(F_i,N)$ is MCM. In the first case, assume that $C=0$. Then $\operatorname{Hom}_R(M,N)\cong\operatorname{Hom}_R(F_0,N)$ is MCM. In the second case, we have $C\neq 0$. Applying the depth lemma to the exact sequences (2.4) and (2.3) respectively, we obtain that both C and $\operatorname{Hom}_R(M,N)$ are MCM. Thus, in any case, $\operatorname{Hom}_R(M,N)\cong\omega^n$ for some $n\geqslant 1$, cf. [5, 3.3.28]. Moreover, in both the cases,

$$\operatorname{Ext}_R^1(C,\operatorname{Hom}_R(M,N))\cong\operatorname{Ext}_R^1(C,\omega^n)=0,$$

see, e.g., [5, 3.3.10]. Therefore (2.3) splits. Hence, setting $m:=\operatorname{rank}(F_0)$,

$$N^m\cong\operatorname{Hom}_R(F_0,N)\cong\operatorname{Hom}_R(M,N)\oplus C\cong\omega^n\oplus C.$$

Since N is indecomposable, by the Krull–Schmidt theorem, $N\cong\omega$. In particular, $\operatorname{injdim}_R(N)$ is finite.

Next we show that $M':=M/\Gamma_{\mathfrak{m}}(M)$ is free, i.e., $M'\cong R^r$ for some $r\geqslant 0$. It further implies that the exact sequence $0\to\Gamma_{\mathfrak{m}}(M)\to M\to M'\to 0$ splits, and hence $M\cong\Gamma_{\mathfrak{m}}(M)\oplus R^r$. In order to prove that M' is free, we may assume that $M'\neq 0$, i.e., $\Gamma_{\mathfrak{m}}(M)\neq M$. Particularly, it ensures that $d\geqslant 1$. We take two steps.

(1a) Consider the case $\Gamma_{\mathfrak{m}}(M)=0$. Then the local duality theorem (cf. [5, 3.5.8]) implies that $\operatorname{Ext}_R^d(M,\omega)=0$. Hence, by the given assumption, $\operatorname{Ext}_R^i(M,\omega)\cong\operatorname{Ext}_R^i(M,N)=0$ for all $1\leqslant i\leqslant d$. Therefore M is MCM (cf. [5, 3.1.24]) and

$$M\cong M^{\dagger\dagger}\cong\operatorname{Hom}_R(M,N)^{\dagger}\cong(\omega^n)^{\dagger}\cong R^n,$$

where $(-)^{\dagger}=\operatorname{Hom}_R(-,\omega)$; see, e.g., [5, 3.3.10]. Thus $M'=M$ is free.

(1b) Consider the general case. Since $\Gamma_{\mathfrak{m}}(M)$ has finite length,

$$\operatorname{Ext}_R^i(\Gamma_{\mathfrak{m}}(M),N)\cong\operatorname{Ext}_R^i(\Gamma_{\mathfrak{m}}(M),\omega)=0 \text{ for all } i<d.$$

Hence, from the long exact sequence of Ext modules induced by $0 \to \Gamma_{\mathfrak{m}}(M) \to M \to M' \to 0$, it follows that

$$\operatorname{Hom}_R(M', N) \cong \operatorname{Hom}_R(M, N) \neq 0 \text{ and } \operatorname{Ext}_R^i(M', N) \cong \operatorname{Ext}_R^i(M, N) = 0$$

for all $1 \leqslant i \leqslant d-1$. Since $\Gamma_{\mathfrak{m}}(M') = 0$, we can apply (1a) to see that M' is free. This completes the proof of (2.2) when N is indecomposable.

(2) Next we consider the general case. Since $\operatorname{Hom}_R(M, N) \neq 0$, there exists an indecomposable direct summand N' of N such that $\operatorname{Hom}_R(M, N') \neq 0$. Applying (1) to M and N', we have that $N' \cong \omega$ and $M \cong \Gamma_{\mathfrak{m}}(M) \oplus R^r$ for some $r \geqslant 0$. It remains to show that $\operatorname{injdim}_R(N) < \infty$. We have two possible cases.

(2a) Suppose that $r \geqslant 1$. Note that $\operatorname{Hom}_R(M, N) \cong \operatorname{Hom}_R(\Gamma_{\mathfrak{m}}(M), N) \oplus N^r$. Thus N is a direct summand of $\operatorname{Hom}_R(M, N)$, and hence $\operatorname{injdim}_R(N) < \infty$.

(2b) Assume that $r = 0$. Then $M = \Gamma_{\mathfrak{m}}(M)$ has finite length, and the assumption $\operatorname{Hom}_R(M, N) \neq 0$ ensures that $d = 0$ as $\operatorname{depth}(N) = d$. Let N'' be any indecomposable direct summand of N. In particular, $N'' \neq 0$. If $\operatorname{Hom}_R(M, N'') = 0$, then $\operatorname{Supp}(M) \cap \operatorname{Ass}(N'') = \emptyset$, and hence either M or N'' is zero as $\operatorname{Spec}(R) = \{\mathfrak{m}\}$, which is a contradiction. Thus $\operatorname{Hom}_R(M, N'') \neq 0$. Applying (1) again, we have $N'' \cong \omega$. It follows that $N \cong \omega^u$ for some $u \geqslant 1$, and $\operatorname{injdim}_R(N) < \infty$. □

Remark 2.4. In Theorem 2.3, the assumptions $\operatorname{Hom}_R(M, N) \neq 0$ and $\operatorname{Ext}_R^i(M, N) = 0$ for all $1 \leqslant i \leqslant d-1$ cannot be omitted, see Examples 2.8 and 2.6 respectively.

Now we are in a position to prove our first main result.

Theorem 2.5. *Let M and N be nonzero R-modules such that* $\operatorname{depth}(N) = d$ *and* $\operatorname{Ext}_R^i(M, N) = 0$ *for all* $1 \leqslant i \leqslant d$. *Then* $\operatorname{Hom}_R(M, N)$ *has finite injective dimension if and only if M is free and N has finite injective dimension.*

Proof. The 'if' part is trivial. We show the 'only if' part. Suppose that $\operatorname{Hom}_R(M, N)$ has finite injective dimension. In view of Remark 2.2, $\operatorname{Hom}_R(M, N) \neq 0$. Hence Theorem 2.3 implies that R is CM, N is MCM and of finite injective dimension, and $M \cong \Gamma_{\mathfrak{m}}(M) \oplus R^r$ for some $r \geqslant 0$. It remains to prove that M is free. We consider two possible cases.

(1) Assume that $d \geqslant 1$. Since $M \cong \Gamma_{\mathfrak{m}}(M) \oplus R^r$, the vanishing $\operatorname{Ext}_R^d(M, N) = 0$ yields that $\operatorname{Ext}_R^d(\Gamma_{\mathfrak{m}}(M), N) = 0$. Since N is MCM and of finite injective dimension, and $\Gamma_{\mathfrak{m}}(M)$ has finite length, it follows that $\operatorname{Ext}_R^i(\Gamma_{\mathfrak{m}}(M), N) = 0$ for all $i \neq d$. Thus $\operatorname{Ext}_R^i(\Gamma_{\mathfrak{m}}(M), N) = 0$ for all integers i, which implies that $\Gamma_{\mathfrak{m}}(M) = 0$ since $N \neq 0$ (see Remark 2.2). It follows that $M \cong R^r$.

(2) Suppose that $d = 0$. Set $E := E_R(k)$, the injective hull of k. Since both N and $\operatorname{Hom}_R(M, N)$ are injective, we have that $N \cong E^m$ and $\operatorname{Hom}_R(M, N) \cong E^n$ for some $m, n \geqslant 1$. Therefore, by Matlis duality,

$$M^m \cong (M^{\vee\vee})^m \cong ((M^\vee)^m)^\vee \cong \operatorname{Hom}_R(M, N)^\vee \cong R^n.$$

This implies that M is free. □

The examples below show that the assumptions $\operatorname{Ext}_R^{1 \leqslant i \leqslant d-1}(M, N) = 0$ in Theorem 2.3 and $\operatorname{Ext}_R^{1 \leqslant i \leqslant d}(M, N) = 0$ in Theorem 2.5 cannot be removed.

Example 2.6. Let $(R, \mathfrak{m}, k)$ be a CM local ring of dimension $d \geqslant 2$ with a canonical module ω. Then $\operatorname{Hom}_R(\mathfrak{m}, \omega) \cong \omega$ (cf. [5, 1.2.24]) has finite injective dimension.

Note that ω is MCM, but $\mathfrak{m}$ and ω do not satisfy the Ext vanishing condition as

$$\operatorname{Ext}^i_R(\mathfrak{m},\omega) \cong \operatorname{Ext}^{i+1}_R(k,\omega) = \begin{cases} 0 & \text{if } i \geqslant 1 \text{ and } i \neq d-1, \\ k \neq 0 & \text{if } i = d-1. \end{cases}$$

(1) We have $\mathfrak{m} \ncong \Gamma_{\mathfrak{m}}(\mathfrak{m}) \oplus R^r$ for any $r \geqslant 0$. If possible, suppose $\mathfrak{m} \cong \Gamma_{\mathfrak{m}}(\mathfrak{m}) \oplus R^r$ for some $r \geqslant 0$. By [9, Cor. 1.2], only $\Omega^d_R(k)$ may have a nonzero free direct summand. It follows that $r = 0$, otherwise R is a direct summand of $\mathfrak{m} = \Omega^1_R(k)$, which is a contradiction. Therefore $\mathfrak{m} \cong \Gamma_{\mathfrak{m}}(\mathfrak{m})$ has finite length, which implies that R has finite length, that is again a contradiction as $d \geqslant 2$.

(2) The R-module $\mathfrak{m}$ is not free. Indeed, if $\mathfrak{m}$ is free, then $\operatorname{projdim}_R(k) \leqslant 1$, which is a contradiction as $\dim(R) \geqslant 2$. If R is non-regular, $\mathfrak{m}$ does not even have finite projective dimension.

Example 2.7. Let R be a d-dimensional non-Gorenstein CM normal local ring with a canonical module ω. Set $M = \omega^*$ and $N = R$. Then $\operatorname{Hom}_R(M,N) = \omega^{**} \cong \omega$ has finite injective dimension. We also have $\operatorname{depth}(N) = d$, but $\operatorname{injdim}_R(R) = \infty$.

The number of vanishing of Ext in Theorem 2.5 cannot be further improved.

Example 2.8. Let $(R,\mathfrak{m},k)$ be a non-Gorenstein CM local ring of dimension $d \geqslant 1$. Then $\operatorname{Hom}_R(k,R) = 0$, so it has finite injective dimension, $\operatorname{depth}(R) = d$ and $\operatorname{Ext}^i_R(k,R) = 0$ for all $1 \leqslant i \leqslant d-1$, but $\operatorname{projdim}_R(k) = \operatorname{injdim}_R(R) = \infty$.

One may ask the following natural question.

Question 2.9. Does Theorem 2.5 hold true when $\operatorname{depth}(N) < d$?

Now we discuss the consequences of Theorems 2.3 and 2.5.

Corollary 2.10. *Let M be a nonzero R-module such that $\operatorname{Ext}^i_R(M,R) = 0$ for all $1 \leqslant i \leqslant d-1$ and M^* has finite injective dimension.*

(1) *If $M^* \neq 0$, then R is Gorenstein and $M \cong \Gamma_{\mathfrak{m}}(M) \oplus R^r$ for some $r \geqslant 0$.*
(2) *If $\operatorname{Ext}^d_R(M,R) = 0$, then R is Gorenstein and M is free.*

Proof. (1) The statement follows from Theorems 1.6 and 2.3.

(2) Note that $M^* \neq 0$ (cf. Remark 2.2). So, by Theorem 1.6, R is CM. Hence the assertion follows from Theorem 2.5. ☐

Remark 2.11. The existence of an R-module M such that M^* is nonzero and of finite injective dimension does not necessarily imply that R is Gorenstein, see Example 2.7.

Corollary 2.12. *Let R be a commutative Noetherian ring. Let M be a (finitely generated) R-module such that $\operatorname{Hom}_R(M,R)$ has finite injective dimension. Then the Auslander-Reiten conjecture holds true for M.*

Proof. Suppose that $\operatorname{Ext}^{>0}_R(M, M \oplus R) = 0$. What we want to show is that M is projective. It is equivalent to saying that the $R_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$ is free for each prime ideal $\mathfrak{p}$ of R. Replacing R and M with $R_{\mathfrak{p}}$ and $M_{\mathfrak{p}}$ respectively, we may assume that R is local. Hence the assertion follows from Corollary 2.10.(2). ☐

Corollary 2.13. *Let M be an R-module such that $\operatorname{depth}(M) = d$, $\operatorname{Ext}^j_R(M,M) = 0$ for all $1 \leqslant j \leqslant d$ and $\operatorname{injdim}_R(\operatorname{Hom}_R(M,M)) < \infty$. Then, M is free, and R is Gorenstein.*

Proof. The condition $\operatorname{depth}(M) = d$ particularly ensures that M is nonzero. By Theorem 2.5, M is free and $\operatorname{injdim}_R(M) < \infty$. So R is Gorenstein. □

We obtain the following criteria for a module to be free over a Gorenstein local ring in terms of vanishing of Ext and projective dimension of Hom.

Corollary 2.14. *Suppose that R is Gorenstein. For an R-module M, the following are equivalent:*
(1) *M is free.*
(2) *$\operatorname{Hom}_R(M,M)$ is free, and $\operatorname{Ext}^j_R(M,M) = 0$ for all $1 \leqslant j \leqslant d$.*
(3) *$\operatorname{Hom}_R(M,M)$ has finite projective dimension, and*

$$\operatorname{Ext}^i_R(M,R) = \operatorname{Ext}^j_R(M,M) = 0 \textit{ for all } 1 \leqslant i \leqslant d \textit{ and } 1 \leqslant j \leqslant d.$$

Proof. The implication (1) ⇒ (2) is trivial.

(2) ⇒ (3): In view of the proof of [2, Lemma 4.1], we have $\operatorname{depth}(M) = \operatorname{depth}(\operatorname{Hom}_R(M,M))$. Since R is Gorenstein and $\operatorname{Hom}_R(M,M)$ is free, it follows that M is MCM, and hence $\operatorname{Ext}^i_R(M,R) = 0$ for all $1 \leqslant i \leqslant d$.

(3) ⇒ (1): Since R is Gorenstein, M is MCM by [5, 3.5.11]. Moreover, for an R-module L, $\operatorname{projdim}_R(L)$ is finite if and only if $\operatorname{injdim}_R(L)$ is finite, cf. [5, 3.1.25]. Therefore $\operatorname{injdim}_R(\operatorname{Hom}_R(M,M))$ is finite. Hence, by Corollary 2.13, M is free. □

Next we provide an affirmative answer to the question whether the Auslander-Reiten conjecture holds true if $\operatorname{Hom}_R(M,M)$ has finite injective dimension.

Theorem 2.15. *Let M be a nonzero R-module such that $\operatorname{Ext}^i_R(M,R) = 0$ and $\operatorname{Ext}^j_R(M,M) = 0$ for all $1 \leqslant i \leqslant 2d+1$ and $1 \leqslant j \leqslant \max\{1, d-1\}$. Suppose that $\operatorname{Hom}_R(M,M)$ has finite injective dimension. Then, M is free, and R is Gorenstein.*

Proof. Let $\mathbb{F}_M : \cdots \xrightarrow{\partial_3} R^{n_2} \xrightarrow{\partial_2} R^{n_1} \xrightarrow{\partial_1} R^{n_0} \xrightarrow{\partial_0} M \to 0$ be a minimal free resolution of M, and set $\Omega^i M := \operatorname{Image} \partial_i$ for each $i \geqslant 0$. As $\operatorname{Ext}^1_R(M,M) = 0$, an exact sequence

$$0 \longrightarrow \operatorname{Hom}_R(M,M) \xrightarrow{f} M^{n_0} \xrightarrow{g} M^{n_1} \xrightarrow{h} M^{n_2}$$

is induced. Putting $N := \operatorname{Coker} f = \operatorname{Image} g$ and $L := \operatorname{Coker} g = \operatorname{Image} h$, we have the following exact sequences:

$$0 \longrightarrow \operatorname{Hom}_R(M,M) \xrightarrow{f} M^{n_0} \xrightarrow{e} N \longrightarrow 0, \tag{2.5}$$

$$0 \longrightarrow N \longrightarrow M^{n_1} \longrightarrow L \longrightarrow 0 \tag{2.6}$$

Note that N is isomorphic to $\operatorname{Hom}_R(\Omega(M), M)$. As $\operatorname{Hom}_R(M,M)$ is nonzero and of finite injective dimension, by Theorem 1.6, R is CM.

We prove the theorem by induction on d.

(1) First, we deal with the case $d = 0$. We may assume that M is indecomposable. The module $\operatorname{Hom}_R(M,M)$ is nonzero and injective. Hence $\operatorname{Hom}_R(M,M)$ is isomorphic to a finite direct sum of copies of $E := E_R(k)$. The map f is a split monomorphism. Since R is henselian and M is indecomposable, the Krull–Schmidt theorem yields that $M \cong E$. Hence $\operatorname{Hom}_R(M,M) \cong \operatorname{Hom}_R(E,E) \cong R$. As $\operatorname{Hom}_R(M,M)$ is injective, the Artinian ring R is Gorenstein and thus $M \cong E \cong R$.

(2) Second, we handle the case $d = 1$. We may assume that R is complete, and M is indecomposable. As R is complete, it admits a canonical module ω.

(2a) We start with the case $\operatorname{depth}(M) > 0$. In this case, M is MCM, and so are $\operatorname{Hom}_R(M,M)$ and $N \cong \operatorname{Hom}_R(\Omega(M), M)$ as

$$\operatorname{depth}(\operatorname{Hom}_R(X, M)) \geqslant \inf\{2, \operatorname{depth} M\} > 0$$

for any finitely generated R-module X. It follows that $\operatorname{Hom}_R(M, M)$ is isomorphic to a finite direct sum of copies of ω, cf. [5, 3.3.28]. The exact sequence (2.5) splits since it is identified with an element of $\operatorname{Ext}^1_R(N, \operatorname{Hom}_R(M, M)) = 0$. As R is henselian and M is indecomposable, the Krull–Schmidt theorem implies that $M \cong \omega$. We get $\operatorname{Hom}_R(M, M) \cong R$, and see that R is Gorenstein, and $M \cong R$.

(2b) Next we consider the case $\operatorname{depth} M = 0$. From (2.5), an exact sequence

$$\operatorname{Ext}^1_R(L, M^{n_0}) \xrightarrow{\phi} \operatorname{Ext}^1_R(L, N) \longrightarrow \operatorname{Ext}^2_R(L, \operatorname{Hom}_R(M, M)) \tag{2.7}$$

is induced, where $\phi = \operatorname{Ext}^1_R(L, e)$. The short exact sequence (2.6) can be identified with an element β of $\operatorname{Ext}^1_R(L, N)$. As $\operatorname{injdim}_R(\operatorname{Hom}_R(M, M)) = d = 1 < 2$, the module $\operatorname{Ext}^2_R(L, \operatorname{Hom}_R(M, M))$ vanishes. Hence the map ϕ in (2.7) is surjective. So there exists an element $\gamma \in \operatorname{Ext}^1_R(L, M^{n_0})$ such that $\phi(\gamma) = \beta$. We obtain a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \operatorname{Hom}_R(M,M) & = & \operatorname{Hom}_R(M,M) & & & & \\
 & & \downarrow{\scriptstyle f} & & \downarrow & & & & \\
\gamma : 0 & \longrightarrow & M^{n_0} & \longrightarrow & Z & \longrightarrow & L & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle e} & & \downarrow & & \| & & \\
\beta : 0 & \longrightarrow & N & \longrightarrow & M^{n_1} & \longrightarrow & L & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

with exact rows and columns. Taking the mapping cone of the above chain map $\gamma \to \beta$, we get an exact sequence

$$0 \longrightarrow M^{n_0} \longrightarrow N \oplus Z \longrightarrow M^{n_1} \longrightarrow 0, \tag{2.8}$$

which is identified with an element of $\operatorname{Ext}^1_R(M^{n_1}, M^{n_0}) \cong \operatorname{Ext}^1_R(M, M)^{n_1 n_0} = 0$. Thus the exact sequence (2.8) splits, and an isomorphism $N \oplus Z \cong M^{n_0+n_1}$ follows. As R is henselian and M is indecomposable, we have $N \cong M^m$ for some $m \geqslant 0$. Set $r := \operatorname{type}(M) = \dim_k \operatorname{Hom}_R(k, M) > 0$; recall that $\operatorname{depth} M = 0$. There are isomorphisms

$$\begin{aligned} k^{rm} \cong \operatorname{Hom}_R(k, M^m) \cong \operatorname{Hom}_R(k, N) &\cong \operatorname{Hom}_R(k, \operatorname{Hom}_R(\Omega(M), M)) \\ &\cong \operatorname{Hom}_R(k \otimes_R \Omega(M), M) \cong k^{n_1 r}, \end{aligned}$$

whence $m = n_1$. Applying $k \otimes_R (-)$ to the exact sequence (2.5):

$$0 \longrightarrow \operatorname{Hom}_R(M, M) \xrightarrow{f} M^{n_0} \longrightarrow M^{n_1} \longrightarrow 0, \tag{2.9}$$

we observe that $n_0 \geqslant n_1$.

Fix a minimal prime ideal $\mathfrak{p}$ of R. If $M_\mathfrak{p} = 0$, then of course $M_\mathfrak{p}$ is $R_\mathfrak{p}$-free. If $M_\mathfrak{p} \neq 0$, then applying the induction hypothesis to the Artinian local ring $R_\mathfrak{p}$

shows that $M_\mathfrak{p}$ is $R_\mathfrak{p}$-free. Thus, in any case, we have $M_\mathfrak{p} \cong R_\mathfrak{p}^t$ for some $t \geqslant 0$. Localization of (2.9) at $\mathfrak{p}$ gives rise to an exact sequence

$$0 \longrightarrow \operatorname{Hom}_{R_\mathfrak{p}}(M_\mathfrak{p}, M_\mathfrak{p}) \xrightarrow{f_\mathfrak{p}} M_\mathfrak{p}^{n_0} \longrightarrow M_\mathfrak{p}^{n_1} \longrightarrow 0,$$

and we get $tn_0 = \operatorname{rank}_{R_\mathfrak{p}} M_\mathfrak{p}^{n_0} = \operatorname{rank}_{R_\mathfrak{p}} \operatorname{Hom}_{R_\mathfrak{p}}(M_\mathfrak{p}, M_\mathfrak{p}) + \operatorname{rank}_{R_\mathfrak{p}} M_\mathfrak{p}^{n_1} = t^2 + tn_1$. Hence $t \in \{0, n_0 - n_1\}$. Localizing the resolution $\mathbb{F}_M$, there is an exact sequence

$$0 \longrightarrow (\Omega^2 M)_\mathfrak{p} \longrightarrow R_\mathfrak{p}^{n_1} \longrightarrow R_\mathfrak{p}^{n_0} \longrightarrow M_\mathfrak{p} \longrightarrow 0,$$

while $M_\mathfrak{p} \cong R_\mathfrak{p}^t$. An isomorphism $(\Omega^2 M)_\mathfrak{p} \cong R_\mathfrak{p}^{n_1 - n_0 + t}$ follows. We have $n_1 - n_0 + t = 0$ when $t = n_0 - n_1$, while $n_1 - n_0 + t = n_1 - n_0 \leqslant 0$ when $t = 0$. Therefore, $(\Omega^2 M)_\mathfrak{p} = 0$ for every minimal prime ideal $\mathfrak{p}$ of R. Since $\Omega^2(M)$ is a torsion submodule of the torsion-free module R^{n_1}, we have $\Omega^2(M) = 0$. It follows that $\operatorname{pd}_R M \leqslant 1 < \infty$. By assumption, $\operatorname{Ext}_R^i(M, R) = 0$ for all $1 \leqslant i \leqslant 2d + 1$. In general, when $\operatorname{pd}_R M$ is finite, it is equal to the supremum of integers i such that $\operatorname{Ext}_R^i(M, R) \neq 0$. Consequently, the module M is free, so is $\operatorname{Hom}_R(M, M)$, and therefore R is Gorenstein.

(3) Finally, we consider the case $d \geqslant 2$. Fix a nonmaximal prime ideal $\mathfrak{p}$ of R. Applying the induction hypothesis to $R_\mathfrak{p}$, we see that $M_\mathfrak{p}$ is $R_\mathfrak{p}$-free. Note that $\max\{1, d-1\} = d - 1$. We can use [17, Theorem 1.2.(2)] to observe that M is R-free, and so is $\operatorname{Hom}_R(M, M)$, whence R is Gorenstein. □

Corollary 2.16. *Let R be a commutative Noetherian ring. Let M be a (finitely generated) R-module such that $\operatorname{Hom}_R(M, M)$ has finite injective dimension. Then the Auslander-Reiten conjecture holds true for M.*

Proof. Let $\operatorname{Ext}_R^{>0}(M, M \oplus R) = 0$. We show that M is projective. It is equivalent to showing that the $R_\mathfrak{p}$-module $M_\mathfrak{p}$ is free for each prime ideal $\mathfrak{p}$ of R. Replacing R and M with $R_\mathfrak{p}$ and $M_\mathfrak{p}$ respectively, we may assume that R is local. Thus the desired statement follows from Theorem 2.15. □

In view of Theorem 2.15, we may wonder if $\operatorname{Hom}_R(M, M)$ having finite injective dimension implies that M has finite projective dimension. This is not true even when R is a Gorenstein normal local ring. Particularly, it shows that the hypothesis on the vanishing of Ext in Theorem 2.15 cannot be omitted.

Example 2.17. Let R be a Gorenstein normal local ring, and I be a nonzero ideal of R. Then $\operatorname{Hom}_R(I, I) \cong R$. Therefore $\operatorname{Hom}_R(I, I)$ has finite injective dimension, but I does not necessarily have finite projective dimension. For example, one may consider $R = k[[x, y, z]]/(x^2 - yz)$ with k a field, and $I = \mathfrak{m} = (x, y, z)$.

We close this section with the following natural question.

Question 2.18. If there exists a nonzero R-module M such that $\operatorname{Hom}_R(M, M)$ has finite injective dimension, then is R Gorenstein?

3. Characterizations of Gorenstein local rings

In this section, we provide a number of characterizations of Gorenstein local rings in terms of finite injective dimension of certain Hom. We start with answering Question 2.18 affirmatively in some special cases.

Proposition 3.1. *Let M be an R-module. Suppose that* (i) *M is torsion-free,* (ii) *M is locally free in codimension* 1*,* (iii) *M has a rank, and* (iv) $\operatorname{rank}(M)$ *is invertible in R. If $\operatorname{Hom}_R(M,M)$ has finite injective dimension, then R is Gorenstein.*

Proof. Since M is nonzero, so is $\operatorname{Hom}_R(M,M)$. Then Theorem 1.6 yields that R is CM. It follows from [14, A.2 and A.5] that R is a direct summand of $\operatorname{Hom}_R(M,M)$. Hence R has finite injective dimension, i.e., R is Gorenstein. □

Proposition 3.2. *Suppose that R admits a module M such that*

$$\operatorname{depth}(\operatorname{Hom}_R(M,M)) = \operatorname{depth}(R) \text{ and } \operatorname{injdim}_R(\operatorname{Hom}_R(M,M)) < \infty.$$

Then R is Gorenstein.

Proof. Replacing R with its completion, we may assume that R is complete. By Theorem 1.6, R is CM, and hence $\operatorname{Hom}_R(M,M)$ is MCM. Since R is complete CM, it admits a canonical module ω, and we have that $\operatorname{Hom}_R(M,M) \cong \omega^n$ for some $n \geqslant 1$, cf. [5, 3.3.28]. It follows that

$$\begin{aligned} R^{n^2} \cong \operatorname{Hom}_R(\omega^n, \omega^n) &\cong \operatorname{Hom}_R(\operatorname{Hom}_R(M,M), \operatorname{Hom}_R(M,M)) \\ &\cong \operatorname{Hom}_R(M \otimes_R \operatorname{Hom}_R(M,M), M). \end{aligned}$$

Consider the R-module homomorphisms

$$f : M \to M \otimes_R \operatorname{Hom}_R(M,M) \quad \text{and} \quad g : M \otimes_R \operatorname{Hom}_R(M,M) \to M$$

defined by $f(x) = x \otimes \operatorname{id}_M$ and $g(x \otimes h) = h(x)$ respectively. Clearly, the composition $g \circ f = \operatorname{id}_M$, and hence f is a split monomorphism. Therefore M is a direct summand of $M \otimes_R \operatorname{Hom}_R(M,M)$. This implies that $\omega^n \cong \operatorname{Hom}_R(M,M)$ is a direct summand of $\operatorname{Hom}_R(M \otimes_R \operatorname{Hom}_R(M,M), M) \cong R^{n^2}$. So, by the Krull–Schmidt theorem, $\omega \cong R$, and hence R is Gorenstein. □

3.3. A partial positive answer to Question 2.18 is provided in [6, Cor. 4.5], where it is shown that if there exists a nonzero R-module M of depth $\geqslant d-1$ such that the injective dimensions of M, $\operatorname{Hom}_R(M,M)$ and $\operatorname{Ext}^1_R(M,M)$ are finite, then the projective dimension of M is finite, and R is Gorenstein.

3.4. Let R be CM. In [23, Thm. 3.1], Ulrich gave a criterion for R to be Gorenstein: If there is an R-module L of positive rank such that $2\nu(L) > e(R)\operatorname{rank}(L)$ and $\operatorname{Ext}_R^{1\leqslant i\leqslant d}(L,R) = 0$, then R is Gorenstein. Hanes-Huneke and Jorgensen-Leuschke gave some analogous criteria in [12, Thms. 2.5 and 3.4] and [15, Thms. 2.2 and 2.4] respectively. Recently, Lyle and Montaño [18, Thm. D] showed that a generically Gorenstein CM local ring R that has a canonical module is Gorenstein if there is an MCM R-module L such that $\operatorname{Ext}_R^{1\leqslant i\leqslant d+1}(L,R) = 0$ and $e_R(L) \leqslant 2\nu(L)$.

3.5. Like Theorems 1.4 and 1.5, having finite injective dimension of certain modules also ensures that the base ring is regular. For example, R is regular if and only if its residue field k has finite injective dimension, see, e.g., [5, 3.1.26]. More generally, it is shown in [10, Thm. 3.7] that R is regular if and only if some syzygy $\Omega^n_R(k)$ ($n \geqslant 0$) has a nonzero direct summand of finite injective dimension.

Inspired by the results mentioned in 3.4, 3.5 and Theorems 1.4 and 1.5, we obtain the following new characterizations of Gorenstein local rings in terms of vanishing of certain Ext and finite injective dimension of Hom.

Theorem 3.6. *The following statements are equivalent:*

(1) *R is Gorenstein.*
(2) *R admits a module M such that $\mathrm{Ext}^i_R(M,R)=0$ for all $1\leqslant i\leqslant d-1$ and M^* is nonzero and of finite injective dimension.*
(3) *R admits a nonzero module M such that $\mathrm{Ext}^i_R(M,R)=\mathrm{Ext}^j_R(M,M)=0$ for all $1\leqslant i\leqslant 2d+1$ and $1\leqslant j\leqslant \max\{1,d-1\}$, and $\mathrm{injdim}_R(\mathrm{Hom}_R(M,M))<\infty$.*
(4) *R admits a module M such that $\mathrm{depth}(M)=d$, $\mathrm{Ext}^j_R(M,M)=0$ for all $1\leqslant j\leqslant d$ and $\mathrm{injdim}_R(\mathrm{Hom}_R(M,M))<\infty$.*
(5) *R admits a module M such that*

$$\mathrm{depth}(\mathrm{Hom}_R(M,M))=\mathrm{depth}(R) \text{ and } \mathrm{injdim}_R(\mathrm{Hom}_R(M,M))<\infty.$$

Proof. The implications (1) $\Rightarrow$ (2), (3), (4) and (5) are trivial as $M=R$ satisfies the respective conditions. The reverse implications (2) $\Rightarrow$ (1), (3) $\Rightarrow$ (1), (4) $\Rightarrow$ (1) and (5) $\Rightarrow$ (1) follow from 2.10.(1), 2.15, 2.13 and 3.2 respectively. □

Remark 3.7. This is known due to Holm [13] that if there exists a nonzero R-module M of finite Gorenstein dimension and finite injective dimension, then R is Gorenstein. It should be noted that Theorem 3.6.(1) $\Leftrightarrow$ (2) would not follow from the result of Holm. Moreover, we can recover that if M is a nonzero R-module such that $\text{G-dim}_R(M)=0$ and $\mathrm{injdim}_R(M)<\infty$, then M^* satisfies the condition (2) in 3.6 as $\mathrm{Ext}^i_R(M^*,R)=0$ for all $i\geqslant 1$ and $(M^*)^*\cong M$ has finite injective dimension, hence R is Gorenstein.

Acknowledgments

Ghosh was supported by Start-up Research Grant (SRG) from SERB, DST, Govt. of India with the Grant No SRG/2020/000597. Takahashi was partly supported by JSPS Grant-in-Aid for Scientific Research 19K03443.

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