UPPER COMPLETE INTERSECTION DIMENSION RELATIVE TO A LOCAL HOMOMORPHISM

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Abstract. In this note, we introduce a homological invariant for finitely generated modules over commutative noetherian local rings by slightly modifying the definition of complete intersection dimension defined by Avramov, Gasharov, and Peeva [4], and observe it from a relative point of view.

1. Introduction

Throughout this note, we assume that all rings are commutative noetherian rings, and all modules are finitely generated.

Projective dimension and Gorenstein dimension (abbr. G-dimension) have played important roles in the classification of modules and rings. Recently, complete intersection dimension (abbr. CI-dimension) and Cohen-Macaulay dimension (abbr. CM-dimension) were introduced by Avramov, Gasharov, and Peeva [4] and Gerko [6], respectively. The former is defined by using projective dimension and the idea of quasi-deformation, and the latter is defined by using G-dimension and the idea of G-quasideformation.

These dimensions are homological invariants for modules, and share many properties with each other. For example, they satisfy the Auslander-Buchsbaum-type equalities. Every module over a regular (resp. complete intersection, Gorenstein, Cohen-Macaulay) local ring is of finite projective (resp. CI-, G-, CM-) dimension, and a local ring is a regular (resp. complete intersection, Gorenstein, Cohen-Macaulay) ring if the projective (resp. CI-, G-, CM-) dimension of its residue class field is finite. Moreover, among these dimensions, there are inequalities which yield the well-known implications for a local ring R: R is regular $\Rightarrow$ R is a complete intersection $\Rightarrow$ R is Gorenstein $\Rightarrow$ R is Cohen-Macaulay.

In this note, we are interested in CI-dimension. Gulliksen [7] showed that every module over a complete intersection has finite complexity, that is, the Betti numbers are eventually bounded by a polynomial. As a result extending this, Avramov, Gasharov, and Peeva [4] proved that any module of finite CI-dimension has finite complexity. Hence, free resolutions of modules of finite CI-dimension are eventually well-behaved. However, there are a lot of unsolved problems on CI-dimension. For instance, it is unknown whether a module of finite complexity is always of finite CI-dimension. Though we do not discuss these problems in this note, it is important to consider CI-dimension.

Here we recall the definition of the CI-dimension of a module over a local ring R. It is similar to that of virtual projective dimension introduced by Avramov [2]:

Key words and phrases: complete intersection, CI-dimension.

2000 Mathematics Subject Classification: 13D05, 13H10, 14M10.

(1) A local homomorphism $\phi : S \to R$ of local rings is called a *deformation* if ϕ is surjective and the kernel of ϕ is generated by an S-regular sequence.
(2) A diagram $S \overset{\phi}{\to} R' \overset{\alpha}{\leftarrow} R$ of local homomorphisms of local rings is called a *quasi-deformation* of R if α is faithfully flat and ϕ is a deformation.
(3) For an R-module M, the *complete intersection dimension* of M is defined as follows:

$$\text{CI-dim}_R M = \inf \left\{ \begin{array}{r|l} \text{pd}_S(M \otimes_R R') & S \to R' \leftarrow R \text{ is a} \\ -\text{pd}_S R' & \text{quasi-deformation of } R \end{array} \right\}$$

Now, slightly modifying the definition of CI-dimension, we define a homological invariant for a module over a local ring as follows.

Definition 1.1. (1) We call a diagram $S \overset{\phi}{\to} R' \overset{\alpha}{\leftarrow} R$ of local homomorphisms of local rings an *upper quasi-deformation* of R if α is faithfully flat, the closed fiber of α is regular, and ϕ is a deformation.
(2) For an R-module M, we define the *upper complete intersection dimension* (abbr. CI^*-dimension) of M as follows:

$$\text{CI}^*\text{-dim}_R M = \inf \left\{ \begin{array}{r|l} \text{pd}_S(M \otimes_R R') & S \to R' \leftarrow R \text{ is an} \\ -\text{pd}_S R' & \text{upper quasi-deformation of } R \end{array} \right\}$$

Here we itemize several properties of CI^*-dimension, which are analogous to those of CI-dimension. We omit their proofs because we can prove them in the same way as the proofs of the corresponding results of CI-dimension given in [4]. Let R be a local ring with residue field k, $M \neq 0$ an R-module, and $\boldsymbol{x} = x_1, x_2, \cdots, x_n$ a sequence in R. We denote by $\Omega_R^r M$ the rth syzygy module of M.

(1) The following conditions are equivalent.
  i) R is a complete intersection.
  ii) $\text{CI}^*\text{-dim}_R X < \infty$ for any R-module X.
  iii) $\text{CI}^*\text{-dim}_R k < \infty$.
(2) If $\text{CI}^*\text{-dim}_R M < \infty$, then $\text{CI}^*\text{-dim}_R M = \operatorname{depth} R - \operatorname{depth}_R M$.
(3) $\text{CI}^*\text{-dim}_R \Omega_R^r M = \sup\{\text{CI}^*\text{-dim}_R M - r, 0\}$.
(4) $\text{CI}^*\text{-dim}_R M/\boldsymbol{x}M = \text{CI}^*\text{-dim}_R M + n$ if $\boldsymbol{x}$ is M-regular.
(5) $\text{CI}^*\text{-dim}_{R/(\boldsymbol{x})} M/\boldsymbol{x}M \leq \text{CI}^*\text{-dim}_R M$ if $\boldsymbol{x}$ is R-regular and M-regular.
The equality holds if $\text{CI}^*\text{-dim}_R M < \infty$.
(6) $\text{CI}^*\text{-dim}_{R/(\boldsymbol{x})} M \leq \text{CI}^*\text{-dim}_R M - n$ if $\boldsymbol{x}$ is R-regular and $\boldsymbol{x}M = 0$.
The equality holds if $\text{CI}^*\text{-dim}_R M < \infty$.
(7) $\text{CI-dim}_R M \leq \text{CI}^*\text{-dim}_R M \leq \text{pd}_R M$.
If any one of these dimensions is finite, then it is equal to those to its left.

Araya, Takahashi, and Yoshino [1], modifying the definition of CM-dimension, define a homological invariant for modules as a relative version of the modified CM-dimension. This invariant has a lot of properties similar to projective dimension, CI-dimension, G-dimension, and CM-dimension.

Let $\phi : S \to R$ be a local homomorphism of local rings. The main purpose of this note is to define a new homological invariant for an R-module M as a relative version of CI^*-dimension over R, and to study its properties. We will call this the *upper complete intersection dimension* of M *relative* to ϕ, and denote it by $\text{CI}^*\text{-dim}_\phi M$. We shall observe that this invariant has many properties similar to those of the invariant defined by Araya, Takahashi, and Yoshino. For example, we will prove the following. Let k denote the residue class field of R.

Theorem 2.10. *Let M be a non-zero R-module. If $\mathrm{CI}^*\text{-}\mathrm{dim}_\phi M < \infty$, then $\mathrm{CI}^*\text{-}\mathrm{dim}_\phi M = \operatorname{depth} R - \operatorname{depth}_R M$.*

Theorem 2.14. *Suppose that $S = R$ and ϕ is the identity map on R. Then $\mathrm{CI}^*\text{-}\mathrm{dim}_\phi M = \mathrm{pd}_R M$ for every R-module M.*

Theorem 2.15. *The following conditions are equivalent.*

i) *R is a complete intersection and S is a regular ring.*
ii) *$\mathrm{CI}^*\text{-}\mathrm{dim}_\phi M < \infty$ for any R-module M.*
iii) *$\mathrm{CI}^*\text{-}\mathrm{dim}_\phi k < \infty$.*

2. Relative CI*-dimension

Throughout the section, $\phi : (S, \mathfrak{n}, l) \to (R, \mathfrak{m}, k)$ always denotes a local homomorphism of local rings.

In this section, we shall make the precise definition of the upper complete intersection dimension of an R-module relative to ϕ to observe CI*-dimension from a relative point of view. To do this, we need the notion of P-factorization, instead of that of upper quasi-deformation used in the definition of (absolute) CI*-dimension.

Definition 2.1. Let

$$\begin{array}{ccc} S' & \xrightarrow{\phi'} & R' \\ {\scriptstyle\beta}\uparrow & & \uparrow{\scriptstyle\alpha} \\ S & \xrightarrow[\phi]{} & R, \end{array}$$

be a commutative diagram of local homomorphisms of local rings. We call this diagram a *P-factorization* of ϕ if α and β are faithfully flat, the closed fiber of α is regular, and ϕ' is a deformation.

Note that this is an imitation of a G-factorization defined in [1]. The existence of a P-factorization of ϕ transmits several properties of R to S:

Proposition 2.2. *Suppose that there exists a P-factorization of ϕ. Then, if R is a regular (resp. complete intersection, Gorenstein, Cohen-Macaulay) ring, so is S.*

Proof. Let $S \overset{\beta}{\to} S' \overset{\phi'}{\to} R' \overset{\alpha}{\leftarrow} R$ be a P-factorization of ϕ. Suppose that R is a regular (resp. complete intersection, Gorenstein, Cohen-Macaulay) ring. Since α is a faithfully flat homomorphism with regular closed fiber, R' is also a regular (resp. ...) ring. Since ϕ' is a deformation, we easily see that S' is also a regular (resp. ...) ring, and so is S by the flatness of β. □

From now on, we consider the existence of a P-factorization of ϕ. First of all, the above proposition yields the following example which says that ϕ may not have a P-factorization.

Example 2.3. Suppose that $R = l$ is the residue class field of S and ϕ is the natural surjection from S to l. Then ϕ has no P-factorization unless S is regular by Proposition 2.2.

Although there does not necessarily exist a P-factorization of ϕ in general, a P-factorization of ϕ seems to exist whenever the ring S is regular. We are able to show it if in addition we assume the condition that S contains a field:

Theorem 2.4. *Suppose that S is a regular local ring containing a field. Then every local homomorphism $\phi : S \to R$ of local rings has a P-factorization.*

This theorem is essentially proved in [1]. But we shall give here a whole proof of it for this note to be as self-contained as possible. We need the following two lemmas:

Lemma 2.5. [3, Theorem 1.1] *Let $\phi : (S, \mathfrak{n}) \to (R, \mathfrak{m})$ be a local homomorphism of local rings, and α be the natural embedding from R into its $\mathfrak{m}$-adic completion $\widehat{R}$. Then there exists a commutative diagram*

$$\begin{array}{ccc} S' & \xrightarrow{\phi'} & \widehat{R} \\ \beta\uparrow & & \uparrow\alpha \\ S & \xrightarrow[\phi]{} & R \end{array}$$

of local homomorphisms of local rings such that β is faithfully flat, the closed fiber of β is regular, and ϕ' is surjective. (Such a diagram is called a Cohen factorization *of ϕ.)*

Lemma 2.6. *Let $\phi : S \to R$ be a local homomorphism of complete local rings that admit the common coefficient field k. Put $S' = S\widehat{\otimes}_k R$. Let $\lambda : S \to S'$ be the injective homomorphism mapping $b \in S$ to $b\widehat{\otimes}1 \in S'$, and $\varepsilon : S' \to R$ be the surjective homomorphism mapping $b\widehat{\otimes}a \in S'$ to $\phi(b)a \in R$. Suppose that S is regular. Then $S \xrightarrow{\lambda} S' \xrightarrow{\varepsilon} R \xleftarrow{id} R$ is a P-factorization of ϕ.*

Proof. Let $y_1, y_2, \cdots, y_s$ be a minimal system of generators of the unique maximal ideal of S. Put $J = \operatorname{Ker}\varepsilon$ and $dy_i = y_i\widehat{\otimes}1 - 1\widehat{\otimes}\phi(y_i) \in S'$ for each $1 \leqq i \leqq s$.

Claim 1. The ideal J of S' is generated by $dy_1, dy_2, \cdots, dy_s$.

Indeed, put $J_0 = (dy_1, dy_2, \cdots, dy_s)S'$. Let $z = b\widehat{\otimes}a$ be an element in J, and let $b = \sum b_{i_1 i_2 \cdots i_s} y_1^{i_1} y_2^{i_2} \cdots y_s^{i_s}$ be a power series expansion in $y_1, y_2, \cdots, y_s$ with coefficients $b_{i_1 i_2 \cdots i_s} \in k$. Then we have $b\widehat{\otimes}1 = \sum b_{i_1 i_2 \cdots i_s} (y_1\widehat{\otimes}1)^{i_1} (y_2\widehat{\otimes}1)^{i_2} \cdots (y_s\widehat{\otimes}1)^{i_s} \equiv \sum b_{i_1 i_2 \cdots i_s} (1\widehat{\otimes}\phi(y_1))^{i_1} (1\widehat{\otimes}\phi(y_2))^{i_2} \cdots (1\widehat{\otimes}\phi(y_s))^{i_s} = 1\widehat{\otimes}\phi(b)$ modulo J_0. It follows that $z \equiv 1\widehat{\otimes}\phi(b)a$ modulo J_0. Since $\phi(b)a = \varepsilon(b\widehat{\otimes}a) = 0$, we have $z \equiv 0$ modulo J_0, that is, the element $z \in J$ belongs to J_0. Thus, we see that $J = J_0$.

Claim 2. The sequence $dy_1, dy_2, \cdots, dy_s$ is an S'-regular sequence.

Indeed, since S is regular, we may assume that $S = k[[Y_1, Y_2, \cdots, Y_s]]$ and $S' = R[[Y_1, Y_2, \cdots, Y_s]]$ are formal power series rings, and $dy_i = Y_i - \phi(Y_i) \in S'$ for each $1 \leqq i \leqq s$. Note that the endomorphism on S' which sends Y_i to dy_i is an automorphism. Since the sequence $Y_1, Y_2, \cdots, Y_s$ is S'-regular, we see that $dy_1, dy_2, \cdots, dy_s$ also form an S'-regular sequence.

These claims prove that the homomorphism ε is a deformation. On the other hand, it is easy to see that λ is faithfully flat. Thus, the lemma is proved. $\square$

Proof of Theorem 2.4. We may assume that R (resp. S) is complete in its $\mathfrak{m}$-adic (resp. $\mathfrak{n}$-adic) topology. Hence Lemma 2.5 implies that ϕ has a Cohen factorization

$$\begin{tikzcd} & S' \arrow[dr, "\phi'"] & \\ S \arrow[ur, "\beta"] \arrow[rr, "\phi"'] & & R, \end{tikzcd}$$

where β is a faithfully flat homomorphism with regular closed fiber, and ϕ' is a surjective homomorphism. Hence S' is also a regular local ring containing a field. Therefore, replacing S with S', we may assume that ϕ is a surjection. In particular R and S have the common coefficient field, hence Lemma 2.6 implies that ϕ has a P-factorization, as desired. $\square$

Conjecture 2.7. Whenever S is regular, the local homomorphism $\phi : S \to R$ would have a P-factorization.

Now, by using the idea of P-factorization, we define the CI*-dimension of a module in a relative sense.

Definition 2.8. For an R-module M, we put

$$\text{CI}^*\text{-dim}_\phi M = \inf\left\{ \begin{array}{r|l} \text{pd}_{S'}(M \otimes_R R') & S \to S' \to R' \leftarrow R \\ -\text{pd}_{S'} R' & \text{is a P-factorization of } \phi \end{array} \right\}$$

and call it the *upper complete intersection dimension* of M *relative* to ϕ.

By definition, $\text{CI}^*\text{-dim}_\phi M = \infty$ for an R-module M if ϕ has no P-factorization. Supppose that ϕ has at least one P-factorization $S \to S' \to R' \leftarrow R$. Then we have $\text{pd}_{S'}(F \otimes_R R') = \text{pd}_{S'} R'$ $(< \infty)$ for any free R-module F. Therefore the above theorem on the existence of a P-factorization yields the following result:

Proposition 2.9. *If S is a regular local ring that contains a field, then*

$$\text{CI}^*\text{-dim}_\phi F = 0 \ (< \infty)$$

for any free R-module F.

In the rest of this section, we observe the properties of relative CI*-dimension CI*-dim$_\phi$. We begin by proving that relative CI*-dimension also satisfies the Auslander-Buchsbaum-type equality:

Theorem 2.10. *Let M be a non-zero R-module. If $\text{CI}^*\text{-dim}_\phi M < \infty$, then*

$$\text{CI}^*\text{-dim}_\phi M = \operatorname{depth} R - \operatorname{depth}_R M.$$

Proof. Since $\text{CI}^*\text{-dim}_\phi M < \infty$, there exists a P-factorization $S \xrightarrow{\beta} S' \xrightarrow{\phi'} R' \xleftarrow{\alpha} R$ of ϕ such that $\text{CI}^*\text{-dim}_\phi M = \text{pd}_{S'}(M \otimes_R R') - \text{pd}_{S'} R' < \infty$. Hence we see that

$$\begin{array}{rcl} \text{CI}^*\text{-dim}_\phi M & = & \text{pd}_{S'}(M \otimes_R R') - \text{pd}_{S'} R' \\ & = & (\operatorname{depth} S' - \operatorname{depth}_{S'}(M \otimes_R R')) - (\operatorname{depth} S' - \operatorname{depth}_{S'} R') \\ & = & \operatorname{depth}_{S'} R' - \operatorname{depth}_{S'}(M \otimes_R R'). \end{array}$$

Note that ϕ' is surjective. Since α and β are faithfully flat, we obtain

$$\begin{cases} \operatorname{depth}_{S'} R' = \operatorname{depth} R + \operatorname{depth} R'/\mathfrak{m}R', \\ \operatorname{depth}_{S'}(M \otimes_R R') = \operatorname{depth}_R M + \operatorname{depth} R'/\mathfrak{m}R'. \end{cases}$$

It follows that $\text{CI}^*\text{-dim}_\phi M = \operatorname{depth} R - \operatorname{depth}_R M$. $\square$

In view of this theorem, we notice that the value of the relative CI*-dimension of an R-module is given independently of the ring S if it is finite.

Proposition 2.11. *Let M be an R-module. Then*
(1) $\mathrm{CI}^*\text{-}\dim_\phi M \geqq \mathrm{CI}^*\text{-}\dim_R M$.
The equality holds if $\mathrm{CI}^*\text{-}\dim_\phi M < \infty$.
(2) $\mathrm{CI}^*\text{-}\dim_\phi M \leqq \mathrm{pd}_R M$ *if ϕ is faithfully flat.*
The equality holds if in addition $\mathrm{pd}_R M < \infty$.

Proof. (1) Since the inequality holds if $\mathrm{CI}^*\text{-}\dim_\phi M = \infty$, assume that $\mathrm{CI}^*\text{-}\dim_\phi M < \infty$. Let $S \xrightarrow{\beta} S' \xrightarrow{\phi'} R' \xleftarrow{\alpha} R$ be a P-factorization of ϕ such that $\mathrm{pd}_{S'}(M \otimes_R R') - \mathrm{pd}_{S'} R' < \infty$. Then by definition $S' \xrightarrow{\phi'} R' \xleftarrow{\alpha} R$ is a quasi-deformation of R, which shows that $\mathrm{CI}^*\text{-}\dim_R M < \infty$. Hence the assertion follows from Theorem 2.10 and the Auslander-Buchsbaum-type equality for CI*-dimension.

(2) Suppose that ϕ is faithfully flat. Since the inequality holds if $\mathrm{pd}_R M = \infty$, assume that $\mathrm{pd}_R M < \infty$. We easily see that the diagram $S \xrightarrow{\phi} R \xrightarrow{id} R \xleftarrow{\mathrm{id}} R$ is a P-factorization of ϕ. Therefore we have $\mathrm{CI}^*\text{-}\dim_\phi M < \infty$. Hence the assertion follows from Theorem 2.10 and the Auslander-Buchsbaum formula for projective dimension. □

The inequality in the second assertion of the above proposition may not hold without the faithful flatness of ϕ; see Remark 2.17 below.

Now, recall that
$$\mathrm{CI}^*\text{-}\dim_R M \leqq \mathrm{pd}_R M$$
for any R-module M. Hence the above proposition says that relative CI*-dimension is inserted between absolute CI*-dimension and projective dimension if ϕ is faithfully flat.

It is natural to ask when relative CI*-dimension $\mathrm{CI}^*\text{-}\dim_\phi$ coincides with absolute one $\mathrm{CI}^*\text{-}\dim_R$ as an invariant for R-modules. It seems to happen if S is the prime field of R.

Let us consider the case that the characteristic $\mathrm{char}\, k$ of k is zero. Then we easily see that $\mathrm{char}\, R = 0$. It follows that R has the prime field $\mathbb{Q}$. Let $S' \xrightarrow{\phi'} R' \xleftarrow{\alpha} R$ be a quasi-deformation of R. Since α is injective and ϕ' is surjective, the residue class field of R' is of characteristic zero, and so is that of S'. Hence we see that $\mathrm{char}\, S' = 0$, and there exists a commutative diagram
$$\begin{array}{ccc} S' & \xrightarrow{\phi'} & R' \\ {\scriptstyle\beta}\uparrow & & \uparrow{\scriptstyle\alpha} \\ \mathbb{Q} & \xrightarrow[\phi]{} & R, \end{array}$$
where ϕ and β denote the natural embeddings. Note that β is faithfully flat because $\mathbb{Q}$ is a field. Therefore this diagram is a P-factorization of ϕ. Thus, Proposition 2.11(1) yields the following:

Proposition 2.12. *Suppose that k is of characteristic zero. If S is the prime field of R, then*
$$\mathrm{CI}^*\text{-}\dim_\phi M = \mathrm{CI}^*\text{-}\dim_R M$$
for any R-module M.

Conjecture 2.13. If S is the prime field of R, then it would always hold that $\mathrm{CI}^*\text{-}\dim_\phi M = \mathrm{CI}^*\text{-}\dim_R M$ for any R-module M.

As we have observed in Proposition 2.11, the relative CI*-dimension $\mathrm{CI}^*\text{-}\dim_\phi M$ of an R-module M is always smaller or equal to its projective dimension $\mathrm{pd}_R M$, as long as ϕ is faithfully flat. The next theorem gives a sufficient condition for these dimensions to coincide with each other as invariants for R-modules.

Theorem 2.14. *Suppose that $S = R$ and ϕ is the identity map on R. Then*

$$\mathrm{CI}^*\text{-}\dim_\phi M = \mathrm{pd}_R M$$

for every R-module M.

Proof. The assumption in the theorem in particular implies that ϕ is faithfully flat. Hence Proposition 2.11(2) yields one inequality relation in the theorem. Thus we have only to prove the other inequality relation $\mathrm{CI}^*\text{-}\dim_\phi M \geqq \mathrm{pd}_R M$. There is nothing to show if $\mathrm{CI}^*\text{-}\dim_\phi M = \infty$. Hence assume that $\mathrm{CI}^*\text{-}\dim_\phi M < \infty$. Then the identity map ϕ on R has a P-factorization $R \xrightarrow{\beta} S' \xrightarrow{\phi'} R' \xleftarrow{\alpha} R$ such that $\mathrm{pd}_{S'}(M \otimes_R R') < \infty$. Let l' denote the residue class field of S'. Taking an S'-sequence $\boldsymbol{x} = x_1, x_2, \cdots, x_r$ generating the kernel of ϕ', we have $\mathbf{R}\mathrm{Hom}_{S'}(R', l') \cong \mathrm{Hom}_{S'}(\mathrm{K}_\bullet(\boldsymbol{x}), l') \cong \bigoplus_{i=0}^r l'^{\binom{r}{i}}[-i]$, where $\mathrm{K}_\bullet(\boldsymbol{x})$ is the Koszul complex of $\boldsymbol{x}$ over S'. Noting that both α and β are faithfully flat, we see that

$$\begin{aligned}
\mathbf{R}\mathrm{Hom}_{S'}(M \otimes_R R', l') &\cong \mathbf{R}\mathrm{Hom}_{S'}((M \otimes_R^{\mathbf{L}} S') \otimes_{S'}^{\mathbf{L}} R', l') \\
&\cong \mathbf{R}\mathrm{Hom}_{S'}(M \otimes_R^{\mathbf{L}} S', \mathbf{R}\mathrm{Hom}_{S'}(R', l')) \\
&\cong \mathbf{R}\mathrm{Hom}_{S'}(M \otimes_R S', \textstyle\bigoplus_{i=0}^r l'^{\binom{r}{i}}[-i]) \\
&\cong \textstyle\bigoplus_{i=0}^r \mathbf{R}\mathrm{Hom}_{S'}(M \otimes_R S', l')^{\binom{r}{i}}[-i].
\end{aligned}$$

It follows from this that

$$\begin{aligned}
\mathrm{Ext}_{S'}^j(M \otimes_R R', l') &\cong \mathrm{H}^j(\mathbf{R}\mathrm{Hom}_{S'}(M \otimes_R R', l')) \\
&\cong \mathrm{H}^j(\textstyle\bigoplus_{i=0}^r \mathbf{R}\mathrm{Hom}_{S'}(M \otimes_R S', l')^{\binom{r}{i}}[-i]) \\
&\cong \textstyle\bigoplus_{i=0}^r \mathrm{Ext}_{S'}^{j-i}(M \otimes_R S', l')^{\binom{r}{i}}.
\end{aligned}$$

Note that $\mathrm{Ext}_{S'}^j(M \otimes_R R', l') = 0$ for any $j \gg 0$ because $\mathrm{pd}_{S'}(M \otimes_R R') < \infty$. Hence we obtain $\mathrm{Ext}_{S'}^j(M \otimes_R S', l') = 0$ for any $j \gg 0$, which implies that $\mathrm{pd}_{S'}(M \otimes_R S') < \infty$. Thus we get $\mathrm{pd}_R M < \infty$. Then the Auslander-Buchsbaum-type equalities for projective dimension and CI*-dimension yield that $\mathrm{CI}^*\text{-}\dim_\phi M = \mathrm{pd}_R M = \operatorname{depth} R - \operatorname{depth}_R M$. □

We know that $\mathrm{CI}^*\text{-}\dim_R M < \infty$ for any R-module M if R is a complete intersection and that R is a complete intersection if $\mathrm{CI}^*\text{-}\dim_R k < \infty$. We can prove the following result similar to this:

Theorem 2.15. *The following conditions are equivalent.*

i) *R is a complete intersection and S is a regular ring.*
ii) $\mathrm{CI}^*\text{-}\dim_\phi M < \infty$ *for any R-module M.*
iii) $\mathrm{CI}^*\text{-}\dim_\phi k < \infty$.

Proof. i) $\Rightarrow$ ii): It follows from Lemma 2.5 that there is a Cohen factorization $S \xrightarrow{\beta} S' \xrightarrow{\phi'} \widehat{R} \xleftarrow{\alpha} R$ of ϕ. Since both the ring S and the closed fiber of β are regular, so is S' by the faithful flatness of β. On the other hand, since R is a complete intersection, so is its $\mathfrak{m}$-adic completion $\widehat{R}$. Hence the homomorphism ϕ' is a deformation. (A surjective homomorphism from a regular local ring to a local complete intersection must be a deformation; see [5, Theorem 2.3.3].) Thus, we see that the factorization $S \xrightarrow{\beta} S' \xrightarrow{\phi'} \widehat{R} \xleftarrow{\alpha} R$ is a P-factorization of ϕ. The regularity of the ring S' implies that every S'-module is of finite projective dimension over S', from which the condition ii) follows.

ii) $\Rightarrow$ iii): This is trivial.

iii) $\Rightarrow$ i): The condition iii) says that ϕ has a P-factorization $S \xrightarrow{\beta} S' \xrightarrow{\phi'} R' \xleftarrow{\alpha} R$ such that $\mathrm{pd}_{S'}(k \otimes_R R') < \infty$. Put $A = k \otimes_R R'$. Note that A is a regular local ring because it is the closed fiber of α. Let $\boldsymbol{a} = a_1, a_2, \cdots, a_t$ be a regular system of parameters of A. Since $\boldsymbol{a}$ is an A-regular sequence, we have $\mathrm{pd}_{S'} A/(\boldsymbol{a}) = \mathrm{pd}_{S'} A + t < \infty$. Since ϕ' is surjective, we see that the quotient ring $A/(\boldsymbol{a})$ is isomorphic to the residue class field l' of S'. Hence we obtain $\mathrm{pd}_{S'} l' < \infty$, which implies that S' is regular, and so is S. On the other hand, it follows from Theorem 2.11(1) that R is a complete intersection. □

Suppose that R is regular. Then, by Proposition 2.2, S is also regular if ϕ has at least one P-factorization. Thus the above theorem implies the following corollary:

Corollary 2.16. *Suppose that R is regular. If $\mathrm{CI}^*\text{-}\dim_\phi N < \infty$ for some R-module N, then $\mathrm{CI}^*\text{-}\dim_\phi M < \infty$ for every R-module M.*

Remark 2.17. Relating to the second assertion of Proposition 2.11, there is no inequality relation between relative CI*-dimension and projective dimension in a general setting. In fact, the following results immediately follow from Theorem 2.15:

(1) $\mathrm{CI}^*\text{-}\dim_\phi k < \mathrm{pd}_R k$ if R is a complete intersection which is not regular and S is a regular ring.

(2) $\mathrm{CI}^*\text{-}\dim_\phi k > \mathrm{pd}_R k$ if R is regular and S is not regular.

We can calculate the relative CI*-dimension of each of the syzygy modules of an R-module M by using the relative CI*-dimension of M:

Proposition 2.18. *For an R-module M and an integer $n \geqq 0$,*

$$\mathrm{CI}^*\text{-}\dim_\phi \Omega_R^n M = \sup\{\mathrm{CI}^*\text{-}\dim_\phi M - n, 0\}.$$

Proof. We claim that $\mathrm{CI}^*\text{-}\dim_\phi M < \infty$ if and only if $\mathrm{CI}^*\text{-}\dim_\phi \Omega_R^1 M < \infty$. Indeed, let $S \to S' \to R' \leftarrow R$ be a P-factorization of ϕ. There is a short exact sequence

$$0 \to \Omega_R^1 M \to R^m \to M \to 0$$

with some integer m. Since R' is flat over R, we obtain

$$0 \to \Omega_R^1 M \otimes_R R' \to {R'}^m \to M \otimes_R R' \to 0.$$

Note that $\mathrm{pd}_{S'} R' < \infty$. Hence we see that $\mathrm{pd}_{S'}(M \otimes_R R') < \infty$ if and only if $\mathrm{pd}_{S'}(\Omega_R^1 M \otimes_R R') < \infty$. This implies the claim.

It follows from the claim that $\mathrm{CI}^*\text{-}\dim_\phi M < \infty$ if and only if $\mathrm{CI}^*\text{-}\dim_\phi \Omega_R^n M < \infty$. Thus, in order to prove the proposition, we may assume that $\mathrm{CI}^*\text{-}\dim_\phi M < \infty$ and $\mathrm{CI}^*\text{-}\dim_\phi \Omega_R^n M < \infty$. In particular, we have $\mathrm{CI}^*\text{-}\dim_R M < \infty$ by Proposition 2.11(1), hence we also have $\mathrm{CI}\text{-}\dim_R M < \infty$. Therefore [4, (1.9)] gives us the equality

$$\mathrm{depth}_R \Omega_R^n M = \min\{\mathrm{depth}_R M + n, \mathrm{depth}\, R\}.$$

Consequently we obtain

$$\begin{array}{rcl}\mathrm{CI}^*\text{-}\dim_\phi \Omega_R^n M & = & \operatorname{depth} R - \operatorname{depth}_R \Omega_R^n M \\ & = & \max\{\operatorname{depth} R - \operatorname{depth}_R M - n, 0\} \\ & = & \max\{\mathrm{CI}^*\text{-}\dim_\phi M - n, 0\},\end{array}$$

as desired. □

As the last result of this note, we state the relationship between relative CI*-dimension and regular sequences.

Proposition 2.19. *Let $\boldsymbol{x} = x_1, x_2, \cdots, x_m$ (resp. $\boldsymbol{y} = y_1, y_2, \cdots, y_n$) be a sequence in R (resp. S). Denote by $\overline{\phi}$ (resp. $\widetilde{\phi}$) the local homomorphism $S/(\boldsymbol{y}) \to R/\boldsymbol{y}R$ (resp. $S \to R/(\boldsymbol{x})$) induced by ϕ. Then*

(1) $\mathrm{CI}^*\text{-}\dim_\phi M/\boldsymbol{x}M = \mathrm{CI}^*\text{-}\dim_\phi M + m$ *if $\boldsymbol{x}$ is M-regular.*

(2) $\mathrm{CI}^*\text{-}\dim_{\overline{\phi}} M/\boldsymbol{y}M \leqq \mathrm{CI}^*\text{-}\dim_\phi M$ *if $\boldsymbol{y}$ is S-regular, R-regular, and M-regular. The equality holds if* $\mathrm{CI}^*\text{-}\dim_\phi M < \infty$.

(3) $\mathrm{CI}^*\text{-}\dim_{\widetilde{\phi}} M \leqq \mathrm{CI}^*\text{-}\dim_\phi M - m$ *if $\boldsymbol{x}$ is R-regular and R-regular and $\boldsymbol{x}M = 0$. The equality holds if* $\mathrm{CI}^*\text{-}\dim_\phi M < \infty$.

Proof. (1) By Theorem 2.10 we have only to show that $\mathrm{CI}^*\text{-}\dim_\phi M/\boldsymbol{x}M < \infty$ if and only if $\mathrm{CI}^*\text{-}\dim_\phi M < \infty$. Let $S \to S' \to R' \leftarrow R$ be a P-factorization of ϕ. Since R' is R-flat, the sequence $\boldsymbol{x}$ is also $(M \otimes_R R')$-regular. Hence we obtain $\mathrm{pd}_{S'}(M \otimes_R R')/\boldsymbol{x}(M \otimes_R R') = \mathrm{pd}_{S'}(M \otimes_R R') + m$. Note that $(M \otimes_R R')/\boldsymbol{x}(M \otimes_R R') \cong (M/\boldsymbol{x}M) \otimes_R R'$. Therefore we see that $\mathrm{pd}_{S'}(M/\boldsymbol{x}M) \otimes_R R' < \infty$ if and only if $\mathrm{pd}_{S'}(M \otimes_R R') < \infty$. Thus the desired result is proved.

(2) We may assume that $\mathrm{CI}^*\text{-}\dim_\phi M < \infty$ because the assertion immediately follows if $\mathrm{CI}^*\text{-}\dim_\phi M = \infty$. It suffices to prove that the left side of the inequality is also finite, because the equality is implied by Theorem 2.10. There exists a P-factorization $S \to S' \to R' \leftarrow R$ of ϕ such that $\mathrm{pd}_{S'}(M \otimes_R R') < \infty$. Since $\boldsymbol{y}$ is both S-regular and R-regular, it is easy to see that the induced diagram $S/(\boldsymbol{y}) \to S'/\boldsymbol{y}S' \to R'/\boldsymbol{y}R' \leftarrow R/\boldsymbol{y}R$ is a P-factorization of $\overline{\phi}$. As $\boldsymbol{y}$ is M-regular, it is also $(M \otimes_R R')$-regular, and we have $\mathrm{pd}_{S'/\boldsymbol{y}S'}(M/\boldsymbol{y}M) \otimes_R R' = \mathrm{pd}_{S'/\boldsymbol{y}S'}(M \otimes_R R')/\boldsymbol{y}(M \otimes_R R') = \mathrm{pd}_{S'}(M \otimes_R R') < \infty$. Hence we have $\mathrm{CI}^*\text{-}\dim_{\overline{\phi}} M/\boldsymbol{y}M < \infty$.

(3) Suppose that $\mathrm{CI}^*\text{-}\dim_\phi M < \infty$. It is enough to prove that $\mathrm{CI}^*\text{-}\dim_{\widetilde{\phi}} M < \infty$ by Theorem 2.10. Let $S \to S' \to R' \leftarrow R$ of ϕ be a P-factorization of ϕ with $\mathrm{pd}_{S'}(M \otimes_R R') < \infty$. Then we easily see that the induced diagram $S \to S' \to R'/\boldsymbol{x}R' \leftarrow R/(\boldsymbol{x})$ is a P-factorization of $\widetilde{\phi}$. Since $M \otimes_{R/(\boldsymbol{x})} R'/\boldsymbol{x}R' \cong M \otimes_R R'$ has finite projective dimension over S', we have $\mathrm{CI}^*\text{-}\dim_{\widetilde{\phi}} M < \infty$, as desired. □

Acknowledgments. The author wishes to express his hearty thanks to his supervisor Yuji Yoshino for a lot of valuable discussions and suggestions.

References

[1] T. Araya, R. Takahashi, and Y. Yoshino, Relative Cohen-Macaulay dimension, Preprint (2002).

[2] L. L. Avramov, Modules of finite virtual projective dimension, *Invent. Math.* **96** (1989), no. 1, 71–101.

[3] L. L. Avramov, H. -B. Foxby, and B. Herzog, Structure of local homomorphisms, *J. Algebra* **164** (1994), no. 1, 124–145.

[4] L. L. Avramov, V. N. Gasharov, and I. V. Peeva, Complete intersection dimension, *Inst. Hautes Etudes Sci. Publ. Math.* No. 86, (1997), 67–114 (1998).

[5] W. Bruns and J. Herzog, *Cohen-Macaulay rings, revised version*, Cambridge University Press, 1998.

[6] A. A. Gerko, On homological dimensions, *Mat. Sb.* **192** (2001), no. 8, 79–94; translation in *Sb. Math.* **192** (2001), no. 7-8, 1165–1179.

[7] T. H. Gulliksen, On the deviations of a local ring, *Math. Scand.* **47** (1980), no. 1, 5–20.

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