UPPER COMPLETE INTERSECTION DIMENSION RELATIVE TO A LOCAL HOMOMORPHISM

RYO TAKAHASHI

Abstract. In this note, we introduce a homological invariant for finitely generated modules over commutative noetherian local rings by slightly modifying the definition of complete intersection dimension defined by Avramov, Gasharov, and Peeva [4], and observe it from a relative point of view.

1. Introduction

Throughout this note, we assume that all rings are commutative noetherian rings, and all modules are finitely generated.

Projective dimension and Gorenstein dimension (abbr. G-dimension) have played important roles in the classification of modules and rings. Recently, complete intersection dimension (abbr. CI-dimension) and Cohen-Macaulay dimension (abbr. CM-dimension) were introduced by Avramov, Gasharov, and Peeva [4] and Gerko [6], respectively. The former is defined by using projective dimension and the idea of quasi-deformation, and the latter is defined by using G-dimension and the idea of G-quasideformation.

These dimensions are homological invariants for modules, and share many properties with each other. For example, they satisfy the Auslander-Buchsbaum-type equalities. Every module over a regular (resp. complete intersection, Gorenstein, Cohen-Macaulay) local ring is of finite projective (resp. CI-, G-, CM-) dimension, and a local ring is a regular (resp. complete intersection, Gorenstein, Cohen-Macaulay) ring if the projective (resp. CI-, G-, CM-) dimension of its residue class field is finite. Moreover, among these dimensions, there are inequalities which yield the well-known implications for a local ring $R$: $R$ is regular $\Rightarrow$ $R$ is complete intersection $\Rightarrow$ $R$ is Gorenstein $\Rightarrow$ $R$ is Cohen-Macaulay.

In this note, we are interested in CI-dimension. Gulliksen [7] showed that every module over a complete intersection has finite complexity, that is, the Betti numbers are eventually bounded by a polynomial. As a result extending this, Avramov, Gasharov, and Peeva [4] proved that any module of finite CI-dimension has finite complexity. Hence, free resolutions of modules of finite CI-dimension are eventually well-behaved. However, there are a lot of unsolved problems on CI-dimension. For instance, it is unknown whether a module of finite complexity is always of finite CI-dimension. Though we do not discuss these problems in this note, it is important to consider CI-dimension.

Here we recall the definition of the CI-dimension of a module over a local ring $R$. It is similar to that of virtual projective dimension introduced by Avramov [2]:

\textit{Key words and phrases:} complete intersection, CI-dimension.
\textit{2000 Mathematics Subject Classification:} 13D05, 13H10, 14M10.
(1) A local homomorphism \( \phi : S \rightarrow R \) of local rings is called a deformation if \( \phi \) is surjective and the kernel of \( \phi \) is generated by an \( S \)-regular sequence.

(2) A diagram \( S \xrightarrow{\phi} R' \xrightarrow{\alpha} R \) of local homomorphisms of local rings is called a quasi-deformation of \( R \) if \( \alpha \) is faithfully flat and \( \phi \) is a deformation.

(3) For an \( R \)-module \( M \), the complete intersection dimension of \( M \) is defined as follows:

\[
\text{CI-dim}_RM = \inf \left\{ \text{pd}_S(M \otimes_R R') \left| S \rightarrow R' \looparrowleft R \text{ is a quasi-deformation of } R \right. \right\}
\]

Now, slightly modifying the definition of CI-dimension, we define a homological invariant for a module over a local ring as follows.

**Definition 1.1.** (1) We call a diagram \( S \xrightarrow{\phi} R' \xrightarrow{\alpha} R \) of local homomorphisms of local rings an upper quasi-deformation of \( R \) if \( \alpha \) is faithfully flat, the closed fiber of \( \alpha \) is regular, and \( \phi \) is a deformation.

(2) For an \( R \)-module \( M \), we define the upper complete intersection dimension (abbr. CI*-dimension) of \( M \) as follows:

\[
\text{CI}^*\text{-dim}_RM = \inf \left\{ \text{pd}_S(M \otimes_R R') \left| S \rightarrow R' \looparrowleft R \text{ is an upper quasi-deformation of } R \right. \right\}
\]

Here we itemize several properties of CI*-dimension, which are analogous to those of CI-dimension. We omit their proofs because we can prove them in the same way as the proofs of the corresponding results of CI-dimension given in [4]. Let \( R \) be a local ring with residue field \( k \), \( M \neq 0 \) an \( R \)-module, and \( x = x_1, x_2, \ldots, x_n \) a sequence in \( R \). We denote by \( \Omega^n_RM \) the \( r \)th syzygy module of \( M \).

(1) The following conditions are equivalent.
   i) \( R \) is a complete intersection.
   ii) CI*-dim\(_R\)X < \infty for any \( R \)-module X.
   iii) CI*-dim\(_R\)k < \infty.

(2) If CI*-dim\(_R\)M < \infty, then CI*-dim\(_R\)M = depth \( R \) - depth\(_R\)M.

(3) CI*-dim\(_R\)\( \Omega^n_RM \) = sup\{CI*-dim\(_R\)M - r, 0\}.

(4) CI*-dim\(_R\)M/\( x \)M = CI*-dim\(_R\)M + n if \( x \) is \( M \)-regular.

(5) CI*-dim\(_R\)(/\( x \))M/\( x \)M \leq CI*-dim\(_R\)M if \( x \) is \( R \)-regular and \( M \)-regular.

The equality holds if CI*-dim\(_R\)M < \infty.

(6) CI*-dim\(_R\)(/\( x \))M \leq CI*-dim\(_R\)M - n if \( x \) is \( R \)-regular and \( x \)M = 0.

The equality holds if CI*-dim\(_R\)M < \infty.

(7) CI-dim\(_R\)M \leq CI*-dim\(_R\)M \leq \text{pd}_R\)M.

If any one of these dimensions is finite, then it is equal to those to its left.

Araya, Takahashi, and Yoshino [1], modifying the definition of CM-dimension, define a homological invariant for modules as a relative version of the modified CM-dimension. This invariant has a lot of properties similar to projective dimension, CI-dimension, G-dimension, and CM-dimension.

Let \( \phi : S \rightarrow R \) be a local homomorphism of local rings. The main purpose of this note is to define a new homological invariant for an \( R \)-module \( M \) as a relative version of CI*-dimension over \( R \), and to study its properties. We will call this the upper complete intersection dimension of \( M \) relative to \( \phi \), and denote it by CI*-dim\(_{\phi}\)M. We shall observe that this invariant has many properties similar to those of the invariant defined by Araya, Takahashi, and Yoshino. For example, we will prove the following. Let \( k \) denote the residue class field of \( R \).
Theorem 2.10. Let $M$ be a non-zero $R$-module. If $\text{CI}^*\dim_\phi M < \infty$, then $\text{CI}^*\dim_\phi M = \text{depth} R - \text{depth}_R M$.

Theorem 2.14. Suppose that $S = R$ and $\phi$ is the identity map on $R$. Then $\text{CI}^*\dim_\phi M = \text{pd}_R M$ for every $R$-module $M$.

Theorem 2.15. The following conditions are equivalent.

i) $R$ is a complete intersection and $S$ is a regular ring.

ii) $\text{CI}^*\dim_\phi M < 1$ for any $R$-module $M$.

iii) $\text{CI}^*\dim_\phi k < 1$.

2. Relative $\text{CI}^*$-dimension

Throughout the section, $\phi : (S, n, l) \to (R, m, k)$ always denotes a local homomorphism of local rings.

In this section, we shall make the precise definition of the upper complete intersection dimension of an $R$-module relative to $\phi$ to observe $\text{CI}^*$-dimension from a relative point of view. To do this, we need the notion of $P$-factorization, instead of that of upper quasi-deformation used in the definition of (absolute) $\text{CI}^*$-dimension.

Definition 2.1. Let

$$
\begin{array}{ccc}
S' & \xrightarrow{\phi'} & R' \\
\beta & \uparrow & \\
S & \xrightarrow{\phi} & R,
\end{array}
$$

be a commutative diagram of local homomorphisms of local rings. We call this diagram a $P$-factorization of $\phi$ if $\alpha$ and $\beta$ are faithfully flat, the closed fiber of $\alpha$ is regular, and $\phi'$ is a deformation.

Note that this is an imitation of a G-factorization defined in [1]. The existence of a $P$-factorization of $\phi$ transmits several properties of $R$ to $S$:

**Proposition 2.2.** Suppose that there exists a $P$-factorization of $\phi$. Then, if $R$ is a regular (resp. complete intersection, Gorenstein, Cohen-Macaulay) ring, so is $S$.

**Proof.** Let $S \xrightarrow{\beta} S' \xrightarrow{\phi'} R' \xrightarrow{\alpha} R$ be a $P$-factorization of $\phi$. Suppose that $R$ is a regular (resp. complete intersection, Gorenstein, Cohen-Macaulay) ring. Since $\alpha$ is a faithfully flat homomorphism with regular closed fiber, $R'$ is also a regular (resp. ...) ring. Since $\phi'$ is a deformation, we easily see that $S'$ is also a regular (resp. ...) ring, and so is $S$ by the flatness of $\beta$. \qed

From now on, we consider the existence of a $P$-factorization of $\phi$. First of all, the above proposition yields the following example which says that $\phi$ may not have a $P$-factorization.

**Example 2.3.** Suppose that $R = l$ is the residue class field of $S$ and $\phi$ is the natural surjection from $S$ to $l$. Then $\phi$ has no $P$-factorization unless $S$ is regular by Proposition 2.2.

Although there does not necessarily exist a $P$-factorization of $\phi$ in general, a $P$-factorization of $\phi$ seems to exist whenever the ring $S$ is regular. We are able to show it if in addition we assume the condition that $S$ contains a field:
Theorem 2.4. Suppose that $S$ is a regular local ring containing a field. Then every local homomorphism $\phi : S \rightarrow R$ of local rings has a $P$-factorization.

This theorem is essentially proved in [1]. But we shall give here a whole proof of it for this note to be as self-contained as possible. We need the following two lemmas:

**Lemma 2.5.** [3, Theorem 1.1] Let $\phi : (S,n) \rightarrow (R,m)$ be a local homomorphism of local rings, and $\alpha$ be the natural embedding from $R$ into its $m$-adic completion $\widehat{R}$. Then there exists a commutative diagram

$$
\begin{array}{ccc}
S' & \xrightarrow{\phi'} & \widehat{R} \\
\beta \downarrow & & \uparrow \alpha \\
S & \xrightarrow{\phi} & R
\end{array}
$$

of local homomorphisms of local rings such that $\beta$ is faithfully flat, the closed fiber of $\beta$ is regular, and $\phi'$ is surjective. (Such a diagram is called a Cohen factorization of $\phi$.)

**Lemma 2.6.** Let $\phi : S \rightarrow R$ be a local homomorphism of complete local rings that admit the common coefficient field $k$. Put $S' = S \widehat{\otimes}_k R$. Let $\lambda : S \rightarrow S'$ be the injective homomorphism mapping $b \in S$ to $b \widehat{\otimes} 1 \in S'$, and $\varepsilon : S' \rightarrow R$ be the surjective homomorphism mapping $b \widehat{\otimes} a \in S'$ to $\phi(b)a \in R$. Suppose that $S$ is regular. Then $S \xrightarrow{\lambda} S' \xrightarrow{\varepsilon} R \xrightarrow{\phi} R$ is a $P$-factorization of $\phi$.

**Proof.** Let $y_1, y_2, \cdots, y_s$ be a minimal system of generators of the unique maximal ideal of $S$. Put $J = \text{Ker} \varepsilon$ and $dy_i = y_i \widehat{\otimes} 1 - 1 \widehat{\otimes} \phi(y_i) \in S'$ for each $1 \leq i \leq s$.

**Claim 1.** The ideal $J$ of $S'$ is generated by $dy_1, dy_2, \cdots, dy_s$.

Indeed, put $J_0 = (dy_1, dy_2, \cdots, dy_s)S'$. Let $b = \sum b_{i_1 i_2 \cdots i_s} y_1^{i_1} y_2^{i_2} \cdots y_s^{i_s}$ be a power series expansion in $y_1, y_2, \cdots, y_s$ with coefficients $b_{i_1 i_2 \cdots i_s} \in k$. Then we have $b \widehat{\otimes} 1 = \sum b_{i_1 i_2 \cdots i_s} (y_1 \widehat{\otimes} 1)^{i_1} (y_2 \widehat{\otimes} 1)^{i_2} \cdots (y_s \widehat{\otimes} 1)^{i_s} \equiv \sum b_{i_1 i_2 \cdots i_s} (1 \widehat{\otimes} \phi(y_1))^{i_1} (1 \widehat{\otimes} \phi(y_2))^{i_2} \cdots (1 \widehat{\otimes} \phi(y_s))^{i_s} = 1 \widehat{\otimes} \phi(b)$ modulo $J_0$. It follows that $z \equiv 1 \widehat{\otimes} \phi(b)a$ modulo $J_0$. Since $\phi(b)a = \varepsilon(b \widehat{\otimes} a) = 0$, we have $z \equiv 0$ modulo $J_0$, that is, the element $z \in J$ belongs to $J_0$. Thus, we see that $J = J_0$.

**Claim 2.** The sequence $dy_1, dy_2, \cdots, dy_s$ is an $S'$-regular sequence.

Indeed, since $S$ is regular, we may assume that $S = k[[Y_1, Y_2, \cdots, Y_s]]$ and $S' = R[[Y_1, Y_2, \cdots, Y_s]]$ are formal power series rings, and $dy_i = Y_i - \phi(Y_i) \in S'$ for each $1 \leq i \leq s$. Note that the endomorphism on $S'$ which sends $Y_i$ to $dy_i$ is an automorphism. Since the sequence $Y_1, Y_2, \cdots, Y_s$ is $S'$-regular, we see that $dy_1, dy_2, \cdots, dy_s$ also form an $S'$-regular sequence.

These claims prove that the homomorphism $\varepsilon$ is a deformation. On the other hand, it is easy to see that $\lambda$ is faithfully flat. Thus, the lemma is proved.

**Proof of Theorem 2.4.** We may assume that $R$ (resp. $S$) is complete in its $m$-adic (resp. $n$-adic) topology. Hence Lemma 2.5 implies that $\phi$ has a Cohen factorization.
where $\beta$ is a faithfully flat homomorphism with regular closed fiber, and $\phi'$ is a surjective homomorphism. Hence $S'$ is also a regular local ring containing a field. Therefore, replacing $S$ with $S'$, we may assume that $\phi$ is a surjection. In particular $R$ and $S$ have the common coefficient field, hence Lemma 2.6 implies that $\phi$ has a P-factorization, as desired. □

**Conjecture 2.7.** Whenever $S$ is regular, the local homomorphism $\phi : S \rightarrow R$ would have a P-factorization.

Now, by using the idea of P-factorization, we define the CI$^\bullet$-dimension of a module in a relative sense.

**Definition 2.8.** For an $R$-module $M$, we put

$$\text{CI}^\bullet_{\phi} M = \inf \left\{ \text{pd}_{S'}(M \otimes_R R') - \text{pd}_S R' \right\}$$

is a P-factorization of $\phi$

and call it the *upper complete intersection dimension* of $M$ relative to $\phi$.

By definition, $\text{CI}^\bullet_{\phi} M = \infty$ for an $R$-module $M$ if $\phi$ has no P-factorization. Suppose that $\phi$ has at least one P-factorization $S \rightarrow S' \leftarrow R'$. Then we have $\text{pd}_S(F \otimes_R R') = \text{pd}_{S'} R'$ ($< \infty$) for any free $R$-module $F$. Therefore the above theorem on the existence of a P-factorization yields the following result:

**Proposition 2.9.** If $S$ is a regular local ring that contains a field, then

$$\text{CI}^\bullet_{\phi} F = 0 \ (< \infty)$$

for any free $R$-module $F$.

In the rest of this section, we observe the properties of relative CI$^\bullet$-dimension $\text{CI}^\bullet_{\phi}$. We begin by proving that relative CI$^\bullet$-dimension also satisfies the Auslander-Buchsbaum-type equality:

**Theorem 2.10.** Let $M$ be a non-zero $R$-module. If $\text{CI}^\bullet_{\phi} M < \infty$, then

$$\text{CI}^\bullet_{\phi} M = \text{depth } R - \text{depth}_R M.$$

**Proof.** Since $\text{CI}^\bullet_{\phi} M < \infty$, there exists a P-factorization $S \xrightarrow{\beta} S' \xrightarrow{\phi'} R' \leftarrow R$ of $\phi$ such that $\text{CI}^\bullet_{\phi} M = \text{pd}_{S'}(M \otimes_R R') - \text{pd}_S R' < \infty$. Hence we see that

$$\text{CI}^\bullet_{\phi} M = \text{pd}_{S'}(M \otimes_R R') - \text{pd}_S R' = (\text{depth } S' - \text{depth}_{S'}(M \otimes_R R')) - (\text{depth } S' - \text{depth}_{S'} R').$$

Note that $\phi'$ is surjective. Since $\alpha$ and $\beta$ are faithfully flat, we obtain

$$\begin{cases}
\text{depth}_{S'} R' = \text{depth } R + \text{depth } R'/mR', \\
\text{depth}_{S'}(M \otimes_R R') = \text{depth}_R M + \text{depth } R'/mR'.
\end{cases}$$

It follows that $\text{CI}^\bullet_{\phi} M = \text{depth } R - \text{depth}_R M$. □
In view of this theorem, we notice that the value of the relative CI*-dimension of an $R$-module is given independently of the ring $S$ if it is finite.

**Proposition 2.11.** Let $M$ be an $R$-module. Then

1. $\text{CI}^*_\phi M \geq \text{CI}^*_\phi R M$.
   The equality holds if $\text{CI}^*_\phi M < \infty$.
2. $\text{CI}^*_\phi M \leq \text{pd}_R M$ if $\phi$ is faithfully flat.
   The equality holds if in addition $\text{pd}_R M < \infty$.

**Proof.** (1) Since the inequality holds if $\text{CI}^*_\phi M = \infty$, assume that $\text{CI}^*_\phi M < \infty$. Let $S \xrightarrow{\beta} S' \xrightarrow{\phi'} R' \xleftarrow{\alpha} R$ be a P-factorization of $\phi$ such that $\text{pd}_{S'} (M \oplus_R R') - \text{pd}_S R' < \infty$. Then by definition $S' \xrightarrow{\phi'} R' \xleftarrow{\alpha} R$ is a quasi-deformation of $R$, which shows that $\text{CI}^*_\phi M < \infty$. Hence the assertion follows from Theorem 2.10 and the Auslander-Buchsbaum-type equality for CI*-dimension.

(2) Suppose that $\phi$ is faithfully flat. Since the inequality holds if $\text{pd}_R M = \infty$, assume that $\text{pd}_R M < \infty$. We easily see that the diagram $S \xrightarrow{\phi} R \xleftarrow{\alpha} R \xrightarrow{\beta} \phi R$ is a P-factorization of $\phi$. Therefore we have $\text{CI}^*_\phi M < \infty$. Hence the assertion follows from Theorem 2.10 and the Auslander-Buchsbaum formula for projective dimension. \qed

The inequality in the second assertion of the above proposition may not hold without the faithful flatness of $\phi$; see Remark 2.17 below.

Now, recall that

$$\text{CI}^*_\phi R M \leq \text{pd}_R M$$

for any $R$-module $M$. Hence the above proposition says that relative CI*-dimension is inserted between absolute CI*-dimension and projective dimension if $\phi$ is faithfully flat.

It is natural to ask when relative CI*-dimension $\text{CI}^*_\phi$ coincides with absolute one $\text{CI}^*_\phi$ as an invariant for $R$-modules. It seems to happen if $S$ is the prime field of $R$.

Let us consider the case that the characteristic $\text{char } k$ of $k$ is zero. Then we easily see that $\text{char } R = 0$. It follows that $R$ has the prime field $\mathbb{Q}$. Let $S' \xrightarrow{\phi'} R' \xleftarrow{\alpha} R$ be a quasi-deformation of $R$. Since $\alpha$ is injective and $\phi'$ is surjective, the residue class field of $R'$ is of characteristic zero, and so is that of $S'$. Hence we see that $\text{char } S' = 0$, and there exists a commutative diagram

$$
\begin{array}{ccc}
S' & \xrightarrow{\phi'} & R' \\
\beta \uparrow & & \uparrow \alpha \\
\mathbb{Q} & \xrightarrow{\phi} & R,
\end{array}
$$

where $\phi$ and $\beta$ denote the natural embeddings. Note that $\beta$ is faithfully flat because $\mathbb{Q}$ is a field. Therefore this diagram is a P-factorization of $\phi$. Thus, Proposition 2.11(1) yields the following:

**Proposition 2.12.** Suppose that $k$ is of characteristic zero. If $S$ is the prime field of $R$, then

$$\text{CI}^*_\phi M = \text{CI}^*_\phi R M$$

for any $R$-module $M$. 
Conjecture 2.13. If $S$ is the prime field of $R$, then it would always hold that $\text{CI}^*\dim \phi M = \text{CI}^*\dim M$ for any $R$-module $M$.

As we have observed in Proposition 2.11, the relative CI$^*$-dimension CI$^*\dim \phi M$ of an $R$-module $M$ is always smaller or equal to its projective dimension $\text{pd}_R M$, as long as $\phi$ is faithfully flat. The next theorem gives a sufficient condition for these dimensions to coincide with each other as invariants for $R$-modules.

**Theorem 2.14.** Suppose that $S = R$ and $\phi$ is the identity map on $R$. Then $\text{CI}^*\dim \phi M = \text{pd}_R M$ for every $R$-module $M$.

**Proof.** The assumption in the theorem in particular implies that $\phi$ is faithfully flat. Hence Proposition 2.11(2) yields one inequality relation in the theorem. Thus we have only to prove the other inequality relation $\text{CI}^*\dim \phi M \geq \text{pd}_R M$. There is nothing to show if $\text{CI}^*\dim \phi M = \infty$. Hence assume that $\text{CI}^*\dim \phi M < \infty$. Then the identity map $\phi$ on $R$ has a P-factorization $R \xrightarrow{\beta} S' \xrightarrow{\phi'} R' \xrightarrow{\alpha} R$ such that $\text{pd}_{S'}(M \otimes_R R') < \infty$. Let $l'$ denote the residue class field of $S'$. Taking an $S'$-sequence $x = x_1, x_2, \ldots, x_r$ generating the kernel of $\phi'$, we have $R\text{Hom}_{S'}(R', l') \cong \text{Hom}_{S'}(K_\bullet(x), l') \cong \bigoplus_{i=0}^r l'^{(i)}[-i]$, where $K_\bullet(x)$ is the Koszul complex of $x$ over $S'$. Noting that both $\alpha$ and $\beta$ are faithfully flat, we see that

$$R\text{Hom}_{S'}(M \otimes_R R', l') \cong R\text{Hom}_{S'}((M \otimes_R S') \otimes_{S'} R', l')$$

$$\cong R\text{Hom}_{S'}(M \otimes_R S', R\text{Hom}_{S'}(R', l'))$$

$$\cong R\text{Hom}_{S'}(M \otimes_R S', \bigoplus_{i=0}^r l'^{(i)}[-i])$$

$$\cong \bigoplus_{i=0}^r R\text{Hom}_{S'}(M \otimes_R S', l'^{(i)}[-i]).$$

It follows from this that

$$\text{Ext}_{S'}^j((M \otimes_R R', l')) \cong H^j(R\text{Hom}_{S'}(M \otimes_R R', l'))$$

$$\cong H^j(\bigoplus_{i=0}^r R\text{Hom}_{S'}(M \otimes_R S', l'^{(i)}[-i]))$$

$$\cong \bigoplus_{i=0}^r \text{Ext}_{S'}^{j-1}(M \otimes_R S', l'^{(i)}).$$

Note that $\text{Ext}_{S'}^j(M \otimes_R R', l') = 0$ for any $j \gg 0$ because $\text{pd}_{S'}(M \otimes_R R') < \infty$. Hence we obtain $\text{Ext}_{S'}^j(M \otimes_R S', l') = 0$ for any $j \gg 0$, which implies that $\text{pd}_{S'}(M \otimes_R S') < \infty$. Thus we get $\text{pd}_R M < \infty$. Then the Auslander-Buchsbaum-type equalities for projective dimension and CI$^*$-dimension yield that $\text{CI}^*\dim \phi M = \text{pd}_R M = \text{depth } R - \text{depth}_R M$. $\square$

We know that $\text{CI}^*\dim \phi M < \infty$ for any $R$-module $M$ if $R$ is a complete intersection and that $R$ is a complete intersection if $\text{CI}^*\dim \phi k < \infty$. We can prove the following result similar to this:

**Theorem 2.15.** The following conditions are equivalent.

i) $R$ is a complete intersection and $S$ is a regular ring.

ii) $\text{CI}^*\dim \phi M < \infty$ for any $R$-module $M$.

iii) $\text{CI}^*\dim \phi k < \infty$.

**Proof.** i) $\Rightarrow$ ii): It follows from Lemma 2.5 that there is a Cohen factorization $S \xrightarrow{\beta} S' \xrightarrow{\phi'} \tilde{R} \xrightarrow{\alpha} R$ of $\phi$. Since both the ring $S$ and the closed fiber of $\beta$ are regular, so is $S'$ by the faithful flatness of $\beta$. On the other hand, since $R$ is a
complete intersection, so is its $m$-adic completion $\widehat R$. Hence the homomorphism $\phi$ is a deformation. (A surjective homomorphism from a regular local ring to a local complete intersection must be a deformation; see [5, Theorem 2.3.3].) Thus, we see that the factorization $S \sim S' \xrightarrow{\phi'} \widehat R \xrightarrow{\alpha} R$ is a P-factorization of $\phi$. The regularity of the ring $S'$ implies that every $S'$-module is of finite projective dimension over $S'$, from which the condition ii) follows.

ii) $\Rightarrow$ iii): This is trivial.

iii) $\Rightarrow$ i): The condition iii) says that $\phi$ has a P-factorization $S \sim S' \xrightarrow{\phi'} \widehat R \xrightarrow{\alpha} R$ such that $\text{pd}_{S'}(k \otimes_R R') < \infty$. Put $A = k \otimes_R R'$. Note that $A$ is a regular local ring because it is the closed fiber of $\alpha$. Let $a = a_1, a_2, \ldots, a_t$ be a regular system of parameters of $A$. Since $a$ is an $A$-regular sequence, we have $\text{pd}_{S'}A/(a) = \text{pd}_{S'}A + t < \infty$. Since $\phi'$ is surjective, we see that the quotient ring $A/(a)$ is isomorphic to the residue class field $l'$ of $S'$. Hence we obtain $\text{pd}_{S'}l' < \infty$, which implies that $S'$ is regular, and so is $S$. On the other hand, it follows from Theorem 2.11(1) that $R$ is a complete intersection. $\square$

Suppose that $R$ is regular. Then, by Proposition 2.2, $S$ is also regular if $\phi$ has at least one P-factorization. Thus the above theorem implies the following corollary:

**Corollary 2.16.** Suppose that $R$ is regular. If $\text{CI}^*-\dim_{S'}N < \infty$ for some $R$-module $N$, then $\text{CI}^*-\dim_{S}M < \infty$ for every $R$-module $M$.

**Remark 2.17.** Relating to the second assertion of Proposition 2.11, there is no inequality relation between relative CI$^*$-dimension and projective dimension in a general setting. In fact, the following results immediately follow from Theorem 2.15:

1. $\text{CI}^*-\dim_{S'}k < \text{pd}_{S'}k$ if $R$ is a complete intersection which is not regular and $S$ is a regular ring.
2. $\text{CI}^*-\dim_{S'}k > \text{pd}_{S'}k$ if $R$ is regular and $S$ is not regular.

We can calculate the relative CI$^*$-dimension of each of the syzygy modules of an $R$-module $M$ by using the relative CI$^*$-dimension of $M$:

**Proposition 2.18.** For an $R$-module $M$ and an integer $n \geq 0$,

$$\text{CI}^*-\dim_{S'}\Omega^n_R M = \sup\{\text{CI}^*-\dim_{S}M - n, 0\}.$$  

**Proof.** We claim that $\text{CI}^*-\dim_{S'}M < \infty$ if and only if $\text{CI}^*-\dim_{S_0}\Omega^n_R M < \infty$. Indeed, let $S \sim S' \xrightarrow{\phi'} \widehat R \xrightarrow{\alpha} R$ be a P-factorization of $\phi$. There is a short exact sequence

$$0 \to \Omega^n_R M \to R^n \to M \to 0$$

with some integer $m$. Since $R'$ is flat over $R$, we obtain

$$0 \to \Omega^n_R M \otimes_R R' \to R'^n \to M \otimes_R R' \to 0.$$  

Note that $\text{pd}_{S'}R' < \infty$. Hence we see that $\text{pd}_{S}(M \otimes_R R') < \infty$ if and only if $\text{pd}_{S'}(\Omega^n_R M \otimes_R R') < \infty$. This implies the claim.

It follows from the claim that $\text{CI}^*-\dim_{S'}M < \infty$ if and only if $\text{CI}^*-\dim_{S_0}\Omega^n_R M < \infty$. Thus, in order to prove the proposition, we may assume that $\text{CI}^*-\dim_{S'}M < \infty$ and $\text{CI}^*-\dim_{S_0}\Omega^n_R M < \infty$. In particular, we have $\text{CI}^*-\dim_R M < \infty$ by Proposition 2.11(1), hence we also have $\text{CI}-\dim_R M < \infty$. Therefore [4, (1.9)] gives us the equality

$$\text{depth}_RM = \min\{\text{depth}_RM + n, \text{depth} R\}.$$
Consequently we obtain
\[
\text{CI}^\ast \cdot \dim_\phi \Omega_R^M = \text{depth } R - \text{depth}_R \Omega_R^M
\]
\[
= \max\{\text{depth } R - \text{depth}_R M - n, 0\}
\]
\[
= \max\{\text{CI}^\ast \cdot \dim_\phi M - n, 0\},
\]
as desired. \(\Box\)

As the last result of this note, we state the relationship between relative \text{CI}^\ast-

Proposition 2.19. Let \(x = x_1, x_2, \ldots, x_m\) (resp. \(y = y_1, y_2, \ldots, y_n\)) be a se-
quence in \(R\) (resp. \(S\)). Denote by \(\overline{\phi} \) (resp. \(\overline{\phi} \)) the local homomorphism \(S/(y) \to R/yR\) (resp. \(S \to R/(x)\)) induced by \(\phi\). Then

(1) \(\text{CI}^\ast \cdot \dim_\phi M/xM = \text{CI}^\ast \cdot \dim_\phi M + m\) if \(x\) is \(M\)-regular.

(2) \(\text{CI}^\ast \cdot \dim_\phi S/yM \leq \text{CI}^\ast \cdot \dim_\phi M\) if \(y\) is \(S\)-regular, \(R\)-regular, and \(M\)-regular.

The equality holds if \(\text{CI}^\ast \cdot \dim_\phi M < \infty\).

(3) \(\text{CI}^\ast \cdot \dim_\phi M \leq \text{CI}^\ast \cdot \dim_\phi M - m\) if \(x\) is \(R\)-regular and \(R\)-regular and \(xM = 0\).

The equality holds if \(\text{CI}^\ast \cdot \dim_\phi M < \infty\).

Proof. (1) By Theorem 2.10 we have only to show that \(\text{CI}^\ast \cdot \dim_\phi M/xM < \infty\) if and only if \(\text{CI}^\ast \cdot \dim_\phi M < \infty\). Let \(S \to S' \to R' \leftarrow R\) be a P-factorization of \(\phi\). Since \(R'\) is \(R\)-flat, the sequence \(x\) is also \((M \otimes_R R')\)-regular. Hence we obtain \(pd_S(M \otimes_R R')/x(M \otimes_R R') = pd_S(M \otimes_R R') + m\). Note that \((M \otimes_R R')/x(M \otimes_R R') \cong (M/xM) \otimes_R R'\). Therefore we see that \(pd_S(M/xM) \otimes_R R' < \infty\) if and only if \(pd_S(M \otimes_R R') < \infty\). Thus the desired result is proved.

(2) We may assume that \(\text{CI}^\ast \cdot \dim_\phi M < \infty\) because the assertion immediately follows if \(\text{CI}^\ast \cdot \dim_\phi M = \infty\). It suffices to prove that the left side of the inequality is also finite, because the equality is implied by Theorem 2.10. There exists a P-factorization \(S \to S' \to R' \leftarrow R\) of \(\phi\) such that \(pd_{S'}(M \otimes_R R') < \infty\). Since \(y\) is both \(S\)-regular and \(R\)-regular, it is easy to see that the induced diagram \(S/(y) \to S'/yS' \to R'/yR' \leftarrow R/yR\) is a P-factorization of \(\overline{\phi}\). As \(y\) is \(M\)-regular, it is also \((M \otimes_R R')\)-regular, and we have \(pd_{S'/yS'}(M/yM) \otimes_R R' = pd_{S'/yS'}(M \otimes_R R')/y(M \otimes_R R') = pd_{S'}(M \otimes_R R') < \infty\). Hence we have \(\text{CI}^\ast \cdot \dim_\phi S/yM < \infty\).

(3) Suppose that \(\text{CI}^\ast \cdot \dim_\phi M < \infty\). It is enough to prove that \(\text{CI}^\ast \cdot \dim_\phi M < \infty\) by Theorem 2.10. Let \(S \to S' \to R' \leftarrow R\) of \(\phi\) be a P-factorization of \(\phi\) with \(pd_{S'}(M \otimes_R R') < \infty\). Then we easily see that the induced diagram \(S \to S' \to R'/xR' \leftarrow R/(x)\) is a P-factorization of \(\overline{\phi}\). Since \(M \otimes_R (xR) \cong M \otimes_R R'\) has finite projective dimension over \(S'\), we have \(\text{CI}^\ast \cdot \dim_\phi M < \infty\), as desired. \(\Box\)

Acknowledgments. The author wishes to express his hearty thanks to his super-

References


Graduate School of Natural Science and Technology, Okayama University, Okayama 700-8530, Japan

E-mail address: takahasi@math.okayama-u.ac.jp