

ON THE IDEAL CASE OF A CONJECTURE OF AUSLANDER AND REITEN

OLGUR CELIKBAS, KEI-ICHIRO IIMA, ARASH SADEGHI, AND RYO TAKAHASHI

ABSTRACT. A celebrated conjecture of Auslander and Reiten claims that a finitely generated module M that has no extensions with $M \oplus \Lambda$ over an Artin algebra Λ must be projective. This conjecture is widely open in general, even for modules over commutative Noetherian local rings. Over such rings, we prove that a large class of ideals satisfy the extension condition proposed in the aforementioned conjecture of Auslander and Reiten. Along the way we obtain a new characterization of regularity in terms of the injective dimensions of certain ideals.

1. INTRODUCTION

Motivated by a conjecture of Nakayama [27], Auslander and Reiten [6] proposed the following conjecture, which is called the *generalized Nakayama conjecture*:

Conjecture 1.1. If Λ is an Artin algebra, then every indecomposable injective Λ -module occurs as a direct summand in one of the terms in the minimal injective resolution of Λ_Λ .

Auslander and Reiten [6] proved that the Generalized Nakayama Conjecture is true if and only if the following conjecture is true:

Conjecture 1.2. If Λ is an Artin algebra, then every finitely generated Λ -module M that is a generator (i.e., Λ is a direct summand of a finite direct sum of copies of M) and satisfies $\text{Ext}_A^i(M, M) = 0$ for all $i \geq 1$ must be projective.

Auslander, Ding and Solberg [4] formulated the following conjecture, which is equivalent to Conjecture 1.2 over Noetherian rings.

Conjecture 1.3. Let M be a finitely generated left module over a left Noetherian ring R . If $\text{Ext}_R^i(M, M \oplus R) = 0$ for all $i \geq 1$, then M is projective.

The case where the ring in Conjecture 1.3 is an Artin algebra is known as the *Auslander–Reiten Conjecture*. Conjecture 1.3 is known to hold for several classes of rings, for example for Artin algebras of finite representation type [6], however it is widely open in general, even for commutative Gorenstein local rings; see [14].

Recently Goto and Takahashi [18] have studied the ideal case of Conjecture 1.3 over commutative Noetherian local rings. In this paper we also study the ideal case of Conjecture 1.3, however our argument is different from [18]: we characterize regular local rings by the vanishing of only two consecutive $\text{Ext}_R^i(I, I)$ for a certain class of ideals I . More precisely, we exploit a beautiful result of Burch [10] and prove that a large class of *weakly \mathfrak{m} -full* ideals satisfy the vanishing condition proposed in Conjecture 1.3 over commutative Noetherian local rings; see 3.7 for the definition of a weakly \mathfrak{m} -full ideal. The main consequence of our argument can be stated as follows; see Theorem 2.19 and Corollary 3.14.

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Theorem 1.4. Let (R, \mathfrak{m}) be a commutative Noetherian local ring and let I be a weakly \mathfrak{m} -full ideal of R such that $I : \mathfrak{m} \neq I$ (or equivalently $\text{depth}(R/I) = 0$.) If R is not regular, then $\text{Ext}_R^n(I, I)$ and $\text{Ext}_R^{n+1}(I, I)$ do not vanish simultaneously for any $n \geq 1$.

It is clear that, if I is an ideal as in Theorem 1.4 and $\text{Ext}_R^i(I, I) = 0$ for all $i \geq 1$, then I is projective, i.e., $I \cong R$. It is also worth noting that Theorem 1.4 is sharp in the sense that the condition $I : \mathfrak{m} \neq I$ is necessary; see Examples 3.21 and 3.22.

Examples of weakly \mathfrak{m} -full ideals are abundant in the literature: for example, it is established in [11] that, over non-regular Gorenstein local rings (R, \mathfrak{m}) of dimension at least one and codimension at least two, an ideal of the form $(\mathfrak{q} : \mathfrak{m})$, where \mathfrak{q} is an arbitrary parameter ideal of R , is weakly \mathfrak{m} -full but not \mathfrak{m} -full; see also Examples 3.8 and 3.10.

We obtain several results on the vanishing of Ext as applications of Theorem 1.4; see Corollaries 3.1, 3.2 and 3.3. Furthermore, motivated by a result of Goto and Hayasaka [17], we give a characterization of regularity in terms of the injective dimensions of weakly \mathfrak{m} -full ideals: if (R, \mathfrak{m}) is a commutative Noetherian local ring and I is a proper weakly \mathfrak{m} -full ideal of R with $I : \mathfrak{m} \neq I$, then R is regular if and only if I has finite injective dimension; see Corollary 3.19.

2. MAIN RESULT

Throughout R denotes a commutative Noetherian local ring with unique maximal ideal \mathfrak{m} , and residue field $k = R/\mathfrak{m}$. Moreover $\text{mod } R$ denotes the category of finitely generated R -modules.

We record various preliminary results and prepare several lemmas to prove our main result, Theorem 2.19.

2.1. Let $M \in \text{mod } R$ be a module with a projective presentation $P_1 \xrightarrow{f} P_0 \rightarrow M \rightarrow 0$. Then the *transpose* $\text{Tr } M$ of M is the cokernel of $f^* = \text{Hom}(f, R)$ and hence is given by the exact sequence $0 \rightarrow M^* \rightarrow P_0^* \rightarrow P_1^* \rightarrow \text{Tr } M \rightarrow 0$. We will use the following results in the sequel:

(2.1.1) If $\text{Ext}_R^j(M, R) = 0$ for some $j \geq 1$, then $\text{Tr } \Omega^{j-1} M$ is stably isomorphic to $\Omega \text{Tr } \Omega^j M$.

Given an integer $n \geq 0$, there is an exact sequence of functors [3, 2.8]:

$$(2.1.2) \quad \text{Tor}_2^R(\text{Tr } \Omega^n M, -) \rightarrow (\text{Ext}_R^n(M, R) \otimes_R -) \rightarrow \text{Ext}_R^n(M, -) \rightarrow \text{Tor}_1^R(\text{Tr } \Omega^n M, -) \rightarrow 0.$$

2.2. Let $M, N \in \text{mod } R$. Then $\underline{\text{Hom}}(M, N)$ is defined to be the quotient of $\text{Hom}(M, N)$ by the R -homomorphisms $M \rightarrow N$ factoring through some free R -module. It follows that there is an isomorphism $\text{Tor}_1^R(\text{Tr } M, N) \cong \underline{\text{Hom}}(M, N)$; see [30, 3.7 and 3.9].

2.3. (Auslander and Reiten [6, 3.4]) If $M, N \in \text{mod } R$ are modules and $n \geq 1$ is an integer, then there is an isomorphism $\underline{\text{Hom}}(M, \Omega^n N) \cong \underline{\text{Hom}}(\text{Tr } \Omega \text{Tr } M, \Omega^{n-1} N)$.

2.4. (Matsui and Takahashi [25, 2.7(2)]) If $X \in \text{mod } R$ and $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is a short exact sequence in $\text{mod } R$, then there is an exact sequence in $\text{mod } R$ of the form:

$$\underline{\text{Hom}}(X, A) \rightarrow \underline{\text{Hom}}(X, B) \rightarrow \underline{\text{Hom}}(X, C) \rightarrow \text{Ext}_R^1(X, A) \rightarrow \text{Ext}_R^1(X, B) \rightarrow \text{Ext}_R^1(X, C).$$

Lemma 2.5. Let R be a local ring and let $M, N \in \text{mod } R$ be modules. If $n \geq 2$ is an integer, then there is an isomorphism:

$$\text{Tor}_n^R(M, N) \cong \text{Ext}_R^1(\text{Tr } \Omega M, \Omega^{n-1} N).$$

Proof. There are isomorphisms:

$$(2.5.1) \quad \begin{aligned} \text{Tor}_n^R(M, N) &\cong \text{Tor}_1^R(\text{Tr}(\text{Tr } M), \Omega^{n-1} N) \\ &\cong \underline{\text{Hom}}(\text{Tr } M, \Omega^{n-1} N) \\ &\cong \underline{\text{Hom}}(\text{Tr } \Omega \text{Tr}(\text{Tr } M), \Omega^{n-2} N) \end{aligned}$$

The first isomorphism of (2.5.1) follows from dimension shifting and the fact that $\text{Tr}(\text{Tr } M)$ is stably isomorphic to M . The second and the third isomorphisms follow from (2.2) and (2.3), respectively.

Next consider the natural exact sequence: $0 \rightarrow \Omega^{n-1}N \rightarrow R^\oplus \rightarrow \Omega^{n-2}N \rightarrow 0$. This gives, by (2.4), the following exact sequence:

$$\underline{\text{Hom}}(\text{Tr } \Omega M, R^\oplus) \rightarrow \underline{\text{Hom}}(\text{Tr } \Omega M, \Omega^{n-2}N) \rightarrow \text{Ext}_R^1(\text{Tr } \Omega M, \Omega^{n-1}N) \rightarrow \text{Ext}_R^1(\text{Tr } \Omega M, R^\oplus)$$

It follows from the definition that $\underline{\text{Hom}}(\text{Tr } \Omega M, R^\oplus) = 0$. Furthermore, by [3, 2.6], we have $\text{Ext}_R^1(\text{Tr } \Omega M, R^\oplus) = 0$. This establishes the isomorphism:

$$(2.5.2) \quad \underline{\text{Hom}}(\text{Tr } \Omega \text{Tr}(\text{Tr } M), \Omega^{n-2}N) \cong \text{Ext}_R^1(\text{Tr } \Omega M, \Omega^{n-1}N)$$

Now (2.5.1) and (2.5.2) yield the claim. \square

Lemma 2.6. Let R be a local ring and let $M, N \in \text{mod } R$ be modules.

- (i) If $\text{Ext}_R^1(M, \Omega N) = 0$, then $\text{Tor}_2^R(\text{Tr } \Omega M, N) = 0$.
- (ii) If $\text{Ext}_R^2(M, \Omega N) = 0$, then $\text{Tor}_1^R(\text{Tr } \Omega M, N) = 0$.

Proof. (i) It follows from [3, 2.21] that there is an exact sequence:

$$0 \rightarrow R^\oplus \rightarrow \text{Tr } \Omega \text{Tr } \Omega M \oplus R^\oplus \rightarrow M \rightarrow 0.$$

This induces the surjection $\text{Ext}_R^1(M, \Omega N) \rightarrow \text{Ext}_R^1(\text{Tr } \Omega \text{Tr } \Omega M, \Omega N) \rightarrow 0$. As $\text{Ext}_R^1(M, \Omega N)$ vanishes by assumption, so does $\text{Ext}_R^1(\text{Tr } \Omega \text{Tr } \Omega M, \Omega N)$. Therefore, by Lemma 2.5, we have $\text{Tor}_2^R(\text{Tr } \Omega M, N) \cong \text{Ext}_R^1(\text{Tr } \Omega \text{Tr } \Omega M, \Omega N) = 0$.

(ii) It follows from (2.4) that the natural exact sequence $0 \rightarrow \Omega N \rightarrow R^\oplus \rightarrow N \rightarrow 0$ induces the exact sequence:

$$\underline{\text{Hom}}(\Omega M, R^\oplus) \rightarrow \underline{\text{Hom}}(\Omega M, N) \rightarrow \text{Ext}_R^1(\Omega M, \Omega N).$$

We have, by definition, that $\underline{\text{Hom}}(\Omega M, R^\oplus) = 0$. Moreover $\text{Ext}_R^1(\Omega M, \Omega N) \cong \text{Ext}_R^2(M, \Omega N) = 0$ by our assumption. Therefore $\underline{\text{Hom}}(\Omega M, N)$ vanishes and hence the result follows from (2.2). \square

Lemma 2.7. Let R be a local ring, $M, N \in \text{mod } R$ be modules and let $n \geq 1$ be an integer. If $\text{Ext}_R^n(M, \Omega N) = \text{Ext}_R^{n+1}(M, \Omega N) = 0$, then $\text{Tor}_1^R(\text{Tr } \Omega^n M, N) = \text{Tor}_2^R(\text{Tr } \Omega^n M, N) = 0$.

Proof. We replace M with $\Omega^{n-1}M$, and have $\text{Ext}_R^1(\Omega^{n-1}M, \Omega N) = \text{Ext}_R^2(\Omega^{n-1}M, \Omega N) = 0$. Thus the result follows from Lemma 2.6. \square

Definition 2.8. Let $0 \neq N \in \text{mod } R$. We say N is 2-Tor-rigid if, whenever $0 \neq M \in \text{mod } R$ and $\text{Tor}_n^R(M, N) = \text{Tor}_{n+1}^R(M, N) = 0$ for some $n \geq 0$, we have $\text{Tor}_i^R(M, N) = 0$ for all $i \geq n$.

There are various examples of 2-Tor-rigid modules in the literature. For example, we have:

2.9. (Levin and Vasconcelos [24, Lemma, page 316]) Let (R, \mathfrak{m}) be a local ring, $M \in \text{mod } R$ be a module such that $\mathfrak{m}M \neq 0$. If $\text{Tor}_n^R(\mathfrak{m}M, N) = \text{Tor}_{n+1}^R(\mathfrak{m}M, N) = 0$ for some $n \geq 0$ and for some $N \in \text{mod } R$, then $\text{pd}(N) \leq n$. In particular $\mathfrak{m}M$ is 2-Tor-rigid.

2.10. (Murthy [26, 1.9]) If R is a hypersurface ring, then each module in $\text{mod } R$ is 2-Tor-rigid.

2.11. (Dao [15, 6.6]; Huneke, Jorgensen and Wiegand [21, 1.9]) Let R be a complete intersection ring that is quotient of an unramified (e.g., equi-characteristic) regular local ring. Assume R has codimension two. If $M \in \text{mod } R$ is a finite length module, then M is 2-Tor-rigid.

An important class of 2-Tor-rigid modules was determined by Burch [10]. We record her result and use it for our proof of Corollary 3.13.

2.12. (Burch [10, Theorem 5(ii), page 949]) Let $M \in \text{mod } R$ and let I be a proper ideal of R . Assume $\mathfrak{m}(I : \mathfrak{m}) \neq \mathfrak{m}I$. If $\text{Tor}_n^R(M, R/I) = \text{Tor}_{n+1}^R(M, R/I) = 0$ for some positive integer n , then $\text{pd}(M) \leq n$.

The following consequence of Burch's theorem has been established in [13].

2.13. (Celikbas and Sather-Wagstaff [13, 2.3]) Let $M \in \mathbf{mod} R$ and let I be a proper ideal of R . Assume I is integrally closed and $\text{depth}(R/I) = 0$. If $\text{Tor}_n^R(M, R/I) = \text{Tor}_{n+1}^R(M, R/I) = 0$ for some $n \geq 1$, then $\text{pd}(M) \leq n$.

Lemma 2.14. Let R be a local ring and let $M, N \in \mathbf{mod} R$ be modules. Assume N is nonfree and 2-Tor-rigid. If $\text{Ext}_R^n(M, \Omega N) = \text{Ext}_R^{n+1}(M, \Omega N) = 0$ for some $n \geq 1$, then $\text{Tor}_j^R(\text{Tr } \Omega^n M, N) = 0$ for all $j \geq 1$ and $\text{Ext}_R^n(M, \Omega^i N \oplus R) = 0$ for all $i \geq 0$.

Proof. As N is 2-Tor-rigid, Lemma 2.7 implies that $\text{Tor}_j^R(\text{Tr } \Omega^n M, N) = 0$ for all $j \geq 1$. Thus, given an integer $i \geq 0$, we have $\text{Tor}_2^R(\text{Tr } \Omega^n M, \Omega^i N) \cong \text{Tor}_{2+i}^R(\text{Tr } \Omega^n M, N) = 0$ and $\text{Tor}_1^R(\text{Tr } \Omega^n M, \Omega^i N) \cong \text{Tor}_{1+i}^R(\text{Tr } \Omega^n M, N) = 0$. This yields, by the exact sequence (2.1.2), that:

$$(2.14.1) \quad \text{Ext}_R^n(M, R) \otimes_R \Omega^i N \cong \text{Ext}_R^n(M, \Omega^i N) \text{ for all } i \geq 0.$$

Letting $i = 1$, we obtain $\text{Ext}_R^n(M, R) \otimes_R \Omega N = 0$, i.e., $\text{Ext}_R^n(M, R) = 0$. So (2.14.1) shows that $\text{Ext}_R^n(M, \Omega^i N) = 0$ for all $i \geq 0$. \square

We use properties of Gorenstein and complete intersection dimension for the rest of the paper. Therefore we recall the definitions of these homological dimensions.

2.15. (Auslander and Bridger [3]) A module $M \in \mathbf{mod} R$ is said to be *totally reflexive* if the natural map $M \rightarrow M^{**}$ is bijective and $\text{Ext}_R^i(M, R) = 0 = \text{Ext}_R^i(M^*, R)$ for all $i \geq 1$. The infimum of n for which there exists an exact sequence $0 \rightarrow X_n \rightarrow \cdots \rightarrow X_0 \rightarrow M \rightarrow 0$, such that each X_i is totally reflexive, is called the *Gorenstein dimension* of M . If M has Gorenstein dimension n , we write $\text{G-dim}(M) = n$. Note, it follows by convention, that $\text{G-dim}(0) = -\infty$.

2.16. (Avramov, Gasharov and Peeva [9]) A diagram of local ring maps $R \rightarrow R' \leftarrow Q$ is called a *quasi-deformation* provided that $R \rightarrow R'$ is flat and the kernel of the surjection $R' \leftarrow Q$ is generated by a Q -regular sequence. The *complete intersection dimension* of M is:

$$\text{CI-dim}(M) = \inf\{\text{pd}_Q(M \otimes_R R') - \text{pd}_Q(R') \mid R \rightarrow R' \leftarrow Q \text{ is a quasi-deformation}\}.$$

In the following H-dim denotes a homological dimension of modules in $\mathbf{mod} R$ that has the following properties; see [7, 3.1.2, 8.7 and 8.8] for details.

2.17. Let $M \in \mathbf{mod} R$.

(i) $\text{G-dim}(M) \leq \text{H-dim}(M) \leq \text{CI-dim}(M) \leq \text{pd}(M)$.

If one of these dimensions is finite, then it equals the one on its left.

(ii) If $\text{H-dim}(M) < \infty$ and $M \neq 0$, then $\text{H-dim}(M) + \text{depth}(M) = \text{depth}(R)$.

(iii) If $\text{H-dim}(M) < \infty$, then $\text{H-dim}(M) = \sup\{i \in \mathbb{Z} : \text{Ext}_R^i(M, R) \neq 0\}$.

(iv) If $\text{H-dim}(M) = 0$, then $\text{Tr} M = 0$ or $\text{H-dim}(\text{Tr} M) = 0$.

(v) If $\text{H-dim}(M) < \infty$ and $0 \neq N \in \mathbf{mod} R$, then $\text{H-dim}(M) = \text{depth}(N) - \text{depth}(M \otimes_R N)$ provided that $\text{Tor}_i^R(M, N) = 0$ for all $i \geq 1$.

The next result is important for our proof of Theorem 2.19.

2.18. (Celikbas, Gheibi, Sadeghi and Zargar [12, 5.8(1)]) Let $M \in \mathbf{mod} R$ and let $n \geq 1$ be an integer. If $\text{G-dim}(\text{Tr } \Omega^n M) < \infty$, then $\text{Ext}_R^i(\text{Tr } \Omega^n M, R) = 0$ for all $i = 1, \dots, n$.

Now we are ready to prove our main result:

Theorem 2.19. Let R be a local ring, $M, N \in \mathbf{mod} R$ be modules and let $n \geq 1$ be an integer. Assume N is not free and the following conditions hold:

(a) N is 2-Tor-rigid.

(b) $\text{H-dim}(\text{Tr } \Omega^n M) < \infty$.

(c) $\text{Ext}_R^n(M, \Omega N) = \text{Ext}_R^{n+1}(M, \Omega N) = 0$ for some $n \geq \text{depth}(N)$.

Then $\text{H-dim}(M) < n$.

Proof. It follows from (a), (c) and Lemma 2.14 that $\text{Tor}_i^R(\text{Tr } \Omega^n M, N) = 0$ for all $i \geq 1$. Hence, in view of 2.17(v), we deduce from (b) that

$$(2.19.1) \quad \text{H-dim}(\text{Tr } \Omega^n M) = \text{depth}(N) - \text{depth}(\text{Tr } \Omega^n M \otimes_R N).$$

Note, since $\text{H-dim}(\text{Tr } \Omega^n M) < \infty$, we have $\text{G-dim}(\text{Tr } \Omega^n M) < \infty$; see 2.17(i). Therefore, by (2.18), we know that $\text{Ext}_R^i(\text{Tr } \Omega^n M, R) = 0$ for all $i = 1, \dots, n$. Moreover, by (2.19.1), $\text{H-dim}(\text{Tr } \Omega^n M) \leq n$. Consequently we deduce $\text{H-dim}(\text{Tr } \Omega^n M) = 0$; see 2.17(iii).

Since $\text{Tr } \text{Tr } \Omega^n M$ is stably isomorphic to $\Omega^n M$, we see from 2.17(iv) that $\Omega^n M = 0$ or $\text{H-dim}(\Omega^n M) = 0$. In either case, this implies $\text{H-dim}(M) \leq n$. Furthermore we have $\text{Ext}_R^n(M, R) = 0$ by Lemma 2.14. Thus we conclude that $\text{H-dim}(M) < n$; see (2.17)(iii). \square

3. COROLLARIES OF THEOREM 2.19

This section is devoted to various corollaries of Theorem 2.19. In particular we prove Theorem 1.4 which is advertised in the introduction; see Corollary 3.14. To prove Corollaries 3.1, 3.2 and 3.3, we use Theorem 2.19 for the cases where H-dim is the projective dimension pd and the complete intersection dimension CI-dim ; see 2.16. We are allowed to do that since these homological dimensions satisfy the conditions listed in 2.17; see [2, 1.2] and [1, 2.5].

Corollary 3.1. Let R be a local ring and let $M, N \in \text{mod } R$ be nonfree modules. Assume N is 2-Tor-rigid and $\text{depth}(N) \leq 1$. Assume further $\text{pd}(M^*) < \infty$. Then $\text{Ext}_R^1(M, \Omega N)$ and $\text{Ext}_R^2(M, \Omega N)$ do not vanish simultaneously.

Proof. Suppose $\text{Ext}_R^1(M, \Omega N) = \text{Ext}_R^2(M, \Omega N) = 0$. Then it follows from Lemma 2.14 that $\text{Ext}_R^1(M, R) = 0$. Thus $\text{Tr } M$ is stably isomorphic to $\Omega \text{Tr } \Omega M$; see (2.1.1). Hence $\text{pd}(M^*) < \infty$ if and only if $\text{pd}(\text{Tr } M) < \infty$ if and only if $\text{pd}(\Omega \text{Tr } \Omega M) < \infty$ if and only if $\text{pd}(\text{Tr } \Omega M) < \infty$. Consequently Theorem 2.19 shows that M is free, which contradicts our assumption. Therefore $\text{Ext}_R^1(M, \Omega N)$ and $\text{Ext}_R^2(M, \Omega N)$ do not vanish simultaneously. \square

Corollary 3.2. Let R be a local ring, $M, N \in \text{mod } R$ be nonfree modules and let $n \geq 1$ be an integer. Assume N is 2-Tor-rigid and $\text{grade}(M) \geq n$. If either $\text{Hom}(M, N) \neq 0$ or $\text{depth}(N) \leq n$, then $\text{Ext}_R^n(M, \Omega N)$ and $\text{Ext}_R^{n+1}(M, \Omega N)$ do not vanish simultaneously.

Proof. We assume $\text{Ext}_R^n(M, \Omega N) = \text{Ext}_R^{n+1}(M, \Omega N) = 0$ and show that $\text{Hom}(M, N) = 0$ and $\text{depth}(N) > n$.

Note, since $\text{grade}(M) \geq n$, we have $\text{Ext}_R^i(M, R) = 0$ for all $i = 0, \dots, n-1$. Furthermore it follows from Lemma 2.14 that $\text{Tor}_j^R(\text{Tr } \Omega^n M, N) = 0$ for all $j \geq 1$ and $\text{Ext}_R^n(M, R) = 0$. Consequently $\text{Ext}_R^i(M, R) = 0$ for all $i = 0, \dots, n$. This implies $\text{Tr } \Omega^{i-1} M$ is stably isomorphic to $\Omega \text{Tr } \Omega^i M$ for all $i = 1, \dots, n$; see (2.1.1). Using this fact, we deduce the vanishing of $\text{Tor}_j^R(\text{Tr } M, N)$, for all $j \geq 1$, from the vanishing of $\text{Tor}_j^R(\text{Tr } \Omega^n M, N)$. Now (2.1.2) gives $\text{Hom}(M, N) = 0$.

Now let $\dots \rightarrow F_1 \rightarrow F_0 \rightarrow 0$ be a free resolution of M . Since $\text{Ext}_R^i(M, R) = 0$ for all $i = 0, \dots, n$, we obtain an exact sequence of the form:

$$0 \rightarrow (F_0)^* \rightarrow \dots \rightarrow (F_{n+1})^* \rightarrow \text{Tr } \Omega^n M \rightarrow 0.$$

Therefore $\text{pd}(\text{Tr } \Omega^n M) < \infty$. Hence, if $\text{depth}(N) \leq n$, then it follows from Theorem 2.19 that $\text{pd}(M) < n$ so that the fact $n \leq \text{grade}(M) \leq \text{pd}(M)$ gives a contradiction. Consequently $\text{depth}(N) > n$. \square

In passing we obtain a result on the vanishing of Ext over complete intersections:

Corollary 3.3. Let R be a complete intersection ring of codimension two that is quotient of an unramified regular local ring. If $M \in \mathbf{mod} R$ is a module that is not maximal Cohen-Macaulay and I is an \mathfrak{m} -primary ideal of R , then $\mathrm{Ext}_R^1(M, I)$ and $\mathrm{Ext}_R^2(M, I)$ do not vanish simultaneously.

Proof. Assume M is not maximal Cohen-Macaulay and I is \mathfrak{m} -primary. Set $N = R/I$. Then N is a nonfree finite length module that is 2-Tor-rigid; see (2.11). If $\mathrm{Ext}_R^1(M, I) = \mathrm{Ext}_R^2(M, I) = 0$, then it follows from Theorem 2.19 that $\mathrm{Cl}\text{-dim}(M) < 1$, i.e., M is maximal Cohen-Macaulay; see (2.17)(ii). Hence $\mathrm{Ext}_R^1(M, I)$ and $\mathrm{Ext}_R^2(M, I)$ do not vanish simultaneously. \square

Before we give the definition of a weakly \mathfrak{m} -full ideal, we record an example that shows the finiteness hypothesis of $\mathrm{pd}(\mathrm{Tr} \Omega^n M)$ is necessary in Theorem 2.19 to conclude $\mathrm{pd}(M) < n$.

Example 3.4. Let $R = \mathbb{C}[[x, y, z]]/(xz - y^2, xy - z^2)$, $M = R/(x, z)$ and $N = R/(x, y)$. Then R is a codimension two complete intersection ring of dimension one, and M and N are finite length R -modules such that $\mathrm{pd}(M) = \infty = \mathrm{pd}(N)$. Furthermore $\mathrm{Ext}_R^i(M, N) = 0 = \mathrm{Tor}_i^R(M, N)$ for all $i \geq 2$; see [22, 4.2]. Hence, setting $J = \Omega N$, we obtain $\mathrm{Ext}_R^i(M, J) = 0$ for all $i \geq 3$. Let $n \geq 1$ and set $X = \Omega^n M$. Note $X^* \neq 0$. Assume $\mathrm{pd}(\mathrm{Tr} X) < \infty$. Then $\mathrm{pd}(X^*) < \infty$ so that X^* is free. Since $X \cong X^{**}$, we see X is free, i.e., $\mathrm{pd}(M) < \infty$. Therefore $\mathrm{pd}(\mathrm{Tr} \Omega^n M) = \infty$ for all $n \geq 1$.

Remark 3.5. In Example 3.4, we can use either Corollary 3.1 or Corollary 3.3 to see $\mathrm{Ext}_R^1(M, J)$ and $\mathrm{Ext}_R^2(M, J)$ do not vanish simultaneously. In fact we have $\mathrm{Ext}_R^1(M, J) \neq 0 = \mathrm{Ext}_R^2(M, J)$.

Recall that an ideal I of a local ring R is called \mathfrak{m} -full if $\mathfrak{m}I : x = I$ for some $x \in \mathfrak{m}$; see [16, 2.1]. Properties of such ideals were extensively studied in the literature. For example Goto established that \mathfrak{m} -full ideals are closely related to integrally closed ideals:

3.6. (Goto [16, 2.4]) Assume the residue field of R is infinite. If I is an integrally closed ideal of R , then $I = \sqrt{(0)}$ or I is \mathfrak{m} -full.

A weakly version of \mathfrak{m} -full property can be defined as follows:

Definition 3.7. Let I be a proper ideal of a local ring R . We call I a *weakly \mathfrak{m} -full ideal* provided that $\mathfrak{m}I : \mathfrak{m} = I$, i.e., if $\mathfrak{m}x \subseteq \mathfrak{m}I$ for some $x \in R$, then $x \in I$.

Example 3.8. Let $\mathfrak{p} \neq \mathfrak{m}$ be a prime ideal of R . Let $x \in \mathfrak{m}\mathfrak{p} : \mathfrak{m}$, $y \in \mathfrak{m}$ and $y \notin \mathfrak{p}$. Then $xy \in \mathfrak{m}\mathfrak{p} \subseteq \mathfrak{p}$ and hence $x \in \mathfrak{p}$. Therefore \mathfrak{p} is weakly \mathfrak{m} -full.

We should remark that, due to its definition, it can be easily checked whether or not a given ideal is weakly \mathfrak{m} -full by using a computer algebra software such as Macaulay2 [19].

Remark 3.9. Let I be a proper ideal of a local ring R . Then $\mathrm{depth}(R/I) = 0$ if and only if $I : \mathfrak{m} \neq I$. Hence it follows that, if $\mathrm{depth}(R/I) \geq 1$, then I is weakly \mathfrak{m} -full.

Although each \mathfrak{m} -full ideal is weakly \mathfrak{m} -full, many weakly \mathfrak{m} -full ideals that are not \mathfrak{m} -full exist. We record such an example next.

Example 3.10. ([11]) Let $R = \mathbb{C}[[t^4, t^5, t^6]]$ and let $I = (t^4, t^{11})$. Then I is a weakly \mathfrak{m} -full ideal which is not \mathfrak{m} -full such that $\mathrm{depth}(R/I) = 0$.

Let I be a nonzero ideal of a local ring R . Assume $\mathfrak{m}(I : \mathfrak{m}) = \mathfrak{m}I$ and I is weakly \mathfrak{m} -full. Let $x \in I : \mathfrak{m}$. Then $\mathfrak{m}x \in \mathfrak{m}(I : \mathfrak{m}) = \mathfrak{m}I$, i.e., $x \in \mathfrak{m}I : \mathfrak{m}$, which equals to I since I is weakly \mathfrak{m} -full. So we conclude:

3.11. Let I be a nonzero ideal of a local ring R . Assume I is weakly \mathfrak{m} -full and $\mathrm{depth}(R/I) = 0$. Then $\mathfrak{m}(I : \mathfrak{m}) \neq \mathfrak{m}I$.

Next we note that weakly \mathfrak{m} -full ideals I with $\mathrm{depth}(R/I) = 0$ are 2-Tor-rigid; see (2.8).

3.12. Let I be a nonzero ideal of a local ring R . Assume I is weakly \mathfrak{m} -full and $\text{depth}(R/I) = 0$. If $\text{Tor}_n^R(M, R/I) = \text{Tor}_{n+1}^R(M, R/I) = 0$ for some $M \in \text{mod } R$ and for some integer $n \geq 1$, then $\text{pd}(M) \leq n$; see (2.12) and (3.11).

Corollary 3.13. Let $M \in \text{mod } R$ be a nonzero module and let I be a nonzero ideal of R such that $\text{depth}(R/I) = 0$. Assume $\text{Ext}_R^n(M, I) = \text{Ext}_R^{n+1}(M, I) = 0$ for some $n \geq 1$. If I is integrally closed or weakly \mathfrak{m} -full, then $\text{pd}(M) < n$.

Proof. Setting $N = R/I$, we see from Lemma 2.14 that $\text{Tor}_i^R(\text{Tr } \Omega^n M, N) = 0$ for all $i \geq 1$ and $\text{pd}(\text{Tr } \Omega^n M) \leq 1$; see 2.13 and (3.12). So the result follows from Theorem 2.19. \square

We now establish Theorem 1.4, advertised in the introduction.

Corollary 3.14. Let I be a proper ideal of a local ring R such that $\text{depth}(R/I) = 0$, or equivalently $I : \mathfrak{m} \neq I$. Assume I is integrally closed or weakly \mathfrak{m} -full. Then,

- (i) $\text{Ext}_R^1(I, I)$ and $\text{Ext}_R^2(I, I)$ do not vanish simultaneously unless $I \cong R$.
- (ii) $\text{Ext}_R^n(I, I)$ and $\text{Ext}_R^{n+1}(I, I)$ do not vanish simultaneously for any $n \geq 1$ unless R is regular.

Proof. Part (i) is clear from Corollary 3.13. Next we assume $\text{Ext}_R^n(I, I) = \text{Ext}_R^{n+1}(I, I) = 0$ for some $n \geq 1$. Then it follows from Corollary 3.13 that $\text{pd}(R/I) < \infty$. Thus $\text{Tor}_i^R(k, R/I) = 0$ for all $i \gg 0$. This implies $\text{pd}(k) < \infty$, i.e., R is regular; see (2.13) and (3.12). \square

Remark 3.15. The conclusion of Corollary 3.14(i) resembles a result of Jorgensen: if R is a local complete intersection ring and $M \in \text{mod } R$, then $\text{Ext}_R^1(M, M)$ and $\text{Ext}_R^2(M, M)$ do not vanish simultaneously unless M is free; see [23, 2.5]. In that sense, a weakly \mathfrak{m} -full ideal I with $I : \mathfrak{m} \neq I$ behaves like a nonfree module over a complete intersection ring.

The vanishing result obtained in Corollary 3.14 is stronger than what is proposed in the Auslander and Reiten conjecture: if I is an ideal as in Corollary 3.14, then we do not need the vanishing of $\text{Ext}_R^i(I, R)$ for any $i \geq 1$, besides the vanishing of $\text{Ext}_R^1(I, I)$ and $\text{Ext}_R^2(I, I)$, to conclude I is free; c.f., Conjecture (1.3). In fact, for such an ideal I , over a Cohen-Macaulay local ring that is not Gorenstein, there cannot be $\dim R + 2$ consecutive vanishing of $\text{Ext}_R^i(I, R)$; this is recorded in the next corollary.

Corollary 3.16. Let R be a d -dimensional Cohen-Macaulay local ring that is not Gorenstein, and let I be a proper ideal of R such that $\text{depth}(R/I) = 0$, or equivalently $I : \mathfrak{m} \neq I$. Assume I is integrally closed or weakly \mathfrak{m} -full. Then, for all $n \geq 1$, there is an integer i with $n \leq i \leq n + d + 1$ such that $\text{Ext}_R^i(I, R) \neq 0$.

Proof. Let $N \in \text{mod } R$ be a module of finite injective dimension. If, for some $n \geq 1$, we have $\text{Ext}_R^i(I, R) = 0$ for all $i = n, \dots, n + d + 1$, then [8, Corollary B.4] gives the vanishing of $\text{Tor}_n^R(I, N)$ and $\text{Tor}_{n+1}^R(I, N)$. Hence we conclude from 2.13 and 3.12 that $\text{pd}(N) < \infty$ so that R is Gorenstein. \square

The assumption that R is Cohen-Macaulay can be removed from Corollary 3.16 in case we consider the vanishing of higher $\text{Ext}_R^i(I, R)$.

Corollary 3.17. Let R be a local ring that is not Gorenstein, and let I be a proper ideal of R . Assume I is weakly \mathfrak{m} -full and $\text{depth}(R/I) = 0$. Then $\text{Ext}_R^i(I, R) \neq 0$ for infinitely many $i \geq 0$. In particular $\text{G-dim}(I) = \infty$.

Proof. It follows from 3.12 that I is a pd -test module; see [13, 3.1]. Therefore [13, 4.4] gives the required result. \square

Goto and Hayasaka [17, 2.2] proved that if R is a local ring and I and J are two ideals of R such that $I \not\subseteq J$, $I : \mathfrak{m} \not\subseteq J$ and J is \mathfrak{m} -full, then R is regular provided that $\text{id}(I) < \infty$. A special, albeit an important case of this result, namely the case where $I = J$, is:

3.18. (Goto and Hayasaka [17, 2.2]) Let R be a local ring and let I be a proper ideal of R . Assume $\text{depth}(R/I) = 0$ and I is \mathfrak{m} -full. If $\text{id}(I) < \infty$, then R is regular.

We use Corollary 3.13 and improve 3.18 by replacing an \mathfrak{m} -full ideal with a weakly \mathfrak{m} -full one:

Corollary 3.19. Let R be a local ring and let I be a proper ideal of R . Assume $\text{depth}(R/I) = 0$ and I is weakly \mathfrak{m} -full. If $\text{id}(I) < \infty$, then R is regular.

Proof. As $\text{id}_R(I) < \infty$ we have $\text{Ext}_R^i(k, I) = 0$ for all $i \gg 0$. This implies, by Corollary 3.13, that $\text{pd}(k) < \infty$, i.e., R is regular. \square

We can obtain a result similar to Corollary 3.14 that characterizes Gorenstein rings in terms of the *Gorenstein injective dimension* $\text{Gid}(I)$ of weakly \mathfrak{m} -full ideals I ; see [20] for the definition of Gorenstein injective dimension.

Corollary 3.20. Let R be a local ring and let I be a proper ideal. Assume $\text{depth}(R/I) = 0$ and I is weakly \mathfrak{m} -full. If $\text{Gid}(I) < \infty$ and $\dim(R/I) < \dim R$, then R is Gorenstein.

Proof. Note that $\dim(I) = \dim R$. Therefore it follows from [29, 1.3] that R is Cohen-Macaulay and so there exists a module $0 \neq M \in \text{mod } R$ such that $\text{id}(M) < \infty$. Hence $\text{Ext}_R^i(M, I) = 0$ for all $i \gg 0$ [20, 2.22]. Now, by Corollary 3.13, we have $\text{pd}(M) < \infty$ and so R is Gorenstein. \square

Now we present several examples to prove the necessity of the assumptions of our results.

Example 3.21. Let $R = \mathbb{C}[[x, y, z]]/(xy - z^2)$ and let $I = (x)$ be the ideal of R . Then R is a two-dimensional integrally closed Gorenstein domain that is not regular. This implies, since I is principal, that I is integrally closed. Therefore I is \mathfrak{m} -full and hence weakly \mathfrak{m} -full; see (3.6). Moreover $\text{id}(I) = 2$ and $\text{depth}(R/I) = 1$.

Example 3.22. Let $S = k[[t, u]]$ be a formal power series ring over a field k , and let $R = k[[t^3, t^4, t^5, u]]$ be a subring of S . Then the ideal $I = (t^3, t^4)$ of R is a canonical module of R , which implies $\text{id}(I) = 2$ and $\text{Ext}_R^i(I, I) = 0$ for all $i \geq 1$. Since $\text{depth}(R/I) = 1$, the ideal I is weakly \mathfrak{m} -full; see (3.9). In fact we can show I is \mathfrak{m} -full but not integrally closed. Since $(t^5)^2 - (t^3)^2 t^4 = 0$, it is clear that $I = (t^3, t^4)$ is not an integrally closed ideal of $R = k[[t^3, t^4, t^5, u]]$. Considering $R/\mathfrak{m}I$, we can easily check that the equality $I = (\mathfrak{m}I : u)$ holds. Hence I is \mathfrak{m} -full.

The ideals I in Examples 3.21 and 3.22 show the necessity of the depth assumption in Corollaries 3.13, 3.14 and 3.19, as well as the necessity of $I : \mathfrak{m} \neq I$ in Theorem 1.4.

Example 3.23. Let $S = k[[t]]$ be a formal power series ring over a field k , and let $R = k[[t^5, t^6, t^8, t^9]]$ be a subring of S . Then R is a non-Gorenstein Cohen-Macaulay local domain of dimension 1. Denote by \mathfrak{m} the maximal ideal of R . Then \mathfrak{m} is integrally closed, and is (weakly) \mathfrak{m} -full. The ideal $I = (t^5, t^8, t^9)$ of R is a canonical module of R , and we have $\text{id}(I) = 1$ and $\text{depth}(R/I) = 0$. We have $\text{id}(\mathfrak{m}) = \infty$; this follows from the fact that \mathfrak{m} is an indecomposable maximal Cohen-Macaulay R -module, which is not isomorphic to the canonical module I . The equality $(t^6)^3 - (t^5)^2(t^8) = 0$ shows that I is not integrally closed. Calculating the socle of $R/\mathfrak{m}I$, we can verify that I is not weakly \mathfrak{m} -full.

The ideal I in Example 3.23 guarantees that the assumption in Corollaries 3.13, 3.14 and in Theorem 1.4 that I is either integrally closed or weakly \mathfrak{m} -full is necessary. It also assures that the weak \mathfrak{m} -fullness assumption in Corollary 3.19 cannot be removed. Moreover the maximal ideal \mathfrak{m} in Example 3.23 shows that the finiteness assumption on the injective dimension in Corollary 3.19 cannot be extracted. Consequently each of the three assumptions in Corollary 3.19 is necessary.

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REFERENCES

- [1] T. Araya and Y. Yoshino. Remarks on a depth formula, a grade inequality and a conjecture of Auslander. *Comm. Algebra*, 26, 3793 – 3806, 1998. 5
- [2] M. Auslander. Modules over unramified regular local rings. *Illinois J. Math.*, 5, 631 – 647, 1961. 5
- [3] M. Auslander and M. Bridger. *Stable module theory*. Mem. of the AMS 94, Amer. Math. Soc., Providence, 1969. 2, 3, 4
- [4] M. Auslander, S. Ding and Ø. Solberg. Liftings and weakly liftings of modules. *J. Algebra*, 156, 273 – 317, 1969. 1
- [5] M. Auslander and I. Reiten. On a generalized version of the Nakayama conjecture. *Proc. Amer. Math. Soc.*, 52, 69 – 74, 1975. 1, 2
- [6] M. Auslander and I. Reiten. Syzygy modules for Noetherian rings. *J. Algebra*, 183, 167 – 185, 1996. 1, 2
- [7] L. L. Avramov. Homological dimensions and related invariants of modules over local rings. In *Representations of algebra*. Vol. I, II, pages 1–39. Beijing Norm. Univ. Press, Beijing, 2002. 4
- [8] L. L. Avramov, R. O. Buchweitz and L. M. Şega. Extensions of a dualizing complex by its ring: commutative versions of a conjecture of Tachikawa. *J. Pure Appl. Algebra*, 201, 218 – 239, 2005. 7
- [9] L. L. Avramov, V. Gasharov, and I. Peeva. Complete intersection dimension. *Inst. Hautes Études Sci. Publ. Math.*, (86):67 – 114, 1998. 1997. 4
- [10] L. Burch. On ideals of finite homological dimension in local rings. *Proc. Cambridge Philos. Soc.*, 64, 941 – 948, 1968. 1, 3
- [11] O. Celikbas, S. Goto, R. Takahashi and N. Taniguchi. On the ideal case of a conjecture of Huneke and Wiegand. preprint 2017. 2, 6
- [12] O. Celikbas, M. Gheibi, A. Sadeghi and M. R. Zargar. Homological dimensions of rigid modules. to appear in *Kyoto J. Math.*; posted at [arXiv:1405.5188](https://arxiv.org/abs/1405.5188). 4
- [13] O. Celikbas and S. Sather-Wagstaff. Testing for the Gorenstein property. *Collect. Math.*, 67, 555 – 568, 2016. 4, 7
- [14] O. Celikbas and R. Takahashi. Auslander-Reiten conjecture and Auslander-Reiten duality. *J. Algebra*, 382, 100 – 114, 2013. 1
- [15] H. Dao. Asymptotic behavior of Tor over complete intersections and applications. Preprint; posted at [arXiv:0710.5818](https://arxiv.org/abs/0710.5818). 3
- [16] S. Goto. Integral closedness of complete-intersection ideals. *J. Algebra*, 108, 151 – 160, 1987. 6
- [17] S. Goto and F. Hayasaka. Finite homological dimension and primes associated to integrally closed ideals II. *J. Math. Kyoto Univ.*, 42, 631 – 639, 2002. 2, 7, 8
- [18] S. Goto and R. Takahashi. On the Auslander-Reiten conjecture for Cohen-Macaulay local rings. *Proc. Amer. Math. Soc.* 145 (2017), no. 8, 3289–3296. 1
- [19] D. R. Grayson and M. E. Stillman. *Macaulay2*, a software system for research in algebraic geometry. Available at <http://www.math.uiuc.edu/Macaulay2/>. 6
- [20] H. Holm. Gorenstein homological dimensions. *J. Pure Appl. Algebra*, 189, 167 – 193, 2004. 8
- [21] C. Huneke, D. A. Jorgensen, and R. Wiegand. Vanishing theorems for complete intersections. *J. Algebra*, 238, 684 – 702, 2001. 3
- [22] D. Jorgensen. Complexity and Tor on a Complete Intersection. *J. Algebra*, 211, 578 – 598, 1999. 6
- [23] D. Jorgensen. Finite projective dimension and the vanishing of $\text{Ext}_R(M, M)$. *Comm. Algebra*, 36, 4461 – 4471, 2008. 7
- [24] Levin, G.; Vasconcelos, W. V. Homological dimensions and Macaulay rings. *Pacific J. Math.*, 25, 315 – 323, 1968. 3
- [25] H. Matsui and R. Takahashi. Singularity categories and singular equivalences for resolving subcategories. *Math. Z.* 285 (2017), no. 1-2, 251 – 286. 2
- [26] M. P. Murthy. Modules over regular local rings. *Illinois J. Math.*, 7, 558 – 565, 1963. 3
- [27] T. Nakayama. On algebras with complete homology. *Abh. Math. Sem. Univ. Hamburg*, 22, 300 – 307, 1958. 1
- [28] J. Rotman. *An Introduction to Homological Algebra*. Academic Press, New York, 1979.
- [29] S. Yassemi. A generalization of a theorem of Bass. *Comm. Algebra*, 35(1), 249 – 251, 2007. 8

- [30] Y. Yoshino. Cohen–Macaulay modules over Cohen–Macaulay rings. London Mathematical Society Lecture Note Series, 146, Cambridge University Press, Cambridge, 1990. 2

DEPARTMENT OF MATHEMATICS, WEST VIRGINIA UNIVERSITY, MORGANTOWN, WV 26506 U.S.A
E-mail address: `olgur.celikbas@math.wvu.edu`

DEPARTMENT OF LIBERAL STUDIES, NATIONAL INSTITUTE OF TECHNOLOGY, NARA COLLEGE, 22 YATA-CHO,
YAMATOKORIYAMA, NARA 639-1080, JAPAN
E-mail address: `iima@libe.nara-k.ac.jp`

SCHOOL OF MATHEMATICS, INSTITUTE FOR RESEARCH IN FUNDAMENTAL SCIENCES (IPM), P.O. Box: 19395-
5746, TEHRAN, IRAN.
E-mail address: `sadeghiarash61@gmail.com`

GRADUATE SCHOOL OF MATHEMATICS, NAGOYA UNIVERSITY, FUROCHO, CHIKUSAKU, NAGOYA 464-8602,
JAPAN
E-mail address: `takahashi@math.nagoya-u.ac.jp`
URL: <http://www.math.nagoya-u.ac.jp/~takahashi/>