SOLID GENERATORS IN MODULE CATEGORIES AND APPLICATIONS

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ABSTRACT. Let R be a commutative noetherian ring. Denote by $\operatorname{mod} R$ the category of finitely generated R-modules. In the present paper, we introduce the notion of solid subcategories of $\operatorname{mod} R$ and investigate it. The main result of this paper not only recovers results of Schoutens, Krause and Stevenson, and Takahashi on thick subcategories, but also unifies and extends them to solid subcategories. Moreover, it provides some contributions to the study of the question asking when a thick subcategory is Serre.

1. Introduction

Let $\mathcal{A}$ be an abelian category. A *thick subcategory* of $\mathcal{A}$ is defined to be a full subcategory closed under direct summands and satisfying the 2-out-of-3 property with respect to short exact sequences. Various works on thick subcategories of abelian categories have been done so far; see [1, 6, 7, 9, 11] for instance.

Stanley and Wang [8] defined a *narrow subcategory* of $\mathcal{A}$ to be a full subcategory closed under extensions and cokernels. In the present paper, we shall define a *solid subcategory* of $\mathcal{A}$ as a full subcategory closed under direct summands, extensions and cokernels of monomorphisms. By definition, the notion of a solid subcategory is a common generalization of those of a thick subcategory and a narrow subcategory.

Now, let R be a commutative noetherian ring, and $\operatorname{mod} R$ the category of finitely generated R-modules. (It is shown in [8] that any narrow subcategory of $\operatorname{mod} R$ is Serre, and in particular it is thick.) For each collection $\mathcal{S}$ of objects of $\operatorname{mod} R$, we denote respectively by $\operatorname{thick} \mathcal{S}$ and $\operatorname{solid} \mathcal{S}$ the *thick closure* and the *solid closure* of $\mathcal{S}$, that is to say, the smallest thick and solid subcategories of the abelian category $\operatorname{mod} R$ containing $\mathcal{S}$. Our main result is the following theorem that provides an equality of solid closures.

Theorem 1.1. *Let R be a commutative noetherian ring. Let M be a finitely generated R-module. Then*

$$\operatorname{solid}\{R/\mathfrak{p}, M \mid \mathfrak{p} \in \operatorname{Sing} R \cup \operatorname{NF}(M)\} = \operatorname{solid}\{R/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Sing} R \cup \operatorname{Supp} M\}.$$

Here, $\operatorname{Sing} R$ denotes the singular locus of the ring R, while $\operatorname{NF}(M)$ and $\operatorname{Supp} M$ respectively stand for the nonfree locus and the support of the R-module M. The meaning of Theorem 1.1 becomes clearer if we ignore the parts coming from the singular locus: if we put $\langle \mathcal{S} \rangle = \operatorname{solid}(\mathcal{S} \cup \{R/\mathfrak{p}\}_{\mathfrak{p} \in \operatorname{Sing} R})$ for each collection $\mathcal{S}$ of modules, then Theorem 1.1 asserts that there is an equality

$$\langle R/\mathfrak{p}, M \mid \mathfrak{p} \in \operatorname{NF}(M) \rangle = \langle R/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Supp} M \rangle.$$

This equality means that, up to the singular locus of R, the support of M can be reconstructed from M itself and the nonfree locus of M, by taking direct summands, extensions and cokernels of monomorphisms.

Theorem 1.1 yields Corollary 1.2 below, which implies Corollary 1.3 below. The latter corollary is the combination of a result of Schoutens [7] and Krause and Stevenson [6], and a result of Takahashi [11]. The only relationship between (1) and (2) of Corollary 1.3 that has been found out so far seems to be the fact that (1) follows from (2) in the case where R is a local ring with an isolated singularity. Theorem 1.1 provides a common generalization of (1) and (2) of Corollary 1.3.

Corollary 1.2. *Let R be a commutative noetherian ring. One then has an equality*

$$\operatorname{mod} R = \operatorname{solid}\{R, R/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Sing} R\}.$$

If R is an isolated singularity with residue field k and M is a nonzero finitely generated R-module which is locally free on the punctured spectrum of R, then there is an equality

$$\operatorname{solid}\{k, M\} = \operatorname{solid}\{R/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Supp} M\}.$$

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Corollary 1.3. *Let R be a commutative noetherian ring. Then the following two statements hold true.*

(1) (Schoutens, Krause–Stevenson) *There is an equality* $\mathsf{mod}\,R = \mathsf{thick}\{R, R/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Sing} R\}$.

(2) (Takahashi) *If R is an isolated singularity with residue field k and $M \neq 0$ is a finitely generated R-module which is locally free on the punctured spectrum, then* $\mathsf{thick}\{k, M\} = \mathsf{thick}\{R/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Supp} M\}$.

This paper is organized as follows. In Section 2, we give the precise definitions of thick, narrow and solid subcategories, and prove the above theorem and two corollaries. In Section 3, as further applications of the above theorem, we give answers to the question asking when a solid subcategory is Serre.

We close the section by stating our convention.

Convention. Throughout the remainder of this paper, we assume that all rings are commutative and noetherian, that all modules are finitely generated, and that all subcategories are nonempty and strictly full. Let R be a (commutative noetherian) ring. Denote by $\mathsf{mod}\,R$ the category of (finitely generated) R-modules. Whenever we are concerned with R, we take $\mathsf{mod}\,R$ as the ambient abelian category.

2. Our theorem and direct applications

In this section, we state and prove the main result of this paper, and provide immediate applications. We first recall the definitions of thick and narrow subcategories, and give that of solid subcategories.

Definition 2.1. Let $\mathcal{A}$ be an abelian category, and let $\mathcal{X}$ be a subcategory of $\mathcal{A}$.

(1) We say that $\mathcal{X}$ is *thick* if it satisfies the following two conditions.
  (a) $\mathcal{X}$ is closed under direct summands, that is, all direct summands of objects in $\mathcal{X}$ are also in $\mathcal{X}$.
  (b) $\mathcal{X}$ satisfies the 2-out-of-3 property, that is, for any short exact sequence $0 \to L \to M \to N \to 0$ in $\mathcal{A}$, if two of the objects L, M and N belong to $\mathcal{X}$, then so does the third.

(2) We say that $\mathcal{X}$ is *narrow* if it satisfies the following two conditions.
  (a) $\mathcal{X}$ is closed under extensions, namely, for every short exact sequence $0 \to L \to M \to N \to 0$ in $\mathcal{A}$, if L and N belong to $\mathcal{X}$, then so does M.
  (b) $\mathcal{X}$ is closed under cokernels, namely, for every exact sequence $L \to M \to N \to 0$ in $\mathcal{A}$, if L and M belong to $\mathcal{X}$, then so does N.

(3) We say that $\mathcal{X}$ is *solid* if it satisfies the two conditions below.
  (a) $\mathcal{X}$ is closed under direct summands and extensions.
  (b) $\mathcal{X}$ is closed under cokernels of monomorphisms, namely, for each short exact sequence $0 \to L \to M \to N \to 0$ in $\mathcal{A}$, if L and M are in $\mathcal{X}$, then so is N.

(4) By $\mathsf{thick}\,\mathcal{X}$ and $\mathsf{solid}\,\mathcal{X}$ respectively we denote the *thick closure* and the *solid closure* of $\mathcal{X}$ in $\mathcal{A}$, i.e., the smallest thick and solid subcategories of $\mathcal{A}$ containing $\mathcal{X}$.

Remark 2.2. (1) As properties of subcategories of an abelian category, the implications

$$\text{thick} \implies \text{solid} \impliedby \text{narrow}$$

hold. Indeed, the former implication is obvious, while the latter follows from the fact that closedness under extensions and cokernels implies closedness under direct summands, which is shown by splicing the exact sequences $M \oplus N \to N \to 0$ and $0 \to N \to M \oplus N \to M \to 0$ in the abelian category.

(2) For each subcategory $\mathcal{X}$ of an abelian category, the equality

$$\mathsf{thick}(\mathsf{solid}\,\mathcal{X}) = \mathsf{thick}\,\mathcal{X}$$

holds. In fact, the inclusion $\mathcal{X} \subseteq \mathsf{solid}\,\mathcal{X}$ induces the inclusion $\mathsf{thick}\,\mathcal{X} \subseteq \mathsf{thick}(\mathsf{solid}\,\mathcal{X})$. The inclusion $\mathsf{solid}\,\mathcal{X} \subseteq \mathsf{thick}\,\mathcal{X}$ that comes from (1) induces the inclusion $\mathsf{thick}(\mathsf{solid}\,\mathcal{X}) \subseteq \mathsf{thick}\,\mathcal{X}$.

(3) Let M be an R-module. Then one has the inclusion

$$\mathsf{solid}\{M\} \subseteq \mathsf{solid}\{R/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Supp} M\}.$$

In fact, there is a filtration $0 = M_0 \subsetneq \cdots \subsetneq M_n = M$ of submodules of M such that each subquotient M_i/M_{i-1} is isomorphic to $R/\mathfrak{p}_i$ for some $\mathfrak{p}_i \in \operatorname{Spec} R$. Then the $\mathfrak{p}_i$ are all in $\operatorname{Supp} M$, while the exact sequences $0 \to M_{i-1} \to M_i \to R/\mathfrak{p}_i \to 0$ show that M belongs to the solid closure of $\{R/\mathfrak{p}_i\}_{1 \le i \le n}$.

(4) (i) Let $\mathcal{A}$ be an abelian category with enough injective (resp. projective) objects. Recall that a subcategory of $\mathcal{A}$ is said to be *coresolving* (resp. *resolving*) if it contains all the injective (resp. projective) objects of $\mathcal{A}$, and is closed under direct summands, extensions and cokernels of monomorphisms (resp. kernels of epimorphisms). By definition, every coresolving subcategory is solid.

(ii) Set $(-)^* = \mathrm{Hom}_R(-, R)$. Suppose that R is an artinian Gorenstein ring and $\mathcal{X}$ is a resolving (resp. thick) subcategory of $\mathsf{mod}\, R$. Then the subcategory of $\mathsf{mod}\, R$ consisting of modules of the form X^* with $X \in \mathcal{X}$ is coresolving (resp. thick). This follows from the fact that $(-)^*$ gives a duality of $\mathsf{mod}\, R$.

(iii) A solid subcategory of an abelian category $\mathcal{A}$ is not necessarily thick even when $\mathcal{A} = \mathsf{mod}\, R$. Indeed, let R be an artinian Gorenstein local ring which does not satisfy the so-called uniform Auslander condition (UAC); such a ring exists by [5, Theorem in §0]. Then by [3, Proposition 6.1] there exists a non-thick resolving subcategory of $\mathsf{mod}\, R$. It follows from the above (i) and (ii) that there exists a non-thick solid subcategory of $\mathsf{mod}\, R$.

In order to prove our theorem, the following two lemmas are necessary. The first one is stated in [11, Lemma 3.1], but its proof contains a gap in the induction step which can easily be corrected by experts. As for the second one, the version where solidity is replaced with thickness is stated in [11, Lemma 2.3(5)] without a proof. For the convenience of the general reader, we give proofs of those two lemmas.

Lemma 2.3. *Let $\mathfrak{p}$ be a prime ideal of R with height n such that the local ring $R_\mathfrak{p}$ is regular. Then there exists an exact sequence $0 \to R/(\boldsymbol{x}) \to R/\mathfrak{p} \oplus R/\mathfrak{q} \to R/\mathfrak{r} \to 0$ of R-modules, where $\boldsymbol{x} = x_1, \dots, x_n$ is a sequence of elements of R such that $\mathrm{ht}(\boldsymbol{x}) = n$, $\mathfrak{q}$ is an ideal of R, and $\mathfrak{r}$ is an ideal of R which strictly contains $\mathfrak{p}$.*

Proof. We claim that there is a sequence $\boldsymbol{x} = x_1, \dots, x_n$ of elements in $\mathfrak{p}$ with $\mathrm{ht}(\boldsymbol{x}) = n$ and $\boldsymbol{x}R_\mathfrak{p} = \mathfrak{p}R_\mathfrak{p}$. Indeed, if $n = 0$, then $R_\mathfrak{p}$ is a field, and $\mathfrak{p}R_\mathfrak{p} = 0$. Let $n \geq 1$. As $\mathfrak{p}$ has positive height, it is not contained in any $\mathfrak{q} \in \mathrm{Min}\, R$. Also, if $\mathfrak{p} \subseteq \mathfrak{p}^2R_\mathfrak{p} \cap R$, then $\mathfrak{p}R_\mathfrak{p} = \mathfrak{p}^2R_\mathfrak{p}$, which implies $\mathfrak{p}R_\mathfrak{p} = 0$ by Nakayama's lemma, and we get a contradiction. Hence $\mathfrak{p} \nsubseteq \mathfrak{p}^2R_\mathfrak{p} \cap R$. By prime avoidance, we can choose an element $x_1 \in \mathfrak{p}$ with $x_1 \notin (\bigcup_{\mathfrak{q} \in \mathrm{Min}\, R} \mathfrak{q}) \cup (\mathfrak{p}^2R_\mathfrak{p} \cap R)$. Krull's height theorem implies $\mathrm{ht}(x_1) = 1$, and the image of x_1 in $R_\mathfrak{p}$ is part of a minimal system of generators of $\mathfrak{p}R_\mathfrak{p}$. As $R_\mathfrak{p}$ is regular, so is $R_\mathfrak{p}/x_1R_\mathfrak{p} = (R/(x_1))_{\mathfrak{p}/(x_1)}$, and we have $\mathrm{ht}\, \mathfrak{p}/(x_1) = \dim R_\mathfrak{p}/x_1R_\mathfrak{p} = \dim R_\mathfrak{p} - 1 = n - 1$. Let $n \geq 2$. Applying the above argument to $\mathfrak{p}/(x_1)$ yields an element $x_2 \in \mathfrak{p}$ which is outside any $\mathfrak{q} \in \mathrm{Min}_R\, R/(x_1)$ and whose image in $R_\mathfrak{p}/x_1R_\mathfrak{p}$ is part of a minimal system of generators of $\mathfrak{p}R_\mathfrak{p}/x_1R_\mathfrak{p}$. We see that $\mathrm{ht}(x_1, x_2) = 2$ and the image of the sequence x_1, x_2 in $R_\mathfrak{p}$ is part of a minimal system of generators of $\mathfrak{p}R_\mathfrak{p}$. Iterating this procedure if $n \geq 3$, we finally choose a sequence $\boldsymbol{x} = x_1, \dots, x_n$ in $\mathfrak{p}$ such that $\mathrm{ht}(\boldsymbol{x}) = n$ and the image of $\boldsymbol{x}$ in $R_\mathfrak{p}$ is part of a minimal system of generators of $\mathfrak{p}R_\mathfrak{p}$. Note that $\mathfrak{p}R_\mathfrak{p}$ is the maximal ideal of the n-dimensional regular local ring $R_\mathfrak{p}$, whose minimal number of generators is n. We get $\boldsymbol{x}R_\mathfrak{p} = \mathfrak{p}R_\mathfrak{p}$, and the claim follows.

Choose a sequence $\boldsymbol{x}$ as in the claim. Then $\boldsymbol{x}R_\mathfrak{p} \cap R = \mathfrak{p}$ and $\mathfrak{p} \in \mathrm{Min}_R\, R/(\boldsymbol{x})$. Hence the $\mathfrak{p}$-primary component of the ideal $(\boldsymbol{x})$ of R coincides with $\mathfrak{p}$. Letting $\mathfrak{q}$ be the intersection of the other primary components, we see that there is an equality $(\boldsymbol{x}) = \mathfrak{p} \cap \mathfrak{q}$ and that $\mathfrak{r} := \mathfrak{p} + \mathfrak{q}$ strictly contains $\mathfrak{p}$. The natural exact sequence $0 \to R/\mathfrak{p} \cap \mathfrak{q} \to R/\mathfrak{p} \oplus R/\mathfrak{q} \to R/\mathfrak{p} + \mathfrak{q} \to 0$ completes the proof of the lemma. ■

Lemma 2.4. *Let $\mathcal{X}$ be a solid subcategory of $\mathsf{mod}\, R$. Let $X = (0 \to X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 \to 0)$ be a complex of R-modules in $\mathcal{X}$. If $\mathrm{H}_i(X)$ belongs to $\mathcal{X}$ for all $1 \leq i \leq n$, then so does $\mathrm{H}_0(X)$.*

Proof. For each integer i, there are exact sequences

$$0 \to B_i \to Z_i \to H_i \to 0, \qquad 0 \to Z_i \to X_i \to B_{i-1} \to 0,$$

where Z_i, B_i, H_i are respectively the ith cycle, the ith boundary and the ith homology of X. By assumption, X_i is in $\mathcal{X}$ for all $0 \leq i \leq n$ and H_i is in $\mathcal{X}$ for all $1 \leq i \leq n$. Note that $B_n = 0$ and $Z_0 = X_0$ belong to $\mathcal{X}$. As $\mathcal{X}$ is closed under extensions and cokernels of monomorphisms, we inductively get $H_0 \in \mathcal{X}$. ■

We denote by $\mathrm{Sing}\, R$ the *singular locus* of R, that is, the set of prime ideals $\mathfrak{p}$ of R such that the local ring $R_\mathfrak{p}$ is singular (i.e., nonregular). For an R-module M we denote by $\mathrm{NF}(M)$ the *nonfree locus* of M, which is by definition the set of prime ideals $\mathfrak{p}$ of R such that the $R_\mathfrak{p}$-module $M_\mathfrak{p}$ is nonfree. Now we can prove the theorem below, which is the main result of this paper and the same as Theorem 1.1. The proof of the theorem given here is obtained by making some modifications to that of [11, Theorem 3.3].

Theorem 2.5. *For any R-module M one has the equality*

$$\mathsf{solid}\{R/\mathfrak{p}, M \mid \mathfrak{p} \in \mathrm{Sing}\, R \cup \mathrm{NF}(M)\} = \mathsf{solid}\{R/\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Sing}\, R \cup \mathrm{Supp}\, M\}.$$

Proof. It is immediately seen from Remark 2.2(3) that the inclusion ($\subseteq$) holds. In what follows, we prove the opposite inclusion ($\supseteq$). It suffices to verify that $R/\mathfrak{q}$ is in $\mathcal{X} := \mathsf{solid}\{R/\mathfrak{p}, M \mid \mathfrak{p} \in \operatorname{Sing} R \cup \operatorname{NF}(M)\}$ for all $\mathfrak{q} \in \operatorname{Supp} M$. We show the stronger statement that $R/I \in \mathcal{X}$ for all ideals I of R with $\operatorname{V}(I) \subseteq \operatorname{Supp} M$.

Suppose that this statement does not hold true. Then the set $\mathcal{I}$ of ideals I of R with $\operatorname{V}(I) \subseteq \operatorname{Supp} M$ and $R/I \notin \mathcal{X}$ is nonempty. Since R is noetherian, there exists a maximal element P of $\mathcal{I}$ with respect to the inclusion relation. Here, let us verify that P is a prime ideal of R. Take a filtration $0 = L_0 \subsetneq \cdots \subsetneq L_m = R/P$ of submodules of the R-module R/P such that $L_i/L_{i-1} \cong R/\mathfrak{p}_i$ with $\mathfrak{p}_i \in \operatorname{Spec} R$ for each $1 \le i \le m$. Then we have $\mathfrak{p}_i \in \operatorname{Supp}_R R/P = \operatorname{V}(P)$, so that $P \subseteq \mathfrak{p}_i$ and $\operatorname{V}(\mathfrak{p}_i) \subseteq \operatorname{V}(P) \subseteq \operatorname{Supp} M$. If P is not a prime ideal of R, then each $\mathfrak{p}_i$ strictly contains P, and the maximality of P shows $R/\mathfrak{p}_i \in \mathcal{X}$ for all $1 \le i \le m$, which and Remark 2.2(3) imply that R/P is in $\mathcal{X}$, a contradiction. Thus P is a prime ideal.

Put $n = \operatorname{ht} P$. As $R/P \notin \mathcal{X}$, we must have $P \notin \operatorname{Sing} R$. Thus the localization R_P is a regular local ring. Lemma 2.3 gives an exact sequence $\sigma : 0 \to R/(\boldsymbol{x}) \to R/P \oplus R/Q \to R/J \to 0$, where $\boldsymbol{x} = x_1, \dots, x_n$ is a sequence in R with $\operatorname{ht}(\boldsymbol{x}) = n$ and J is an ideal of R which strictly contains P. We establish two claims.

Claim 1. If N is an R-module such that $\operatorname{Supp} N$ is contained in $\operatorname{V}(P) \setminus \{P\}$, then N belongs to $\mathcal{X}$.

Proof of Claim 1. There is a filtration $0 = N_0 \subsetneq \cdots \subsetneq N_m = N$ of submodules of N such that $N_i/N_{i-1} \cong R/\mathfrak{p}_i$ with $\mathfrak{p}_i \in \operatorname{Spec} R$ for each $1 \le i \le m$. We then have $\mathfrak{p}_i \in \operatorname{Supp} N \subseteq \operatorname{V}(P) \setminus \{P\}$, so that $P \subsetneq \mathfrak{p}_i$ and $\operatorname{V}(\mathfrak{p}_i) \subseteq \operatorname{V}(P) \subseteq \operatorname{Supp} M$. The maximality of P shows $R/\mathfrak{p}_i \in \mathcal{X}$. Remark 2.2(3) implies $N \in \mathcal{X}$. □

Claim 2. For all $i > 0$, the support of the Koszul homology $\operatorname{H}_i(\boldsymbol{x}, M)$ is contained in $\operatorname{Sing} R \cup \operatorname{NF}(M)$.

Proof of Claim 2. Suppose that $\operatorname{Supp} \operatorname{H}_i(\boldsymbol{x}, M)$ is not contained in $\operatorname{Sing} R \cup \operatorname{NF}(M)$ for some positive integer i, and choose a prime ideal $\mathfrak{p} \in \operatorname{Supp} \operatorname{H}_i(\boldsymbol{x}, M)$ with $\mathfrak{p} \notin \operatorname{Sing} R \cup \operatorname{NF}(M)$. Then $R_\mathfrak{p}$ is a regular local ring, $M_\mathfrak{p}$ is isomorphic to $R_\mathfrak{p}^{\oplus \ell}$ for some $\ell \ge 0$, and we have $0 \ne \operatorname{H}_i(\boldsymbol{x}, M)_\mathfrak{p} \cong \operatorname{H}_i(\boldsymbol{x}, M_\mathfrak{p}) \cong \operatorname{H}_i(\boldsymbol{x}, R_\mathfrak{p})^{\oplus \ell}$. In particular, $\ell > 0$ and $\operatorname{H}_i(\boldsymbol{x}, R_\mathfrak{p}) \ne 0$. As the sequence $\boldsymbol{x}$ annihilates $\operatorname{H}_i(\boldsymbol{x}, M)$, the set $\operatorname{Supp} \operatorname{H}_i(\boldsymbol{x}, M)$ is contained in $\operatorname{V}(\boldsymbol{x})$, and hence $\mathfrak{p}$ contains $\boldsymbol{x}$. It holds that $n \ge \operatorname{ht}(\boldsymbol{x} R_\mathfrak{p}) \ge \operatorname{ht}(\boldsymbol{x}) = n$, where the first inequality follows from Krull's height theorem. Hence $\operatorname{ht}(\boldsymbol{x} R_\mathfrak{p}) = n$. Since $R_\mathfrak{p}$ is a regular local ring, $\boldsymbol{x}$ is a regular sequence on $R_\mathfrak{p}$. This implies that $\operatorname{H}_i(\boldsymbol{x}, R_\mathfrak{p}) = 0$ as $i > 0$, which is a contradiction. □

Claim 2 and Remark 2.2(3) deduce that $\operatorname{H}_i(\boldsymbol{x}, M)$ is in $\mathcal{X}$ for all $i > 0$. The Koszul complex $\operatorname{K}(\boldsymbol{x}, M)$ has the form $(0 \to M \to M^{\oplus n} \to \cdots \to M^{\oplus n} \to M \to 0)$, all of whose homogeneous components are in $\mathcal{X}$. Lemma 2.4 yields that $M/\boldsymbol{x}M = \operatorname{H}_0(\boldsymbol{x}, M)$ is in $\mathcal{X}$. The exact sequence σ induces an exact sequence

$$\operatorname{Tor}_1^R(R/J, M) \xrightarrow{f} M/\boldsymbol{x}M \xrightarrow{g} M/PM \oplus M/QM \to M/JM \to 0.$$

Let F and G be the images of the maps f and g, respectively. There are inclusions

$$\operatorname{Supp} F \subseteq \operatorname{Supp} \operatorname{Tor}_1^R(R/J, M) \subseteq \operatorname{V}(J) \supseteq \operatorname{Supp} M/JM \quad \text{and} \quad \operatorname{V}(J) \subseteq \operatorname{V}(P) \setminus \{P\},$$

which and Claim 1 show that F and M/JM are in $\mathcal{X}$. From the exact sequences $0 \to F \to M/\boldsymbol{x}M \to G \to 0$ and $0 \to G \to M/PM \oplus M/QM \to M/JM \to 0$, we see that $M/PM \in \mathcal{X}$. Putting $r = \operatorname{rank}_{R/P} M/PM$, we have $(M/PM)_P \cong \kappa(P)^{\oplus r}$. Since $P \in \operatorname{V}(P) \subseteq \operatorname{Supp} M$, Nakayama's lemma shows $(M/PM)_P \ne 0$, whence $r > 0$. There is an exact sequence $0 \to K \to M/PM \xrightarrow{h} (R/P)^{\oplus r} \to C \to 0$ of R/P-modules such that the localized map h_P is an isomorphism. Let H be the image of h. We see that the supports of K, C are contained in $\operatorname{V}(P) \setminus \{P\}$, and Claim 1 shows $K, C \in \mathcal{X}$. It follows from the exact sequences $0 \to K \to M/PM \to H \to 0$ and $0 \to H \to (R/P)^{\oplus r} \to C \to 0$ that $(R/P)^{\oplus r}$ is in $\mathcal{X}$, and so is its direct summand R/P. This is a contradiction, and the proof of the theorem is completed. ■

Let R be a local ring with maximal ideal $\mathfrak{m}$. We put $\operatorname{Spec}_0 R = \operatorname{Spec} R \setminus \{\mathfrak{m}\}$; this is called the *punctured spectrum* of R. Recall that R is said to be an *isolated singularity* if it is locally regular on $\operatorname{Spec}_0 R$. We denote by $\mathsf{mod}_0 R$ the subcategory of $\mathsf{mod} R$ consisting of modules that are locally free on $\operatorname{Spec}_0 R$. By definition, every artinian local ring R is an isolated singularity and satisfies $\mathsf{mod}_0 R = \mathsf{mod} R$.

From now on to the end of this section, we provide a couple of direct applications of the above theorem. We begin with the following corollary. In the situation of the first assertion of the corollary, the specialized term $\operatorname{NF}(M)$ in the left-hand side of the equality of Theorem 2.5 disappears. In the situation of the second assertion, the terms $\operatorname{Sing} R$ in both sides of the equality of Theorem 2.5 are removed.

Corollary 2.6. (1) *Assume either that M is a maximal Cohen–Macaulay R-module, or that R is singular and $M \in \mathsf{mod}_0 R$. One then has* $\mathsf{solid}\{R/\mathfrak{p}, M \mid \mathfrak{p} \in \operatorname{Sing} R\} = \mathsf{solid}\{R/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Sing} R \cup \operatorname{Supp} M\}$.

(2) *Suppose either that R is a regular ring, or that R is an isolated singularity and M is a nonfree R-module. Then there is an equality* $\mathsf{solid}\{R/\mathfrak{p}, M \mid \mathfrak{p} \in \mathrm{NF}(M)\} = \mathsf{solid}\{R/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Supp} M\}$.

Proof. One has $\mathrm{NF}(M) \subseteq \operatorname{Sing} R$ for (1) and $\operatorname{Sing} R \subseteq \mathrm{NF}(M) \subseteq \operatorname{Supp} M$ for (2). Use Theorem 2.5. ■

Next we deduce the following result from Theorem 2.5.

Corollary 2.7. *There is an equality* $\mathsf{mod}\, R = \mathsf{solid}\{R, R/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Sing} R\}$. *If R is an isolated singularity with residue field k and $0 \neq M \in \mathsf{mod}_0\, R$, then the equality* $\mathsf{solid}\{k, M\} = \mathsf{solid}\{R/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Supp} M\}$ *holds.*

Proof. We prove the first and second assertions of the corollary in (1) and (2) below, respectively.

(1) Let $M = R$ in Theorem 2.5. Since $\mathrm{NF}(R) = \emptyset$ and $\operatorname{Supp} R = \operatorname{Spec} R$, we have $\mathsf{solid}\{R/\mathfrak{p}, R \mid \mathfrak{p} \in \operatorname{Sing} R\} = \mathsf{solid}\{R/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R\}$. By Remark 2.2(3) the right-hand side coincides with $\mathsf{mod}\, R$.

(2) The equality $\mathsf{solid}\{R/\mathfrak{p}, M \mid \mathfrak{p} \in \operatorname{Sing} R \cup \mathrm{NF}(M)\} = \mathsf{solid}\{R/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Sing} R \cup \operatorname{Supp} M\}$ follows by Theorem 2.5, while $\operatorname{Sing} R \cup \mathrm{NF}(M) \subseteq \{\mathfrak{m}\} \subseteq \operatorname{Supp} M$ by assumption. The right-hand side of the equality is $\mathsf{solid}\{R/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Supp} M\}$. The left-hand side is $\mathsf{solid}\{M\}$ if $\operatorname{Sing} R \cup \mathrm{NF}(M) = \emptyset$, and is $\mathsf{solid}\{k, M\}$ if $\operatorname{Sing} R \cup \mathrm{NF}(M) = \{\mathfrak{m}\}$. Let us consider the former case. Then R is regular and M is free. The R-module k has finite projective dimension. As M is nonzero, k is in $\mathsf{solid}\{M\}$. Hence $\mathsf{solid}\{M\} = \mathsf{solid}\{k, M\}$. ■

By Corollary 2.7 (and Remark 2.2(2)), we immediately recover the two results below due to Schoutens [7, Theorem VI.8], Krause and Stevenson [6, Proposition 9], and Takahashi [11, Theorem 1.1(i)].

Corollary 2.8 (Schoutens, Krause–Stevenson)**.** *There is an equality* $\mathsf{mod}\, R = \mathsf{thick}\{R, R/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Sing} R\}$.

Corollary 2.9 (Takahashi)**.** *Let R be an isolated singularity with maximal ideal $\mathfrak{m}$ and residue field k. Let $0 \neq M \in \mathsf{mod}_0\, R$. Then the equality* $\mathsf{thick}\{k, M\} = \mathsf{thick}\{R/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Supp} M\}$ *holds.*

Following [2], we say that a module N over a local ring S is *deep* if $\operatorname{depth} N \geq \operatorname{depth} S$ holds. Using this notion, we can define the *nondeep locus* $\mathrm{ND}(M)$ of each module M over a ring R as the set of prime ideals $\mathfrak{p}$ of R such that the module $M_\mathfrak{p}$ over the local ring $R_\mathfrak{p}$ is not deep. It is evident that $\mathrm{ND}(M) \subseteq \mathrm{NF}(M)$, so that $\operatorname{Sing} R \cup \mathrm{ND}(M) \subseteq \operatorname{Sing} R \cup \mathrm{NF}(M)$. This gives rise to the first inclusion in the following, while the second one is the obvious part of Theorem 2.5; see the first sentence in the proof of the theorem.

$$\mathsf{solid}\{R/\mathfrak{p}, M\}_{\mathfrak{p} \in \operatorname{Sing} R \cup \mathrm{ND}(M)} \subseteq \mathsf{solid}\{R/\mathfrak{p}, M\}_{\mathfrak{p} \in \operatorname{Sing} R \cup \mathrm{NF}(M)} \subseteq \mathsf{solid}\{R/\mathfrak{p}\}_{\mathfrak{p} \in \operatorname{Sing} R \cup \operatorname{Supp} M}.$$

The following corollary says that the above two inclusions are actually equalities. Thus, the corollary can be regarded as a refinement of Theorem 2.5.

Corollary 2.10. *Let M be an R-module. Then there is an equality* $\mathsf{solid}\{R/\mathfrak{p}, M \mid \mathfrak{p} \in \operatorname{Sing} R \cup \mathrm{ND}(M)\} = \mathsf{solid}\{R/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Sing} R \cup \operatorname{Supp} M\}$.

Proof. By the arguments discussed before the corollary and Theorem 2.5, it suffices to show that $\operatorname{Sing} R \cup \mathrm{ND}(M)$ contains $\operatorname{Sing} R \cup \mathrm{NF}(M)$. Assume that there is a prime ideal $\mathfrak{p} \in \operatorname{Sing} R \cup \mathrm{NF}(M)$ with $\mathfrak{p} \notin \operatorname{Sing} R \cup \mathrm{ND}(M)$. Then $R_\mathfrak{p}$ is a regular local ring, and $M_\mathfrak{p}$ is a nonfree deep $R_\mathfrak{p}$-module. The Auslander–Buchsbaum formula implies that $M_\mathfrak{p}$ is a free $R_\mathfrak{p}$-module, a contradiction. The proof is completed. ■

3. Applications to a basic question

In this section, we provide further applications of our Theorem 2.5. We begin with stating two lemmas on the subcategory $\mathsf{mod}_0\, R$ of $\mathsf{mod}\, R$. The first one gives criteria for the equality $\mathsf{mod}_0\, R = \mathsf{mod}\, R$.

Lemma 3.1. *Let R be a local ring with maximal ideal $\mathfrak{m}$ and residue field k. The following are equivalent.*

(1) *One has the equality* $\mathsf{mod}_0\, R = \mathsf{mod}\, R$.
(2) *The local ring R is locally a field on the punctured spectrum* $\operatorname{Spec}_0 R$.
(3) *The local ring R is an isolated singularity of dimension at most one.*

Proof. The implications (3) ⇒ (2) ⇒ (1) obviously hold. Assume that $\dim R \geq 2$. Then there exist prime ideals $\mathfrak{p} \subsetneq \mathfrak{q} \subsetneq \mathfrak{m}$. If $(R/\mathfrak{q})_\mathfrak{q}$ is $R_\mathfrak{q}$-free, then $\mathfrak{q}R_\mathfrak{q} = 0$ and $\mathfrak{q} \in \operatorname{Min} R$, which is a contradiction. Hence $R/\mathfrak{q}$ is not in $\mathsf{mod}_0\, R$. Suppose $\mathsf{mod}_0\, R = \mathsf{mod}\, R$. Then for each $\mathfrak{p} \in \operatorname{Spec}_0 R$, the $R_\mathfrak{p}$-module $(R/\mathfrak{p})_\mathfrak{p}$ is free. This implies $\mathfrak{p}R_\mathfrak{p} = 0$, which means that $R_\mathfrak{p}$ is a field. We have shown the implication (1) ⇒ (3). ■

For a subset Φ of $\operatorname{Spec} R$, we denote by $\mathrm{IPD}^{-1}\, \Phi$ the subcategory of $\mathsf{mod}\, R$ consisting of modules that are locally of finite projective dimension outside Φ. Note that $\mathrm{IPD}^{-1}\, \Phi$ is a thick subcategory of $\mathsf{mod}\, R$.

Lemma 3.2. *Let R be a d-dimensional local ring with maximal ideal $\mathfrak{m}$ and residue field k. Then:*
(1) *One has the equality* $\mathrm{IPD}^{-1}\{\mathfrak{m}\} = \mathsf{thick}\{k, R\}$.
(2) *If $d \le 1$, then the equality* $\mathsf{mod}_0 R = \mathsf{thick}\{k, R\}$ *holds and in particular,* $\mathsf{mod}_0 R$ *is a thick subcategory.*

Proof. (1) Clearly, $\mathrm{IPD}^{-1}\{\mathfrak{m}\}$ contains k and R, whence it contains $\mathsf{thick}\{k, R\}$. Let M be an R-module in $\mathrm{IPD}^{-1}\{\mathfrak{m}\}$. Then M locally has finite projective dimension on $\mathrm{Spec}_0 R$. By the Auslander–Buchsbaum formula the dth syzygy N of M is locally free on $\mathrm{Spec}_0 R$, that is, $N \in \mathsf{mod}_0 R$. The first equality in [10, Corollary 4.3(3)] shows that $\mathsf{mod}_0 R$ is contained in $\mathsf{thick}\{k, R\}$. Thus N is in $\mathsf{thick}\{k, R\}$, and so is M.

(2) It is obvious that $\mathsf{mod}_0 R$ is contained in $\mathrm{IPD}^{-1}\{\mathfrak{m}\}$, which coincides with $\mathsf{thick}\{k, R\}$ by (1). Let M be an R-module which does not belong to $\mathsf{mod}_0 R$. Then $M_{\mathfrak{p}}$ is not $R_{\mathfrak{p}}$-free for some $\mathfrak{p} \in \mathrm{Spec}_0 R$. We must have that $d = 1$ and $\mathfrak{p} \in \mathrm{Min}\, R$. Over the artinian local ring $R_{\mathfrak{p}}$, finite projective dimension is equivalent to freeness. Thus M is not in $\mathrm{IPD}^{-1}\{\mathfrak{m}\}$, and we conclude that $\mathsf{mod}_0 R = \mathrm{IPD}^{-1}\{\mathfrak{m}\} = \mathsf{thick}\{k, R\}$. ∎

Next, we recall the definition of a Serre subcategory of $\mathsf{mod}\, R$.

Definition 3.3. A subcategory $\mathcal{X}$ of $\mathsf{mod}\, R$ is said to be *Serre* provided that it is closed under subobjects, quotients and extensions, that is to say, that for each short exact sequence $0 \to L \to M \to N \to 0$ of R-modules, one has $M \in \mathcal{X}$ if and only if $L, N \in \mathcal{X}$.

For a subcategory $\mathcal{X}$ of $\mathsf{mod}\, R$, let $\mathrm{Supp}\, \mathcal{X}$ denote the union of the supports of modules in $\mathcal{X}$. For a subset Φ of $\mathrm{Spec}\, R$, denote by $\mathrm{Supp}^{-1} \Phi$ the subcategory of $\mathsf{mod}\, R$ consisting of modules whose supports are contained in Φ. It is seen that $\mathrm{Supp}\, \mathcal{X}$ is specialization-closed and $\mathrm{Supp}^{-1} \Phi$ is Serre. A well-known theorem of Gabriel [4] asserts that the assignments $\mathcal{X} \mapsto \mathrm{Supp}\, \mathcal{X}$ and $\Phi \mapsto \mathrm{Supp}^{-1} \Phi$ give a one-to-one correspondence between the Serre subcategories of $\mathsf{mod}\, R$ and the specialization-closed subsets of $\mathrm{Spec}\, R$.

It is evident that a Serre subcategory of $\mathsf{mod}\, R$ is thick. Thus it is natural to ask the following question.

Question 3.4. When is a thick subcategory of $\mathsf{mod}\, R$ a Serre subcategory?

This natural question is studied in [11]. Although he knows no other direct reference for this question, the author has learnt through oral communication that several people have the question in mind.

Theorem 2.5 gives an answer to the more general question asking when a solid subcategory is Serre. For a subcategory $\mathcal{X}$ of $\mathsf{mod}\, R$, we denote the union of the nonfree loci of modules in $\mathcal{X}$ by $\mathrm{NF}(\mathcal{X})$.

Corollary 3.5. *A solid subcategory $\mathcal{X}$ of $\mathsf{mod}\, R$ is Serre if $R/\mathfrak{p} \in \mathcal{X}$ for every $\mathfrak{p} \in \mathrm{Sing}\, R \cup \mathrm{NF}(\mathcal{X})$.*

Proof. It suffices to deduce $\mathcal{X} = \mathrm{Supp}^{-1}(\mathrm{Supp}\, \mathcal{X})$, whose inclusion ($\subseteq$) is clear. To show the opposite inclusion ($\supseteq$), let $M \in \mathrm{Supp}^{-1}(\mathrm{Supp}\, \mathcal{X})$. Take a filtration $0 = M_0 \subsetneq \cdots \subsetneq M_n = M$ of submodules of M such that for each $1 \le i \le n$ the module M_i/M_{i-1} is isomorphic to $R/\mathfrak{p}_i$, where $\mathfrak{p}_i \in \mathrm{Spec}\, R$. We have $\mathfrak{p}_i \in \mathrm{Supp}\, M \subseteq \mathrm{Supp}\, \mathcal{X}$, and find $X_i \in \mathcal{X}$ with $\mathfrak{p}_i \in \mathrm{Supp}\, X_i$. It follow from Theorem 2.5, the assumption of the corollary and the solidity of $\mathcal{X}$ that $R/\mathfrak{p}_i \in \mathsf{solid}\{R/\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Sing}\, R \cup \mathrm{Supp}\, X_i\} = \mathsf{solid}\{R/\mathfrak{p}, X_i \mid \mathfrak{p} \in \mathrm{Sing}\, R \cup \mathrm{NF}(X_i)\} \subseteq \mathcal{X}$. Remark 2.2(3) implies that M belongs to $\mathcal{X}$. Thus we are done. ∎

As a direct consequence of the above corollary, we get the following result.

Corollary 3.6. *Let R be an isolated singularity with maximal ideal $\mathfrak{m}$ and residue field k. Then, a solid subcategory $\mathcal{X}$ of $\mathsf{mod}\, R$ is Serre if $\mathcal{X}$ contains k and is contained in $\mathsf{mod}_0 R$.*

Proof. The assumptions imply that $\mathrm{Sing}\, R \cup \mathrm{NF}(\mathcal{X}) \subseteq \{\mathfrak{m}\}$ and $R/\mathfrak{m} = k \in \mathcal{X}$. Apply Corollary 3.5. ∎

We denote by $\mathsf{fl}\, R$ the subcategory of $\mathsf{mod}\, R$ consisting of R-modules of finite length.

Remark 3.7. (1) The assumption in Corollary 3.6 that R is an isolated singularity cannot be removed. In fact, let R be a 1-dimensional local ring which is not an isolated singularity, and let $\mathcal{X} = \mathsf{mod}_0 R$. Then $\mathcal{X}$ is a thick subcategory of $\mathsf{mod}\, R$ with $k \in \mathcal{X} \subseteq \mathsf{mod}_0 R$ by Lemma 3.2. If $\mathcal{X}$ is Serre, then $\mathcal{X} = \mathsf{mod}\, R$ as $R \in \mathcal{X}$, and get a contradiction from Lemma 3.1. Therefore, $\mathcal{X}$ is not Serre.

(2) Corollary 3.6 should be compared with the following two results from [11, Theorem 1.2].
   (a) Let R be an isolated singularity of dimension at most two. Then a thick subcategory of $\mathsf{mod}\, R$ is Serre if it contains the residue field of the local ring R.
   (b) Let R be a regular local ring of positive characteristic. Let $\mathcal{X}$ be a nonzero thick subcategory of $\mathsf{mod}\, R$ contained in $\mathsf{fl}\, R$ (hence, $\mathcal{X}$ is contained in $\mathsf{mod}_0 R$). Then $\mathcal{X} = \mathsf{fl}\, R$ (hence, $\mathcal{X}$ is Serre).

From now on, we concentrate our attention on the condition imposed on the subcategory $\mathcal{X}$ in Corollary 3.6. First of all, we show that this condition forces the modules belonging to $\mathcal{X}$ to have low dimension.

Proposition 3.8. *Suppose that R is a local ring with maximal ideal $\mathfrak{m}$ and residue field k. Let $\mathcal{X}$ be a solid subcategory of $\operatorname{mod} R$ such that $k \in \mathcal{X} \subseteq \operatorname{mod}_0 R$. Then one has $\dim X \le 1$ for all modules $X \in \mathcal{X}$.*

Proof. Suppose that there exists a module $X \in \mathcal{X}$ such that $\dim X \ge 2$. Let $M = \Gamma_{\mathfrak{m}}(X)$ be the $\mathfrak{m}$-torsion submodule of X. The inclusion map $M \to X$ induces an exact sequence $0 \to M \to X \to N \to 0$. The module M has finite length, and belongs to $\mathcal{X}$ since $\mathcal{X}$ is closed under extensions and contains k. As $\mathcal{X}$ is closed under cokernels of monomorphisms, N is in $\mathcal{X}$. Note that $\dim N = \dim X \ge 2$ and $\operatorname{depth} N \ge 1$. Replacing X with N, we may assume that there exists an X-regular element $x \in \mathfrak{m}$. The exact sequence $0 \to X \xrightarrow{x} X \to X/xX \to 0$ shows that $L := X/xX$ belongs to $\mathcal{X}$. Since $\dim L = \dim X - 1 \ge 1$, there exists a prime ideal $\mathfrak{p} \in \operatorname{Supp} L$ with $\mathfrak{p} \ne \mathfrak{m}$. As L is in $\mathcal{X}$, the localization $L_{\mathfrak{p}}$ is a nonzero free $R_{\mathfrak{p}}$-module, that is, $L_{\mathfrak{p}} \cong R_{\mathfrak{p}}^{\oplus n}$ for some $n > 0$. Since $L = X/xX$ is annihilated by x, so is $L_{\mathfrak{p}}$, so is $R_{\mathfrak{p}}$, and so is $X_{\mathfrak{p}}$. The exact sequence $0 \to X_{\mathfrak{p}} \xrightarrow{x} X_{\mathfrak{p}} \to L_{\mathfrak{p}} \to 0$ shows $X_{\mathfrak{p}} = 0$, which implies $L_{\mathfrak{p}} = 0$, a contradiction. ■

The following corollary is deduced from the above proposition, which says that a stronger conclusion than Corollary 3.6 can be obtained, if we assume that the local ring R is equidimensional and $\dim R \ge 2$, instead of assuming that R is an isolated singularity. Compare the corollary with Remark 3.7(2)(b).

Corollary 3.9. *Let $(R, \mathfrak{m}, k)$ be an equidimensional local ring of dimension at least two. Let $\mathcal{X}$ be a solid subcategory of $\operatorname{mod} R$ containing k and contained in $\operatorname{mod}_0 R$. Then one has $\mathcal{X} = \operatorname{fl} R$.*

Proof. Since $\mathcal{X}$ contains k and is closed under extensions, it contains $\operatorname{fl} R$. Proposition 3.8 implies $\dim X \le 1$ for all $X \in \mathcal{X}$. If suffices to derive a contradiction by assuming that $\dim X = 1$ for some $X \in \mathcal{X}$. There exists a prime ideal $\mathfrak{p} \in \operatorname{Min} X$ such that $\dim R/\mathfrak{p} = 1$. If $\mathfrak{p}$ is in $\operatorname{Min} R$, then by equidimensionality we get $1 = \dim R/\mathfrak{p} = \dim R \ge 2$, which is a contradiction. Hence there exists a prime ideal $\mathfrak{q}$ of R which is strictly contained in $\mathfrak{p}$. As $X \in \operatorname{mod}_0 R$ and $\mathfrak{m} \ne \mathfrak{p} \in \operatorname{Supp} X$, we have $X_{\mathfrak{p}} \cong R_{\mathfrak{p}}^{\oplus n}$ for some $n > 0$. It follows that $X_{\mathfrak{q}} \cong R_{\mathfrak{q}}^{\oplus n} \ne 0$, which says that $\mathfrak{q} \in \operatorname{Supp} X$. This contradicts the fact that $\mathfrak{p} \in \operatorname{Min} X$. ■

Remark 3.10. (1) We cannot remove the assumption in Corollary 3.9 that R is equidimensional, even if we instead assume that R is an isolated singularity. Indeed, let $R = k[\![x, y, z]\!]/(xy, xz)$ with k a field. Putting $\mathfrak{p} = (x)$, $\mathfrak{q} = (y, z)$ and $\mathfrak{m} = (x, y, z)$, we have that $(R, \mathfrak{m}, k)$ is an isolated singularity with $\dim R = 2$ and $\operatorname{Min} R = \{\mathfrak{p}, \mathfrak{q}\}$. Let $\mathcal{X} = \operatorname{Supp}^{-1} \mathrm{V}(\mathfrak{q})$. Then $\mathcal{X}$ is a Serre subcategory of $\operatorname{mod} R$, whence it is solid, and contains k. The R-module $R/\mathfrak{q}$ is not in $\operatorname{fl} R$ but in $\mathcal{X}$. Assume that there exist $X \in \mathcal{X}$ and $\mathfrak{r} \in \operatorname{Spec}_0 R$ such that $X_{\mathfrak{r}}$ is not $R_{\mathfrak{r}}$-free. We then have $\mathfrak{m} \ne \mathfrak{r} \in \operatorname{Supp} X \subseteq \mathrm{V}(\mathfrak{q}) = \{\mathfrak{q}, \mathfrak{m}\}$, which implies $\mathfrak{r} = \mathfrak{q}$. As $R_{\mathfrak{q}}$ is a field, we get a contradiction. Thus $\mathcal{X}$ is contained in $\operatorname{mod}_0 R$.

(2) We discuss the necessity of the assumption in Corollary 3.9 that R has dimension at least two.
  - (a) Suppose that $\dim R = 0$. Let $\mathcal{X}$ be a subcategory of $\operatorname{mod} R$ such that $k \in \mathcal{X} \subseteq \operatorname{mod}_0 R$. Then $\mathcal{X} = \operatorname{fl} R = \operatorname{mod}_0 R = \operatorname{mod} R$, as R is artinian and $\mathcal{X}$ contains k and is closed under extensions.
  - (b) Suppose that $\dim R = 1$, and assume further that R is an isolated singularity. Then there exists an R-module M of infinite length, and one has $\operatorname{mod}_0 R = \operatorname{mod} R$ by Lemma 3.1. It follows that $\mathcal{X} = \operatorname{solid}\{k, M\}$ is a solid subcategory of $\operatorname{mod} R$ with $k \in \mathcal{X} \subseteq \operatorname{mod} R = \operatorname{mod}_0 R$ and $\mathcal{X} \ne \operatorname{fl} R$.

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References

[1] D. J. Benson; S. B. Iyengar; H. Krause, Stratifying modular representations of finite groups, *Ann. of Math. (2)* **174** (2011), no. 3, 1643–1684.
[2] H. Dao; M. Eghbali; J. Lyle, Hom and Ext, revisited, *J. Algebra* **571** (2021), 75–93.
[3] H. Dao; R. Takahashi, The radius of a subcategory of modules, *Algebra Number Theory* **8** (2014), no. 1, 141–172.
[4] P. Gabriel, Des catégories abéliennes, *Bull. Soc. Math. France* **90** (1962), 323–448.
[5] D. A. Jorgensen; L. M. Şega, Nonvanishing cohomology and classes of Gorenstein rings, *Adv. Math.* **188** (2004), no. 2, 470–490.
[6] H. Krause; G. Stevenson, A note on thick subcategories of stable derived categories, *Nagoya Math. J.* **212** (2013), 87–96.
[7] H. Schoutens, Projective dimension and the singular locus, *Comm. Algebra* **31** (2003), no. 1, 217–239.

[8] D. Stanley; B. Wang, Classifying subcategories of finitely generated modules over a Noetherian ring, *J. Pure Appl. Algebra* **215** (2011), no. 11, 2684–2693.
[9] R. Takahashi, Thick subcategories over Gorenstein local rings that are locally hypersurfaces on the punctured spectra, *J. Math. Soc. Japan* **65** (2013), no. 2, 357–374.
[10] R. Takahashi, Reconstruction from Koszul homology and applications to module and derived categories, *Pacific J. Math.* **268** (2014), no. 1, 231–248.
[11] R. Takahashi, Thick subcategories over isolated singularities, *Pacific J. Math.* **291** (2017), no. 1, 183–211.

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